THE ESSENTIALS OF SCHOOL ALGEBRA

WITH ANSWERS

A. B. MAYNE

209

THE ESSENTIALS OF
SCHOOL ALGEBRA

WORKS BY A. B. MAYNE, M.A.

THE ESSENTIALS OF SCHOOL GEOMETRY

COMPLETE	With or without Answers
PARTS I– II	With or without Answers
PARTS I–III	With or without Answers
PARTS III– V	With or without Answers
PARTS IV– V	With or without Answers

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PART III	With or without Answers
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PARTS II–III	With or without Answers

THE ESSENTIALS OF SCHOOL ARITHMETIC

PART I	With or without Answers
PART II	With or without Answers

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OF
SCHOOL ALGEBRA

BY

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PREFACE

This book contains all the material needed in a school course up to, but not including, Permutations, Combinations and the Binomial Theorem. It has been written to meet the requirements both of the ordinary pupil and of the pupil who will specialise later. With this end in view the number of examples is very large and they are so classified that the teacher will have no difficulty in selecting examples to suit any pupil. In Parts I and II parallel sets of examples of equal difficulty are provided, followed immediately by a harder set where necessary. Thus, Exs. 8**a** and 8**b** are parallel sets of equal difficulty, and Ex. 8**c** contains harder examples. It is suggested that the normal pupil should do the **a** exercises together with selected examples from the **c** exercises, and that the **b** exercises should be kept in reserve for extra practice. The weaker pupils may do both the **a** and the **b** exercises, and leave out the **c** exercises ; for really able pupils the number of **c** examples chosen may be considerably increased at discretion. In Part III the **a** and **b** exercises have been combined ; thus, Ex. 96 contains straightforward easy examples, and Ex. 96**c** is harder.

The needs of beginners have constantly been kept in mind. In Part I great care has been taken to choose simple examples illustrating the various principles ; all examples requiring skill in manipulation or involving heavy working have been postponed. In the early chapters the work is based on the pupil's knowledge of Arithmetic and great care has been taken to include only work with which the average pupil may reasonably be expected to be familiar. In particular, the work on symbolical expression, which usually proves so difficult to beginners, has been simplified by excluding all examples which involve the *simplification* of algebraic fractions. Harder questions of this type have been collected together in Ch. XX. This is an important chapter, which may be taken in the

middle of the course, as soon as the corresponding arithmetical processes have been mastered.

Factors have been treated in great detail and special attention is called to the treatment of trinomials. All the standard methods have been given and the work has been so arranged that the teacher is free to choose which method to adopt, but it is strongly urged that the pupil should be taught to *rely* upon the method of splitting up the middle term. Almost all School examining bodies have commented on a widespread failure to factorise quadratics which occur in the solution of problems ; such quadratics often contain large numbers and it seems to be a fact that those who have only been taught to obtain factors of trinomials by inspection often fail when the numbers involved are large. Although it is reasonable to encourage factorisation by inspection in simple cases, it seems essential that a method should be adopted which can be *relied* upon to give the result. The method of splitting up the middle term is a sure and certain method, especially if the preliminary work (see pp. 184-186) is done thoroughly. It is also noticeable that pupils who are accustomed to factorise trinomials by splitting up the middle term very rarely have any difficulty in factorising by grouping terms ; in general, such pupils acquire real confidence in factorisation. There is the further advantage that the method can be applied to the general expression of the second degree in x and y (see p. 312).

All the sections covering Equations, Problems, Graphs, Functionality and Variation are very thoroughly treated, and a choice of method is given in the Logarithm chapter. In the chapter on Series, general series have been introduced first and the traditional special cases—the progressions—follow. In dealing with H.C.F. by the long method (see pp. 325 to 328) stress has been laid on the use of the remainders which occur at each stage ; it is very rarely necessary to complete the traditional process. There are nine sets of Test Papers. Throughout the book stress has been laid on the importance of checking results.

In deciding what work should be included beyond the stage of General Certificate (Ordinary Level), consideration has been given to the needs of the able pupil who will afterwards specialise in other subjects as well as to the needs of the future mathematical

specialist. At the Fifth Form stage it is more important to introduce all able pupils to new ideas than to anticipate Sixth Form specialist work ; it has therefore been thought best to select questions in which the stress is laid on simple applications of new ideas rather than on skill in manipulation. It is hoped that the last chapter will prove a suitable introduction to the ideas of the Calculus, although the notation of the Calculus has not been introduced.

Acknowledgements are due to Mr. T. Grantley Powell, M.A., of the Cowley School, St. Helens, to Mr. Jacob Morgan, M.A., Headmaster of the Boys' County School, Brecon, and to my colleagues, Mr. C. Kingsley Dove, M.A., and Mr. T. Marsden, M.A., B.Sc., for most valuable assistance and suggestions at all stages of the work. Thanks are also due to the Senate of the University of London, the Cambridge Local Examinations Syndicate, the Delegates of the Oxford Local Examinations, the Joint Matriculation Board, the Oxford and Cambridge Schools Examination Board, and the Central Welsh Board for permission to use questions from the printed papers set in their examinations. I also have to thank Messrs. Macmillan for permission to reproduce the tables of Logarithms, Antilogarithms and Square Roots which appear at the end of the book.

TABLE OF CONTENTS

PART I

PART II

PART I

CHAPTER I

THE USE OF LETTERS. GENERALISED ARITHMETIC

1. Algebra in its simplest applications may be regarded as an extension of Arithmetic. In Arithmetic we deal with certain definitions and processes involving numbers, each of which has a definite value. In Algebra, besides the ordinary arithmetical numbers we use symbols which usually have not a single definite value. In some instances the symbols may stand for any numerical values we choose to give them ; in others the value or values of any symbol may be restricted by the conditions of the problem under consideration. The symbols generally used are the letters of our alphabet, $a, b, c, \dots A, B, \dots x, y, z \dots$; the Greek letters $\alpha, \beta, \gamma \dots$ are also occasionally used.

2. In Algebra, in the initial stages, we use all the definitions and processes used in Arithmetic. In the later stages these definitions and processes are extended in such a way as to make them of more general use, so that they may be applied to numbers and quantities which have no place in ordinary Arithmetic. Letters should be used to represent **numbers.** They should **not** be used to represent quantities. Do not say that the length of a pencil is l, but say that its length is l inches or l cm., etc.

3. Notation. The signs $+$, $-$, $\times$, $\div$, $=$ have the same meanings as in Arithmetic.

(i) $8+2=10$ means that by adding the numbers 8 and 2 we obtain 10 as the sum. In Algebra $a+b=c$ means that the **sum** of the two numbers denoted by the symbols a and b is equal to a number denoted by the symbol c. Thus if c stands for 20, a and b may stand for any pair of numbers whose sum is 20, such as 18 and 2, 15 and 5, 11 and 9, $8\frac{1}{2}$ and $11\frac{1}{2}$, and so on

PART I

CHAPTER I

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John has p apples and Henry has q apples; they together have p apples $+ q$ apples. By inserting brackets we can abbreviate this to read $(p+q)$ apples. The contents of a bracket are regarded as a single number.

The above statement is an example of an algebraical result which has a *general* value, and is true for any values we choose to give to p and q.

Thus if $p=4,\ q=5,$ then $p+q=4+5=9$;

if $p=7,\ q=3,$ then $p+q=7+3=10$.

From this we see that *single definite* values can be obtained from the *general* algebraic value. When we say "let $p=4$", we do not mean that p must always have the value 4, but that 4 is the value to be given to p in the particular example we are considering. We may also work with symbols without giving them any particular value; it is with such operations that Algebra is chiefly concerned.

(ii) $9-2=7$ means that the **difference** between 9 and 2 is 7. In Algebra $a-b=c$ means that the difference between the two numbers denoted by the symbols a and b is equal to a number denoted by the symbol c. As above, we may consider the case when $c=20$; then a and b may stand for any pair of numbers whose difference is 20, such as 28 and 8, 35 and 15, $53\frac{1}{3}$ and $33\frac{1}{3}$, and so on.

(iii) $6\times 5=30$ means that by multiplying the numbers 6 and 5 we obtain 30 as the **product.** In Algebra $a\times b=c$ means that the product of the two numbers denoted by the symbols a and b is equal to a number denoted by the symbol c. We may consider the case when $c=60$; then a and b may stand for any pair of numbers whose product is 60, such as 6 and 10, 50 and $1\frac{1}{5}$, and so on.

It is usual in Algebra to abbreviate by leaving out multiplication signs. Thus we abbreviate $4\times q$ to $4q$, and $x\times y$ to xy. **But notice most carefully that 2×3 must not be written 23** (for this means twenty-three); we must leave it as 2×3 or else write 6. The number whose units digit is b and whose tens digit is a is $10a+b$. In Arithmetic 65 means $6\times 10+5$ and $6\frac{2}{3}$ means $6+\frac{2}{3}$; but in Algebra $6N$ *always* means "six times N" or "N times six"

We know that $7\times 8=8\times 7$, that $11\times 13=13\times 11$, and that, when any two numbers are multiplied together, it does not matter which is multiplied by which. In Algebra letters stand for numbers;

hence $t \times 14 = 14 \times t$, and **each of these expressions is written 14t. The number is always placed first.** Also $x \times y = y \times x$, and each expression may be written xy or yx. It is more usual to write xy.

(iv) $12 \div 4 = 3$ means that by dividing 12 by 4 we obtain 3 as the **quotient.** The pupil will now have no difficulty in interpreting the statement $a \div b = c$. In Algebra "a divided by b" is usually written $\frac{a}{b}$ and not $a \div b$. It is sometimes written a/b. The sign / is called the "**solidus**".

4. The following symbols are frequently used:

$>$ means "is greater than";

$<$ means "is less than";

$\sim$ means "the result of taking the smaller number from the larger"; thus $7 \sim 5 = 2$, $8 \sim 13 = 5$;

$\equiv$ means "is identically equal to";

$\doteqdot$ means "is approximately equal to"

$\neq$ means "is not equal to";

$\ngtr$ means "is not greater than";

$\nless$ means "is not less than";

$\therefore$ means "therefore". **Do not confuse it with = ; use the** symbol = as a verb.

Any collection of symbols denoting numbers and operations to be performed on them is called an **algebraic expression.**

5. The pupil cannot make any real progress in Algebra until he has learnt to handle letters as confidently as he handles numbers. The following examples are designed to give the necessary practice. In working them the pupil should notice particularly

(i) the way in which letters may be used just as if they were the ordinary numbers of Arithmetic;

(ii) the points in which algebraical usage differs from that of Arithmetic;

(iii) the way in which a symbol, or collection of symbols, may have different numerical equivalents according to the numerical values the symbols are taken to represent. The process of finding the numerical value of an expression, when the letters it contains stand for given numbers, is called **substitution.**

The following worked examples should be discussed orally :

Example 1. *Find a number which exceeds p by 15.*

If the answer is not immediately obvious, the pupil should first think of a few cases in which p is replaced by an ordinary number. The process of solution in these cases will give the clue required, e.g., if $p=2$, 3, 7, 9, etc., it is easily seen that the required answer is $2+15$, $3+15$, $7+15$, $9+15$, etc. In each case the required answer is the sum of the chosen number and 15, so that the answer to the question is seen to be $p+15$.

Example 2. *How many pence are there in y shillings?*

First consider the cases $y=2$, 4, 7, 11, etc. In each case it will be seen that the required answer is obtained by multiplying the given number by 12. The process is quite general, so that the required answer is $12y$.

Example 3. *What is the meaning of $5x-4$? What is its value, if x stands for 2?*

$5x$ means "multiply x by 5" or "multiply 5 by x".

To obtain the value of $5x-4$, multiply 5 by x and then subtract 4 from the result.

If x stands for 2, $5x-4=5\times 2-4=10-4=6$

EXERCISE 1. a

Use of Symbols

(Many of the following examples may be taken orally)

State in words the following :

1. $5\times 3=15$. **2.** $9>7$. **3.** $2\frac{1}{5}=2\cdot 2$. **4.** $1\cdot 3<1\frac{1}{3}$.

5. $x>5$. **6.** $y<13$. **7.** $s=7$. **8.** $z \not> 1\cdot 8$.

9. $15\cdot 99 \doteqdot 16$. **10.** $x\equiv y$. **11.** $c\neq d$. **12.** $11 \sim 15=4$.

13. $l \not< m$ **14.** $g \doteqdot h$. **15.** $ab=c$. **16.** $\frac{r}{s}\equiv t$.

Write in symbols the following :

17. 9 is greater than 5. **18.** Q is equal to 8.

19. Y is not less than 7. **20.** r is greater than 3.

21. t is not equal to 3. **22.** p and q are identically equal.

23. a is not greater than b.

24. Three times y equals twelve, therefore y equals four.

25. Z plus four equals thirty, therefore Z equals twenty-six

26. X minus twenty equals sixteen, therefore X equals thirty-six.

Write, without multiplication or division signs, the following:

27. $q \times 3$, $3 \times q$, $25 \times c$, $c \times 25$, 25×5, $x \times y$, $y \times x$.

28. $q \div 3$, $3 \div q$, $25 \div c$, $c \div 25$, $x \div y$, $y \div x$.

Correct, if necessary, the following statements:

29. $30 \times 4 = 304$. **30.** $xy \times z = xyz$.

31. $304 \times 7 = 2128$. **32.** $113 \times 7 = 1137$.

State in words the meaning of the following, and *afterwards* state their values if $x=4$, $y=3$, $z=2$:

33. $5x$. **34.** $x+7$. **35.** $\frac{x}{2}$. **36.** $x-1$.

37. $\frac{1}{2}x$. **38.** $\frac{8}{x}$. **39.** $\frac{3x}{4}$. **40.** $\frac{10}{x}$.

41. $\frac{2x}{3}$. **42.** $2xz$. **43.** $2z+x$. **44.** $2x-3z$.

45. xyz. **46.** $\frac{z}{y}$. **47.** $z \sim y$. **48.** $x+yz$.

EXERCISE 1. b

(*Many of the following examples may be taken orally*)

State in words the following:

1. $11 > 9$. **2.** $5 < 6$. **3.** $6 \times 4 = 24$. **4.** $3\frac{2}{5} = 3 \cdot 4$.

5. $2 \cdot 6 < 2\frac{2}{3}$. **6.** $v \not< 1 \cdot 9$. **7.** $y < 11$. **8.** $s = 14$.

9. $q \neq 5$. **10.** $\frac{8}{2} = 4$. **11.** $m \simeq n$. **12.** $c \not< d$.

13. $5 \sim 8 = 3$. **14.** $e \neq f$. **15.** $p \equiv q$. **16.** $cd = f$.

Write in symbols the following:

17. z is not less than 4. **18.** X is equal to 7.

19. 2 is less than 7. **20.** e is not greater than f.

21. q is approximately equal to r.

22. s is not equal to zero. **23.** c is not greater than d.

24. One-third of Z equals four, therefore Z equals twelve.

25. Twice X equals eighteen, therefore X equals nine.

26. Y minus twelve equals eight, therefore Y equals twenty.

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37. $\frac{1}{2}x$.

38. $\frac{8}{x}$.

39. $\frac{3x}{4}$.

40. $\frac{10}{x}$.

41. $\frac{2x}{3}$.

42. $2xz$.

43. $2z + x$.

44. $2x - 3z$.

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Write, without multiplication or division signs, the following:

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28. $r \div 5$, $5 \div r$, $23 \times d$, $d \times 23$, $p \div s$, $s \div p$.

Correct, if necessary, the following statements:

29. $112 \times 3 = 1123$.

30. $207 \times 5 = 1035$.

31. $ab \times c = abc$.

32. $20 \times 3 = 203$.

State in words the meaning of the following, and *afterwards* state their values, if $p = 6$, $q = 5$, $r = 4$.

33. $4p$. **34.** $p + 5$. **35.** $\frac{2p}{3}$. **36.** $\frac{p}{2}$.

37. $p - 4$. **38.** $3p + 1$. **39.** $\frac{1}{3}p$. **40.** $\frac{3p}{5}$.

41. $3q + 2r$. **42.** $\frac{15}{p}$. **43.** pr. **44.** $3qr$.

45. $2pqr$. **46.** $3p - 2q$. **47.** $q + 2pr$. **48.** $\frac{2q - r}{p}$.

EXERCISE 2. a

Generalised Arithmetic

(Many of the following examples may be taken orally)

1. How many pence are equal to (*a*) 3 shillings, (*b*) 7 shillings, (*c*) a shillings, (*d*) x shillings?

2. How many inches are equal to (*a*) 3 feet, (*b*) 7 feet, (*c*) a feet, (*d*) x feet?

3. How many feet are equal to (*a*) 24 inches, (*b*) 96 inches, (*c*) a inches, (*d*) h inches?

4. How many pounds are equal to (*a*) 64 ounces, (*b*) 80 ounces, (*c*) x ounces, (*d*) f ounces?

5. How many minutes are equal to (*a*) 4 hours, (*b*) 7 hours, (*c*) b hours, (*d*) $7m$ hours?

6. How many hours are equal to (*a*) 480 minutes, (*b*) 720 minutes, (*c*) a minutes, (*d*) n minutes?

7. How many gallons are equal to (*a*) 16 pints, (*b*) 56 pints, (*c*) x pints, (*d*) $5q$ pints?

8. How many centimes are equal to (*a*) 7 francs, (*b*) 23 francs, (*c*) q francs, (*d*) r francs?

9. Part of a post, of length 3 feet, is painted black; the rest is green. Find the length of the black portion, if the length of the green is (*a*) 2 feet, (*b*) 9 inches, (*c*) a feet, where $a < 3$.

10. Repeat Ex. 9 for a post of length x feet, where $x > a > 2$.

11. There are two unequal weights in the scale-pans of a weighing machine, the heavier being on the left.

	i	ii	iii	iv	v	vi
Weight in left pan	7 lb.	10 lb.	10 lb	y lb.	a lb.	$2P$ lb.
Weight in right pan	3 lb.	6 lb.	x lb.	4 lb.	b lb.	$3Q$ lb.

In each case find what weight must be placed in the right-hand scale-pan to make them balance.

12. A man walks p miles and then q miles. How many miles has he walked altogether?

13. Find a number which is (i) greater than 6 by 7, (ii) greater than p by q.

14. Find a number which (i) added to 13 gives 20, (ii) added to r gives s.

15. Find a number which (i) taken from 30 leaves 19, (ii) taken from x leaves y.

16. If s is any number, write down (*a*) three times the number, (*b*) the number increased by 7, (*c*) the number diminished by 8, (*d*) the number less than s by 12, (*e*) half of the number, (*f*) two-thirds of the number.

17. How many cauliflowers cost (i) two shillings, if each costs fourpence, (ii) C shillings, if each costs threepence, (iii) five shillings, if each costs D pence, (iv) x shillings, if each costs z pence?

18. How far shall I travel (i) in 3 days, if I travel 24 miles a day, (ii) in 5 days, if I travel k miles a day, (iii) in N days, if I travel 30 miles a day, (iv) in a days, if I travel b miles a day?

19. A man is 24 years old. How old was he (i) y years ago, (ii) last year, (iii) x years ago? Answer the same questions, if the man is z years old.

20. A farmer takes r pigs to market and returns with $2s$ of them. How many has he sold?

21. If x is any whole number, write down (i) the number next below x, (ii) the number next above x.

22. If 12 is greater than a by 7, what is a?

23. If 23 is less than b by 11, what is b?

24. What number must be subtracted from $x+11$ in order to obtain x? If $x+11=19$, what is the value of x?

25. What number must be added to $x-4$ in order to obtain x? If $x-4=7$, what is the value of x?

If x stands for an unknown number, state its value when:

26. $x+3=14$. **27.** $4+x=11$. **28.** $x-3=13$.

29. $23-x=8$. **30.** $14=x+8$. **31.** $33=x-7$.

32. $x\times 5=30$. **33.** $x\times 7=7$. **34.** $8\times x=40$.

35. $x-3=9-3$. **36.** $x+7=6+8$. **37.** $x\div 4=6$.

38. $8=\frac{x}{5}$. **39.** $\frac{x}{7}=6$.

40. A book costs three shillings. What are the receipts, in shillings, if (i) 80, (ii) 3000, (iii) p, (iv) $3q$ copies are sold?

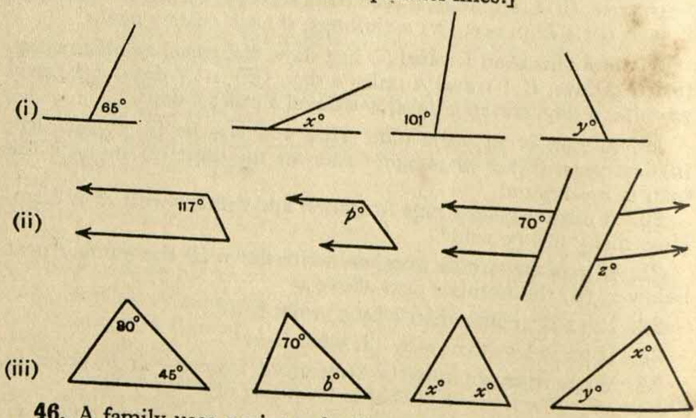
41 If N is any number, write down the results of the following operations: (i) halve the sum of N and 8, (ii) multiply N by 5 and then add 3, (iii) divide N by 3 and subtract the result from 12, (iv) halve N and then subtract 6.

42. A tram-ticket costs 2 pence; what is the cost, in pence, of (i) 6 tickets, (ii) c tickets?

43. I can walk 4 miles an hour. How far can I walk in (i) 2 hours, (ii) x hours?

44. What is the cost of the following: (i) 3 lb. of cheese at k pence per lb., (ii) 2 oz. of rennet at z pence per oz., (iii) c yd. of calico at 3 shillings per yd., (iv) x gallons of milk at w pence per gallon, (v) W dozen pencils at b pence per pencil?

45. Find the sizes of the unmarked angles in the following figures: [The arrowheads denote parallel lines.]



46. A family uses 5 pints of milk a day. How many pints will be used in (i) x days, (ii) y weeks? How long will K pints last?

47. A bookshelf is $4' \, 6''$ long; how many books, each b inches thick, will it hold?

48. Write down expressions for the numbers whose digits in order from left to right are (i) 4, 6, (ii) p, q, (iii) 2, 3, 7, (iv) p, q, r.

EXERCISE 2. b

(Many of the following examples may be taken orally)

1. How many feet are equal to (*a*) 36 inches, (*b*) 84 inches, (*c*) b inches, (*d*) k inches?

2. How many ounces are equal to (*a*) 4 pounds, (*b*) 6 pounds, (*c*) h pounds, (*d*) $3j$ pounds?

3. How many pounds are equal to (*a*) 80 ounces, (*b*) 128 ounces, (*c*) m ounces, (*d*) n ounces?

4. How many shillings are equal to (*a*) 36 pence, (*b*) 96 pence, (*c*) e pence, (*d*) $7k$ pence?

5. How many hours are equal to (*a*) 360 minutes, (*b*) 540 minutes, (*c*) g minutes, (*d*) k minutes?

6. How many pints are equal to (*a*) 4 gallons, (*b*) 7 gallons, (*c*) h gallons, (*d*) $3p$ gallons?

7. How many centimes are equal to (*a*) 6 francs, (*b*) 21 francs, (*c*) s francs, (*d*) w francs?

8. How many francs are equal to (*a*) 400 centimes, (*b*) 750 centimes, (*c*) x centimes, (*d*) $7n$ centimes?

9. A man walks s miles and then t miles. How many miles has he walked altogether?

10. Find a number which is (i) greater than 3 by 9, (ii) greater than a by b.

11. Part of a post, of length 5 feet, is painted blue; the rest is red. Find the length of the red portion, if the length of the blue is (*a*) 3 feet, (*b*) 11 inches, (*c*) b feet, where $b < 5$.

12. Repeat Ex. 11 for a post of length y feet, where $y > b > 3$.

13. A rectangular piece of cloth has its length and breadth given by the table:

	i	ii	iii	iv	v
Length - -	6 ft.	16 ft.	5 yd.	m ft.	d yd.
Breadth - -	2 ft.	15 ft.	4 ft.	n ft.	c in.

Find in each case (i) its perimeter, (ii) its area.

14. How many cabbages cost (i) four shillings, if each costs three pence, (ii) G shillings, if each costs fourpence, (iii) six shillings, if each costs K pence, (iv) P shillings, if each costs Q pence?

15. Find the cost, in pence, of (i) 9 stamps at 4 pence each, (ii) c stamps at 2 pence each, (iii) 3 stamps at d pence each, (iv) l stamps at n pence each.

16. Find a number which (i) taken from 27 leaves 8, (ii) taken from e leaves f.

17. Find a number which (i) added to 11 gives 30, (ii) added to c gives d.

18. How long will it take to travel (i) 200 miles at the rate of 25 miles a day, (ii) 250 miles at the rate of a miles a day, (iii) h miles at the rate of 35 miles a day, (iv) s miles at the rate of t miles a day?

19. A girl is 17 years old. How old was she (i) 3 years ago, (ii) $2k$ years ago? Answer the same question, if the girl is t years old.

20. Any even number may be represented by $2x$, where x represents an integer. Write down (i) the even number next below $2x$, (ii) the even number next above $2x$.

21. A farmer takes l pigs to market and sells m of them. How many has he left?

22. If 19 is less than d by 14, what is d?

23. What number must be added to $x-11$ in order to obtain x? If $x-11=23$, what is the value of x?

24. What number must be subtracted from $x+17$ in order to obtain x? If $x+17=29$, what is the value of x?

25. If 14 is greater than c by 3, what is c?

If x stands for an unknown number, state its value when:

26. $x+7=15$. **27.** $3+x=10$. **28.** $x\times 11=11$.

29. $43=x-17$. **30.** $28-x=18$. **31.** $x\times 3=18$.

32. $18=x+3$. **33.** $x-5=17$ **34.** $x-5=11-5$.

35. $x+9=10+11$. **36.** $10=\dfrac{x}{4}$. **37.** $4\times x=32$.

38. $\dfrac{x}{7}=12$. **39.** $8x=32$.

40. A family uses 8 ounces of sugar per day. How many ounces will be used in (i) k days, (ii) s weeks? How long will z ounces last?

41. A train ticket costs 3 shillings; what is the cost of (i) 4 tickets, (ii) h tickets?

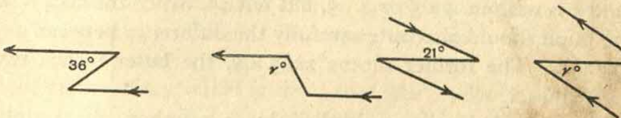
42. If Q is any number, write down the results of the following operations : (i) three times the sum of Q and 4, (ii) divide Q by 3 and then add 7, (iii) multiply Q by 4 and then subtract 6, (iv) double Q and subtract the result from 180.

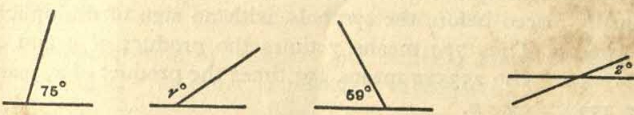
43. Reserved tickets for a football match cost five shillings each. What are the receipts, in shillings, from the sale of (i) 500, (ii) 2000, (iii) x, (iv) $3y$ such tickets?

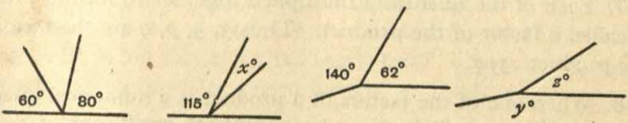
44. What is the cost of the following : (i) 4 lb. of grapes at c shillings per lb., (ii) 6 cwt. of potatoes at t shillings per cwt., (iii) m pints of lemonade at 4 pence per pint, (iv) X feet of rope at l pence per foot, (v) Z dozen nibs at e pence per dozen?

45. A row of bungalows is 120 yards long ; each bungalow is w feet wide. How many bungalows are there in the row?

46. Find the sizes of the unmarked angles in the following figures : [The arrowheads denote parallel lines.]

(i)


(ii)


(iii)


47. Write down expressions for the numbers whose digits in order from left to right are (i) 9, 3, (ii) x, y, (iii) 7, 5, 8, (iv) x, y, z.

48. A man motors at 25 miles per hour. How far does he go in (i) 5 hours, (ii) v hours?

CHAPTER II

PRODUCTS AND POWERS

6. When two or more numbers are multiplied together the result is called the **product.** In Algebra the product of two numbers x and y may be written in any of the forms $x \times y$, $y \times x$, $x \,.\, y$, $y \,.\, x$, xy or yx. The form xy is the most usual. Similarly the product of x, y and z may be written xyz or xzy or yxz or yzx or zxy or zyx, but it is usual to write the factors in alphabetical order, i.e. xyz.

The beginner should note carefully that this differs considerably from the usage in Arithmetic. In Arithmetic the product of 4 and 5 is written 4×5 or $4 \cdot 5$, **but not** 45, which means $4 \times 10 + 5$. The pupil should also note carefully the difference between $2 \,.\, 3 \,.\, x$ and $23x$. The former means $2 \times 3 \times x$, the latter means twenty-three times x.

When symbols are multiplied by a number, the number is usually placed before the symbols, with no sign of multiplication between. Thus $7pq$ means 7 times the product of p and q, or $7 \times p \times q$. Also $253xyz$ means 253 times the product of x, y and z, or $253 \times x \times y \times z$.

7. Each of the quantities multiplied together to form a product is called a **factor** of the product. Thus 3, 5, p, q are the factors of the product $15pq$.

8. When one of the factors of a product is a numerical quantity it is called the **coefficient** or the **numerical coefficient** of the remaining factors. Thus, in the product $7pq$, 7 is the coefficient of pq. Similarly in the product $24abc$, 24 is the coefficient of abc. It is sometimes convenient to consider any factor, or factors, of a product as the coefficient of the remaining factors. Thus, in the product $7pq$, $7p$ is the coefficient of q or $7q$ is the coefficient of p. A coefficient which involves letters is called a **literal coefficient.**

When the coefficient is unity it is usually omitted. Thus we do not write $1x$, but simply x.

9. The product obtained by multiplying together several factors all equal to the same number is called a **power** of that number.

Thus 3×3 is called the **second power** (or **square**) of 3; $5 \times 5 \times 5$ is called the **third power** (or **cube**) of 5; $x \times x \times x \times x \times x$ is called the **fifth power** of x, and so on.

The following notation is used $3 \times 3 = 3^2$; $5 \times 5 \times 5 = 5^3$; $x \times x \times x \times x \times x = x^5$; and the small figure which indicates the number of equal factors is called the **index** of the power.

Thus in 4^2, 7^3, x^5, the **indices** are 2, 3, 5 respectively. 3^2 is usually read "3 squared"; 5^3 is read "5 cubed"; x^5 is read "x to the fifth"; and so on.

The **first power** of a number is the number itself. We do not usually write x^1, but simply x. Thus x, $1x$, x^1, $1x^1$, all have the same meaning. It should be noted that **every power of 1 is 1.**

The pupil must distinguish between **coefficient** and **index.**

10. Fractional coefficients which are greater than unity are usually kept in the form of improper fractions. Thus $\frac{10}{7}xy$ is written more frequently than $1\frac{3}{7}xy$.

If one factor of a product is equal to 0, the product must equal 0, whatever values the other factors have. It follows that every power of 0 is 0. A factor 0 is usually called a zero factor.

11. Any collection of numbers and symbols connected by the signs $+$, $-$, $\times$, $\div$ is called an **algebraic expression.**

Parts of an expression separated by the signs $+$ or $-$ are called **terms.** The signs $\times$ and $\div$ do not separate terms

Thus $3a + 4c \times p - 7k + qr + s \div 2t$ is an expression of five terms. It should be noted that $4c \times p$ is a single term. So are qr and $s \div 2t$.

When no sign precedes a term the sign $+$ is understood.

12. An expression which consists of one term, e.g. $8b$, is called **simple** (or **monomial**) **expression.** An expression which consists of **two or more** terms is called a **compound expression.** An expression of two terms, as $3c - 4d$, is called a **binomial** expression; one of three terms, as $3x + 4y - 2z$, a **trinomial**; one of more than three terms a **multinomial,** or **polynomial.**

A term which consists of the product of a number of letters or numbers, so that only multiplication and neither addition nor subtraction nor division occurs, is called an **integral** term. An expression containing a number of integral terms separated only by the signs $+$ and $-$ is called an **integral algebraic expression.**

An expression in which the letters occur under a root sign (e.g., $\sqrt{x}$, $\sqrt[3]{xy}$) is called an **irrational expression.** If the letters do not occur under a root sign, the expression is called **rational.** We shall be chiefly concerned with expressions which are both rational and integral.

13. In the case of expressions which contain more than one term, each term can be dealt with singly by the rules already given, and by combining the terms the numerical value of the whole expression is obtained, as in Example 4 below. When brackets are used they have the same meaning as in Arithmetic, indicating that the terms enclosed within them are to be considered as one quantity.

Example 1. *If* $a=2$, $b=3$, $c=7$, *find the value of* (i) $2abc$, (ii) $24ab$, (iii) $2 \cdot 4ab$.

$$\begin{aligned}
&\text{(i)}\ \ 2abc = 2 \times a \times b \times c = 2 \times 2 \times 3 \times 7 = 84;\\
&\text{(ii)}\ \ 24ab = 24 \times a \times b = 24 \times 2 \times 3 = 144;\\
&\text{(iii)}\ \ 2 \cdot 4ab = 2 \times 4 \times a \times b = 2 \times 4 \times 2 \times 3 = 48.
\end{aligned}$$

Example 2. *What is the difference in meaning between* $5x$ *and* x^5?

By $5x$ we mean the product of 5 and x. By x^5 we mean the fifth power of x; that is, the product of the quantities x, x, x, x, x. Thus, if $x=2$, $5x = 5 \times x = 5 \times 2 = 10$;

$$x^5 = x \times x \times x \times x \times x = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

Example 3. *What is the difference in meaning between* $6x^2$ *and* $(6x)^2$?

By $6x^2$ we mean the product of 6 and x^2, or the product of 6, x and x.

By $(6x)^2$ we mean the product of $6x$ and $6x$, or the product of 6, x, 6, x, which is the same as the product of 6, 6, x, x, and is equal to $36x^2$.

Similarly $\quad xy^2 = x \times y \times y$; $\quad (xy)^2 = x \times x \times y \times y$;

$$2yz^3 = 2 \times y \times z \times z \times z;$$

$$(2yz)^3 = 2 \times y \times z \times 2 \times y \times z \times 2 \times y \times z,$$

which is the same as $8y^3z^3$.

The pupil should be very careful to notice this difference.

Example 4. *If* $a=2$, $b=3$, $c=0$, *find the value of* $3a^2bc + 2ab^2$.

$$\begin{aligned}
3a^2bc + 2ab^2 &= 3 \times a \times a \times b \times c + 2 \times a \times b \times b\\
&= 3 \times 2 \times 2 \times 3 \times 0 + 2 \times 2 \times 3 \times 3 = 0 + 36 = 36.
\end{aligned}$$

Example 5. *Write in its simplest form* $x \times 3 \times x^2 \times 2 \times y$.

The expression equals $x \times 3 \times x \times x \times 2 \times y$.

As in Arithmetic, we may take the factors in any order we please when multiplying. We therefore write the expression

$$3 \times 2 \times x \times x \times x \times y = 6x^3y.$$

EXERCISE 3. a

(*Many of the following examples may be taken orally*)

1. Explain the difference between (i) 54 and $5 \cdot 4$, (ii) $63pq$ and $6 \cdot 3pq$.

2. Which is the greater, 345 or $3 \cdot 4 \cdot 5$?

3. Write down the product of a, b and c in six ways, without using the sign of multiplication.

4. If a day's work consists of 7 lessons, what is a boy's total mark for the day, if in each lesson he obtains (i) 9 marks, (ii) 0 marks, (iii) s marks?

5. Explain the difference between (i) $4c$ and c^4, (ii) $3c^2$ and $2c^3$.

6. Express algebraically the result of multiplying together (i) 6 factors each equal to h, (ii) q factors each equal to n.

7. Write in their simplest form: (i) $x \times 3 \times z$, (ii) $c \times c \times 7$, (iii) $3a \times 3a$, (iv) $(2b)^3$, (v) $n \times n \times 3n$, (vi) $(3s)^2$, (vii) $x \times (2x)^2$.

8. What is the difference between $3k^2$ and $(3k)^2$ when (i) $k=2$, (ii) $k=0$?

If $a=3$, $b=4$, $c=1$, $q=0$, $x=5$, $y=6$, $z=10$, find the values of:

9. $4a$. **10.** ab **11.** abc. **12.** $4zq$. **13.** $2a^3$. **14.** $\frac{yz}{x}$.

15. $\frac{3}{4}bc$. **16.** $\frac{3}{8}by$. **17.** $\frac{7}{12}ab$. **18.** $\frac{13}{48}qxyz$.

19. $3a^2$. **20.** $2c^4$. **21.** $14c^{12}$. **22.** $5q^3$.

23. $4x^2$. **24.** $7z^3$. **25.** $10a^4$. **26.** $(2c)^2$.

27. $3a^c$. **28.** $4q^a$. **29.** $y^2 - 5xq$. **30.** $z^2 - x^2$.

If $l=2$, $m=\frac{1}{2}$, $n=\frac{1}{3}$, find the values of:

31. $l-m-n$. **32.** lmn. **33.** $l-1-n$.

34. $3l^2+2n^2$. **35.** $6lmn^2$. **36.** $lm+n$.

EXERCISE 3. b

(*Many of the following examples may be taken orally*)

1. Which is the greater, 724 or $7 \cdot 2 \cdot 4$?

2. Write down the product of p, q, r in six ways without using the sign of multiplication.

3. Explain the difference between (i) 72 and 7 . 2, (ii) $85rs$ and $8 . 5rs$.

4. Express algebraically the result of multiplying together (i) 5 factors each equal to k, (ii) x factors each equal to v.

5. Explain the difference between (i) $5d$ and d^5, (ii) $4e^3$ and $3e^4$.

6. How many pints of milk does a man buy in a week, if each day he buys (i) 3 pints, (ii) t pints, (iii) 0 pints?

7. Write in their simplest form: (i) $y \times 4 \times w$, (ii) $4c \times 4c$, (iii) $m \times m \times m \times 3$, (iv) $(3d)^2$, (v) $t \times t \times 4t$, (vi) $y \times (3y)^3$, (vii) $(5t)^3$.

8. What is the difference between $2k^3$ and $(2k)^3$ when (i) $k=1$, (ii) $k=3$, (iii) $k=0$?

If $a=2$, $b=3$, $c=1$, $l=0$, $m=4$, $n=5$, $p=10$, find the values of:

9. am. **10.** lmn. **11.** $4ln$. **12.** $5a$. **13.** $\frac{3}{20}mp$.

14. $\frac{7}{12}lp$. **15.** $\frac{7}{10}mn$. **16.** $\frac{2}{3}ab$. **17.** $\frac{bm}{a}$. **18.** $4n^3$.

19. $6a^4$. **20.** $3p^3$. **21.** $4m^2$. **22.** $3a^4$. **23.** $22c^{17}l^4$.

24. $12c^{27}$. **25.** $14l^b$. **26.** n^2+b^2. **27.** n^3-11pl.

28. $10n^3$. **29.** $(4c)^3$. **30.** $5m^a$.

If $r=3$, $s=\frac{1}{4}$, $t=\frac{1}{2}$, find the values of the following:

31. $r-2s-1$. **32.** $2r^2+4st$. **33.** $rt-2s$.

34. $8r^2st$. **35.** $r-s-t$. **36.** $3rst$.

EXERCISE 3. c

If $a=3$, $b=4$, $c=1$, $q=0$, $x=5$, $y=6$, $z=10$, find the values of:

1. $(2x-y)^2$. **2.** z^2-xy. **3.** $3c^2+4bq$. **4.** z^3-5bx.

5. b^c+c^b. **6.** $(3b-2a)^3$. **7.** $(b-3c)^x$. **8.** c^3+q^3.

9. $3c^a a^c$. **10.** $3abc+qx^2+5q^2yz$. **11.** $2ax+b^2z$.

12. $5b^3$. **13.** $x^2+y^2+z^2$. **14.** a^b+b^a. **15.** $3z^c+4q^2b^2$.

16. $5y^a-3z^2$. **17.** $12abcqxyz$. **18.** $5y^4q^3$. **19.** $\frac{3y^2}{2a^3}$.

20. $\frac{4y^2}{9b^3}$. **21.** $\frac{qy}{30b^4}$. **22.** $\frac{z^b}{x^a}$. **23.** $\frac{25z^2}{b^2x^3}$. **24.** $\frac{3x^bb^2}{z^a}$.

If $l=2$, $m=\frac{1}{2}$, $n=\frac{1}{3}$, find the values of:

25. $l^2+m^2+n^2$. **26.** $l^2+2m^2+3n^2$. **27.** $12lm^2n$.

28. $6lm^2n^2$. **29.** $5-l^2+mn$. **30.** m^l+9n^2.

31. $\frac{l}{mn}$. **32.** $\frac{1}{mn}$. **33.** $\frac{3}{l}-3m$. **34.** $\frac{3}{m}-\frac{2}{n}$.

CHAPTER III

LIKE AND UNLIKE TERMS. ASCENDING AND DESCENDING POWERS

14. When the terms of an algebraical expression either do not differ or differ only in their coefficients they are called **like**; otherwise they are called **unlike**.

Thus $5a$, $9a$; $3a^2$, $8a^2$; $4ab^2$, $7ab^2$ are pairs of like terms; and $5a$, $9b$; $4a^2$, $3ac$ are pairs of unlike terms.

An expression consisting only of like terms can be reduced to a single term.

Example 1. *Simplify the expression* $6x - 3x + 4x$.

The beginner will find it helpful to consider first similar expressions such as the following:

6 baskets − 3 baskets + 4 baskets,
6 eggs − 3 eggs + 4 eggs,
6 shillings − 3 shillings + 4 shillings.

The process of reasoning is the same in each case.

Thus 6 baskets − 3 baskets + 4 baskets = 3 baskets + 4 baskets = 7 baskets. Similarly the other expressions equal 7 eggs and 7 shillings respectively. Likewise $6x - 3x + 4x = 3x + 4x = 7x$.

The argument may be more generally stated as follows: whatever number x stands for, if we take 6 times the number and then subtract 3 times the number, we get 3 times the number, just as 6 tens less 3 tens equals 3 tens. If we then add 4 times the number, we get 7 times the number.

Similarly $\quad 8y + 7y + 11y = 15y + 11y = 26y,$

$$3x^2y^2 + 4x^2y^2 - 5x^2y^2 = 7x^2y^2 - 5x^2y^2 = 2x^2y^2,$$

$$l - \frac{2l}{3} + \frac{4l}{3} = \frac{l}{3} + \frac{4l}{3} = \frac{5l}{3}.$$

When simplifying always work from the left, unless brackets or × and ÷ signs show that operations must be performed in a different order.

EXERCISE 4. a

(*Many of the following examples may be taken orally*)

Simplify the following expressions :

1. $x+x+x+x$.	**2.** $y+y+y+y+y$.	**3.** $4x+2x$.
4. $6p+p$.	**5.** $3m+3m$.	**6.** $3m\times 2$.
7. $3m\times 5$.	**8.** $4s\times 3$.	**9.** Three times $4z$.
10. Five times $2t$.	**11.** Half of $4h$.	**12.** One-quarter of $12X$.
13. One-third of $2Y$.	**14.** $3x+4x$.	**15.** $5w+5w$.
16. $5v+7v$.	**17.** $5t+t$.	**18.** $t+7t$.
19. $3d-2d$.	**20.** $4d-d$.	**21.** $6r-6r$.
22. $2w-w$.	**23.** $x+x+4x$.	**24.** $2z+4z-5z$.
25. $8r-2r-6r$.	**26.** $4w-0-2w$.	**27.** $3x+7x-5x$.
28. $4b+7b-2b-2b$	**29.** $3d+6d-4d+2d$.	**30.** $2p-\frac{1}{2}p$.
31. $h-\frac{1}{4}h$.	**32.** $2k+k\times 5$.	**33.** $3n\times 4-5n$.
34. $10\times 17-8\times 17$.	**35.** $2\times 4d+d$.	**36.** $4r\times 5-5r\times 4$.

EXERCISE 4. b

(*Many of the following examples may be taken orally*)

Simplify the following expressions :

1. $3t+4t$.	**2.** $p+p+p$.	**3.** $w+w+w+w$.
4. $5l\times 2$.	**5.** $7s+s$.	**6.** $4n+4n$.
7. Half of $6c$.	**8.** $3k\times 4$.	**9.** $5v\times 3$.
10. One-fifth of $10Z$.	**11.** Four times $3u$.	**12.** Seven times $3s$.
13. One-sixth of $5X$.	**14.** $7c+c$.	**15.** $2x+9x$.
16. $d+13d$.	**17.** $4l+9l$.	**18.** $d+3d+5d$.
19. $3x-2x$.	**20.** $8s-8s$.	**21.** $8d-d$.
22. $5d-4d$.	**23.** $7c-3c$.	**24.** $7m-3m-4m$.
25. $3w-w+4w$.	**26.** $3b+6b-8b$.	**27.** $5h-0-h$.
28. $3x-\frac{1}{2}x$.	**29.** $\frac{7}{2}d+\frac{9}{2}d$.	**30.** $3t+t\times 4$.
31. $3t+9t-4t-4t$.	**32.** $5v-4v+3v-v$.	**33.** $3v\times 4-4v\times 3$.
34. $4c^2+5c^2-2c^2$.	**35.** $7\times 53+3\times 53$.	**36.** $4s\times 5-6s$.

EXERCISE 4. c

Simplify the following expressions :

1. $47\times 6+47\times 4$.	**2.** $6x^2+3x^2+2x^2$.
3. $4lm-lm$.	**4.** $2cd+3dc$.
5. $3w^2+4w^2-7w^2$.	**6.** $3x^2y+4yx^2$.
7. $a^2b^2+4a^2b^2+9a^2b^2-3a^2b^2$.	**8.** $4c^3d+6c^3d-8c^3d+10c^3d$.

9. $7abcd + 11abcd - 9abcd$. **10.** $4xyz + 3yzx + 2zyx$.

11. There were $20x$ people in a train. At successive stations $2x$, $3x$, $5x$, $2x$ people got out. How many were left?

12. A basket contains $4s$ apples. How many apples are contained in 6 baskets?

13. What is the total bill for d lb. of tea at 2/- per lb. and $2d$ lb. of coffee at 3/- per lb.?

14. A man does no work on Sunday; he works for t hours on Saturday and for $2t$ hours on each of the other days of the week. How many hours does he work each week?

15. There are 18 forms in a school. In each of the 10 junior forms there are $3x$ boys; in each of the remainder there are $2x$ boys. How many boys are there in the school?

16. A man walks $4z$ miles East, then $2z$ miles West, then again $7z$ miles East. How far is he then from his starting point?

17. A man bought five books costing $2n$ shillings each, four costing $3n$ shillings each and eight costing n shillings each. How much did he spend?

18. A man bought 144 pencils at $3x$ pence per dozen, and one dozen blue pencils at x pence each. How much did he spend?

19. A line is divided into 5 pieces measuring respectively $2a$ in., $3a$ in., $7a$ in., $5a$ in., $4a$ in. What is the length of the whole line?

20. A man receives during a week $3x$ shillings, $5x$ florins and $2x$ half-crowns. He then pays out $7x$ shillings and $4x$ half-crowns. How much money has he left? Give the answer in shillings.

15. Addition of unlike terms. When two or more **like** terms are to be added together we have seen that they may be collected and the result expressed as a single term. If, however, the terms are **unlike**, this cannot be done. Consider the sum of the quantities: 4 plums, 5 pears, 3 plums, 2 bananas and 7 pears. We can say that we have 7 plums, 12 pears and 2 bananas, but we cannot carry the process of simplification any further. Similarly in Algebra, the sum of the quantities: $4a$, $5b$, $3a$, $2c$ and $7b$ may be written $4a + 3a + 5b + 7b + 2c$, or $7a + 12b + 2c$.

There is no shorter way of writing this expression, if a, b, c represent any numbers whatever.

Similarly $6c + 2 + c + 7b - 2c - 3b + 5$ may be written

$$7b - 3b + 6c + c - 2c + 2 + 5 = 4b + 5c + 7.$$

This process of simplification is called " collecting like terms "

EXERCISE 5. a

Simplify, if possible, the following expressions. If it is not possible, say so:

1. $a+b+a$. **2.** $a+b+2b+3a$. **3.** $x+3+3x$.
4. $x+y+y+x+y$. **5.** $2c+3-c$. **6.** $a+3b+3a+1$.
7. $2c+d+3d-c$. **8.** $l+m+l$. **9.** $lm+lm+ml$.
10 $lm+2$. **11.** $cd-c$. **12.** $xy+x+y$.
13. $1+a-b-1$. **14.** $a+4+b+5+a$. **15.** $5+3e$.
16. $4x+3x+2$. **17.** $d+4+d+2d+3$. **18.** $s+t-s+2t$.
19. $8k-5k-3$. **20.** $4s-t+?s$. **21.** $z+3-z$.
22. $7m-3m-2$. **23.** $2b+4+3b+4b$. **24.** $4x+2y-2x+y$.
25. $3u-3v$. **26.** $4s+7t-2s-4t$. **27.** $4x\times 5-5y\times 4$.
28. $2\times 2s+3\times 2t-s-4\times t$. **29.** $a+b+c-a-b+c$.
30. $l+4m+4+4m-3$. **31.** $2xy+2xz$.
32. $4xy+3yz-2yx$. **33.** $5ab+4ba$. **34.** $xy+7+y$.
35. $3ab+3b-2ba$. **36.** $4ab+3ba-ac-2ab$.

EXERCISE 5. b

Simplify, if possible, the following expressions. If it is not possible, say so:

1. $2c+d+c+2d$. **2.** $3a+4+2a$. **3.** $u+v+2u$.
4. $r+4s+2r+5$. **5.** $l+m+2m+2l+m$. **6.** $3x+4-x$.
7. $rs+rs+sr$. **8.** $3u+2v+4v-2u$. **9.** $p+2q+p$.
10. $lm+m-l$. **11.** $pq-4$. **12.** $lm-m$.
13. $m+m+5n+4m$. **14.** $7+5h$. **15.** $7k+3k+4$.
16. $u+5+2v-2+3u$. **17.** $3+2x-y-3$. **18.** $2z+5-2z$.
19. $7u+7v$. **20.** $9k-6k-5$. **21.** $5m-2n+3m$.
22. $2m+n-2m+4n$. **23.** $3l+9m-2l-5m$. **24.** $2u+3v-2u$.
25. $7t\times 6-6s\times 7$. **26.** $4m+3n-m-2n$. **27.** $4x-4y$.
28. $r+s+t-r+s+t$. **29.** $3cd+3ce$. **30.** $5cd+4da-3dc$.
31. $2u+3v+4+5v-1$. **32.** $r+s+t-r-s-t$.
33. $3ml+6lm-ls-5lm$. **34.** $5cd+5c-3dc$.
35. $12a^2-5a^2$. **36.** $bc+cd+db-dbc$.

EXERCISE 5. c

Simplify, if possible, the expressions in Nos. 1–20. If it is not possible, say so:

1. $xy+yz+zx+2xyz$. **2.** $14x^2-7x^2$.

3. $5x^2+3x^2+0-2x^2$. **4.** $3y^2-2$.

5. $7u^2-7u^2$. **6.** $2w^2+3w^2-5w^2$. **7.** $7X-3x$.

8. $2u+\frac{u}{3}$. **9.** $x-\frac{x}{4}$. **10.** $\frac{k}{2}+\frac{5k}{2}$.

11. $4x^3-4x$. **12.** $2\times 5st-3\times 3ts$.

13. $4\times 4A+2\times 2A-3\times 3A$. **14.** $12z^3-z^3$.

15. $x^2y+2xy^2+yx^2$. **16.** $3st+2s+3t+2ts$.

17. $ab+bc-ba+ca-cb-ac$. **18.** $4bc-3ac+cb-ab$.

19. $4pqrs+12rsqp-13qrps$. **20.** $xyxy+xxyy+yxxy$.

21. There were $8a+6b$ people in a train. At successive stations $2a$, a, $4b$ people got out. How many were left?

22. What is the total bill for a lb. of tea at 2/- per lb. and b lb. of coffee at 3/- per lb.?

23. A man does no work on Sunday; he works for $c+d$ hours on Saturday and for $2c$ hours on each of the other days of the week. How many hours does he work each week?

24. A man walks $3a$ miles East, then $\frac{3a}{2}$ miles West, then again $2b$ miles East. How far is he then from his starting point?

25. A man bought four books each costing $2l$ shillings, seven each costing $3m$ shillings and nine each costing $4n$ shillings. How much did he spend?

26. A man bought 144 pencils at $2l$ pence per dozen, and two dozen red pencils at m pence each. How much did he spend?

27. A man receives during a week $3x$ shillings, $6y$ florins, and $8z$ half-crowns. He then pays out x florins, $3y$ shillings and $6z$ half-crowns. How much money (in shillings) has he left?

28. What is the length of a fence formed by 8 hurdles each $2r$ feet long and 10 hurdles each $3s$ feet long?

29. Subtract $5x$ pence from $2x$ shillings. Give answer in pence.

30. The sides XY, YZ, ZX of a triangle are respectively $2x$ in., y in., 3 in. What is the perimeter?

31. The two adjacent sides of a rectangle are $(a+4)$ in. and $(b-2)$ in. What is the perimeter?

32. A box, weighing X lb. when empty, contains $5x$ lb. of sugar; if $3x$ lb. of sugar are sold, what is the weight of the box and the remaining sugar? What is the answer if $X=6$, $x=24$?

33. How many pence have I left, if I spend (i) 8 pence out of 3/-, (ii) 7 pence out of x shillings, (iii) t pence out of z shillings?

34. By selling a table for $(2z+7)$ shillings I gain z shillings; how much did it cost me?

35. A taxi-driver is paid $3c$ shillings a week and receives in tips d shillings a day. Out of this he has to pay for oil etc. $c+2d$ shillings a week. If he works each day except Sunday, how much has he left each week?

36. How much change is there out of a ten-shilling note, if s cakes, costing 2 pence each, are bought?

DIMENSION AND DEGREE. ASCENDING AND DESCENDING POWERS

16. The number of times that any particular letter occurs by way of multiplication in any term is called the **dimension** (or **degree**) of the term in that particular letter; and the degree of the term in any specified letters is the sum of its degrees in each of these letters. Thus the product xyz is said to be of **three dimensions** or of the **third degree** in x, y and z, but it is also considered to be of **one dimension** or of the **first degree** in x or y or z considered separately. Similarly, x^2y^4 is said to be of **six dimensions** or of the **sixth degree** in x and y. It is also of **two dimensions** or of the **second degree** in x, and of **four dimensions** or of the **fourth degree** in y.

Notice that a numerical coefficient is not counted; $4x^2y^2$ and x^2y^2 are each of the same degree.

17. The degree of an expression is the degree of the term of highest dimension contained in it; thus $x^4+3x^3+4x^2+7$ is an expression of the fourth degree, for x^4 is of the fourth degree, $3x^3$ is of the third degree, $4x^2$ is of the second degree and 7 is of degree 0.

A term which does not contain x is called the **constant** (or **absolute**) **term** or the **term independent of x.**

Similarly $a^2x^2-4a^3x^3$ is an expression of the sixth degree. But we sometimes speak of the degree of an expression with regard to some one of the letters it involves. Thus the expression

$$ax^3+bx^2+cx+d$$

is said to be of the third degree in x. So $3x^3y+2x^2y^2$ is of the third degree in x or of the second degree in y.

18. Different powers of the **same** letter are **unlike** terms; thus the result of adding together $2x^2$ and $5x$ cannot be expressed by a single term, but must be left in the form $2x^2+5x$. Similarly the sum of $3a^2$, $2ab$ and $4b^2$ is $3a^2+2ab+4b^2$. This expression is in its simplest form and cannot be written in any shorter way.

19. It is usual to write algebraic expressions either in **descending powers** of one of the letters, i.e. beginning with the highest power, then the next highest, and so on, or in **ascending powers,** i.e. beginning with the constant term (if any), then the term of the first degree, then the term of the second degree, and so on. Thus $3x^4+5x^3+7x^2-2x+5$ is arranged in descending powers of x. And $5-2x+7x^2+5x^3+3x^4$ is the same expression arranged in ascending powers of x.

Example 2. *Simplify* $3x^2+4x+7-x^2-2x+3$ *and arrange it in ascending powers of* x.

The expression $=3x^2+4x+7-x^2-2x+3$

$=7+3+4x-2x+3x^2-x^2$, placing the like terms together, beginning with the terms of lowest degree

$=10+2x+2x^2$.

Example 3. *Simplify* $4y^2+5y+8-2y^2-y+1$ *and arrange it in descending powers of* y.

The expression $=4y^2-2y^2+5y-y+8+1$, placing the like terms together, beginning with the terms of highest degree

$=2y^2+4y+9$.

EXERCISE 6. a

Simplify, where possible, the following expressions, and arrange in descending powers :

1. $4x+x^3+2x^2$. **2.** $2t^2+3t-t^2$. **3.** $5a^2-2a^2+a$.

4. $4+6m^2+3m+m^3$. **5.** $7c+c^4+3c^2$. **6.** $5l^2+3l+3-3l^2$.

7. $5r+7-2r+r^4$. **8.** $5+6t+3t^3-5t^2$.

Simplify, where possible, the following expressions, and arrange in ascending powers :

9. $2x^2+5x^4+3x$. **10.** $6m^2+4m+5-m^3$.

11. $-2n+3+n^3$. **12.** $3t^3+7t^2+3-2t^2+t$.

13. $2t+3t^2+5+7t$. **14.** $3r+8-r+r^2$.

15. $5d^2+7+9d-4+d$. **16.** $4c^3-c^2+5c+3-3c$.

EXERCISE 6. b

Simplify, where possible, the following expressions, and arrange in descending powers :

1. $7b^2 - 3b^2 + 2b$. **2.** $3t + t^4 + 2t^2$.
3. $4s^2 + 7s - s^2$. **4.** $5d + 2d^4 + d^3$.
5. $4n^2 + 6n + 5 - 2n^2$. **6.** $2 + 3k^2 + k + 4k^3$.
7. $7s^2 + 5s^3 - 4 + 2s^2$. **8.** $7m + 5 - 3m + m^3$.

Simplify, where possible, the following expressions, and arrange in ascending powers :

9. $-4x + 5 + 3x^2$. **10.** $2z^2 + 9z^4 + 3z^3$.
11. $5t^2 + 3t + 8 - t^3$. **12.** $2c + 5c^2 + 3 + 4c$.
13. $5n + 7 - 2n + n^3$. **14.** $5x^3 + 6x^2 + 2 - 3x^2 + 4x$
15. $6b^3 - 2b^2 + 13b + 7 - 8b$. **16.** $4h^2 + 2 + 7h - 1 + 2h$.

EXERCISE 6. c

Simplify, where possible, the following expressions, and arrange in descending powers :

1. $3t^2 + 2t^3 - 3 + t^2$. **2.** $2x^2 + x + 13 + x^2 + 2x - 5 + 3x^2 - x - 3$.
3. $5a^2 + 12a + 3 + 3a^2 - 4a + 5 + 2a^2 + 3a - 4$.
4. $t^4 + 2t^3 + 2 + 3t - t^3$.

Simplify, where possible, the following expressions, and arrange in ascending powers :

5. $y^3 + 3y^2 + 6y^2 + 7y - 4y + 8 + y^3 + y^2 + 1$.
6. $7 + 2a + 3a^3 - 2 + a^3 + a^4 - 5 + a^5 - a^4 - a^3 + 2a$.
7. Write down (i) in ascending powers, (ii) in descending powers :
(*a*) $4s^4 + 11 - 2s^2 + s$, (*b*) $8t^2 + t^3 + 2 + 5t$,
(*c*) $x^4 + 1 + 13x^3$, (*d*) $x^3 + 3 + 11x^2 - 4x$,
(*e*) $5c^3 - 3c + 7 + 2c^2$, (*f*) $3y^2 + 3 + 8y^3 - 2y$,
(*g*) $3h^3 + 4 + 2h^6$, (*h*) $5 - 7x + 3x^4 - 2x^2$.
8. Write down (i) the coefficient of x^2, (ii) the constant term, (iii) the coefficient of the highest power of x in
(*a*) $4 + 2x + 3x^2 + 7x^3 + 4x^4$, (*b*) $2x^4 + 2x^3 + 12x^2 - 7x$,
(*c*) $x^4 + 7x^2 + 11$, (*d*) $7x^3 + 4x$,
(*e*) $3a^3 + 3x^3 + 2x^2a - 5xa^2$, (*f*) $2x^3 + 6x^2b + 5b^3 + 3xb^2$,
(*g*) $x^3 + 4x^2 + 3x^4 + 11 - 5x$, (*h*) $2x^5 + 7x^3 + 5x$.

CHAPTER IV

SIMPLE EQUATIONS

20. We have seen that in Algebra letters are used to represent numbers. It is sometimes possible to obtain two different expressions to represent the same quantity. This enables us to use Algebra to solve problems,

Example 1. *Think of a number, double it and add* 3 *to it; the result is* 17. *Find the number.*

We might say

$$(\text{Twice the number thought of}) + 3 = 17.$$

The pupil will probably have no difficulty in guessing the answer, but in more difficult questions we shall see that such statements have to be written several times. It is rather tedious to write "the number thought of" every time it is necessary to use it, and it is more convenient to represent the number by a letter. Thus if we denote the number by n, we have, in this example,

$$2 \times n + 3 = 17, \quad \text{or} \quad 2n + 3 = 17.$$

The form in which this statement is now written is called an **equation** and n is called the **unknown.** The process of discovering the unknown number is called **solving the equation**; and the value of the unknown number is called the **root** or **solution** of the equation. The parts of an equation separated by the sign of equality are called **sides** of the equation; in the above equation $2n+3$ is called the **left-hand** side of the equation. Similarly 17 is called the **right-hand** side of the equation.

21. Let us now consider the equation $2n+3=17$. The two sides of the equation represent numbers which are equal; if we subtract 3 from each side the expressions so obtained represent numbers which are equal. We then have

$$2n = 14.$$

If we now divide each side by 2, the expressions so obtained represent numbers which are equal. We then have $n = 7$, which means that the unknown number is 7.

It is easily verified that the answer is correct; thus, if $n=7$, the left-hand side is equal to $2\times 7+3=14+3=17$, which is the same as the right-hand side.

In the equation $2n+3=17$, the value 7, which when substituted for n makes both sides equal, is said to **satisfy** the equation. The object of this chapter is to show how to solve equations in which the quantity whose value is sought (e.g. x) occurs only in the first degree (i.e., simply x, not x^2, $\sqrt{x}$ etc.), when reduced to its simplest form. Such equations are called **simple equations.**

Many of the easier types of equation may be solved by inspection or mental calculation. Thus,

if $x+3=9$, x must stand for 6,
if $s-4=7$, s must stand for 11,
if $3x=15$, x must stand for 5,
if $4z=0$, z must stand for 0,
if $\frac{n}{4}=2$, n must stand for 8,
if $\frac{t}{3}=0$, t must stand for 0.

The pupil has already had instances of such examples without any knowledge of the formal definition of an equation. [See Exercises 2 a and 2 b, Nos. 26–39.]

EXERCISE 7. a (*Oral*)

In the following equations the letters stand for unknown numbers; find the numbers and check the result:

1. $n+5=8$. **2.** $4+p=13$. **3.** $2x=8$. **4.** $5x=30$.

5. $7l=21$. **6.** $14+t=25$. **7.** $m-10=4$. **8.** $y-8=0$.

9. $\frac{x}{5}=2$. **10.** $x-5=3$. **11.** $\frac{z}{2}=8$. **12.** $\frac{t}{6}=\frac{1}{2}$.

13. $3x=0$. **14.** $t+7=7$. **15.** $5w=3$. **16.** $11k=77$.

17. $\frac{a}{6}=0$. **18.** $9+x=25$. **19.** $2t+3t=10$. **20.** $7x-5x=3$.

21. $5v+7v=25+11$. **22.** $9y-3y-2y=18-2$.

23. $4x-2x+3x=9-3+4$. **24.** $\frac{1}{3}+d=6$.

EXERCISE 7. b (*Oral*)

In the following equations the letters stand for unknown numbers; find the numbers and check the result:

1. $3t=9$. **2.** $11+w=16$. **3.** $9k=36$. **4.** $w+3=11$.

5. $7+s=13$. **6.** $11m=55$. **7.** $h-3=8$. **8.** $\frac{c}{4}=7$.

9. $d-11=3$. **10.** $\frac{b}{11}=3$. **11.** $\frac{z}{7}=8$. **12.** $s-15=9$.

13. $x+3=3$. **14.** $15n=7$. **15.** $8u=0$. **16.** $\frac{v}{7}=0$.

17. $k-6=17$. **18.** $7v=105$. **19.** $9t-5t=4$. **20.** $5t+9t=7-7$.

21. $2t+11t=26$. **22.** $\frac{1}{2}+k=5\frac{1}{2}$.

23. $7u-3u+u=7-5+3$. **24.** $11x-3x-6x=17-11$.

22. General methods for solving simple equations. In most of the above examples the unknown number has been easily guessed, but in harder examples it is necessary to work more methodically. The solution of a simple equation depends upon the following axioms:

1. If to equals we add equals, the sums are equal. Thus, if $x=y$, $x+2=y+2$, or $x+a=y+a$.

2. If from equals we take equals, the remainders are equal. Thus, if $x=y$, $x-2=y-2$, or $x-a=y-a$.

3. If equals are multiplied by equals, the products are equal. Thus, if $x=y$, $x\times 5=y\times 5$, or $x\times a=y\times a$.

4. If equals are divided by equals, the quotients are equal. Thus, if $x=y$, $x\div 5=y\div 5$, or $x\div a=y\div a$ (provided that $a\neq 0$).

5. The order of an equation may be reversed. Thus, if $5=y$, we can at once say that $y=5$ without using any of the axioms 1–4 above. Similarly, if $0=3x-15$, we may write at once $3x-15=0$; or if $12-r=6$, we may write at once $6=12-r$.

In the early stages of the work the number a in the axioms 1–4 above always means a number whose value is known. Later it will be necessary to discuss the result of multiplying or dividing both

sides of an equation by an expression containing the unknown. It will then be shown that axioms 3 and 4 above have only a restricted application.

We now give examples of the use of each of the above axioms.

Example 2. *Solve* $x - 5 = 16$.

Since the numbers $x - 5$ and 16 are equal, if we *add* 5 to *each*, the sums are equal; $\therefore\ x - 5 + 5 = 16 + 5$,

$$\therefore\ x = 21.$$

Check. If $x = 21$, the left-hand side equals $21 - 5 = 16$, which is the same as the right-hand side.

Example 3. *Solve* $y + 13 = 35$.

Since the numbers $y + 13$ and 35 are equal, if we *subtract* 13 from *each*, the remainders are equal;

$$\therefore\ y + 13 - 13 = 35 - 13,$$

$$\therefore\ y = 22.$$

Check. If $y = 22$, the left-hand side equals $22 + 13 = 35$, which is the same as the right-hand side

Example 4. *Solve* $\frac{t}{5} = 19$.

Since the numbers $\frac{t}{5}$ and 19 are equal, if we *multiply each* by 5, the products are equal; $\therefore\ \frac{t}{5} \times 5 = 19 \times 5$,

$$\therefore\ t = 95.$$

Check. If $t = 95$, the left-hand side equals $\frac{95}{5} = 19$, which is the same as the right-hand side.

Example 5. *Solve* $11l = 88$.

Since the numbers $11l$ and 88 are equal, if we *divide each* by 11, the quotients are equal; $\therefore\ 11l \div 11 = 88 \div 11$,

$$\therefore\ l = 8.$$

Check. If $l = 8$, the left-hand side equals $11 \times 8 = 88$, which is the same as the right-hand side.

Note 1. In future we shall usually write L.H.S. for the left-hand side, and R.H.S. for the right-hand side.

Note 2. The pupil is reminded that it is necessary to distinguish between the symbols $=$ and $\therefore$. It is wrong to put $=$ instead of $\therefore$ at the beginning of each fresh line in the working of an equation.

Example 6. *Solve* $\frac{x}{3}+14=3x-10$.

Add 10 to each side,

$$\therefore \frac{x}{3}+14+10=3x-10+10,$$

$$\therefore \frac{x}{3}+24=3x.$$

Subtract $\frac{x}{3}$ from each side,

$$\therefore \frac{x}{3}+24-\frac{x}{3}=3x-\frac{x}{3}.$$

[Since x represents a *number*, so does $\frac{x}{3}$. Hence we are subtracting the same *number* from each side, even though we do not at present know what the number is.]

$$\therefore 24=3x-\frac{x}{3}.$$

Multiply each side by 3,

$$\therefore 72=9x-x,$$

$$\therefore 72=8x.$$

Divide each side by 8,

$$\therefore 72\div 8=8x\div 8,$$

$$\therefore 9=x,$$

$$\therefore x=9 \text{ (reversing the order of the equation).}$$

Check. When $x=9$, L.H.S. $=\frac{9}{3}+14=3+14=17$;

R.H.S. $=3\times 9-10=27-10=17$;

$\therefore$ when $x=9$, L.H.S. $=$ R.H.S., i.e. $x=9$ satisfies the equation.

Note. When checking your results substitute for the unknown in each side separately, as in Example 6. **It is essential to substitute in the equation as it is given, not in any simplified form of it,** for in simplifying a mistake may have been made.

EXERCISE 8. a

(*Nos.* 1–4 *may be discussed orally*)

1. What must be done to get rid of the term containing the unknown on the right-hand side of each of the following equations?

(i) $4x = 2x + 6$, (ii) $3x = 16 - 5x$,

(iii) $t = 30 - 2t$, (iv) $7x + 13 = 33 + 2x$.

2. What must be done to get rid of the term containing the unknown on the left-hand side of each of the following equations?

(i) $15 + 2x = 5x$, (ii) $24 - 5x = 3x$,

(iii) $17 - 2t = 3t$, (iv) $10 + 2y = 4y - 8$.

3. What must be done to get rid of the term not containing the unknown on the left-hand side of each of the following equations?

(i) $8x + 17 = 52 + 3x$, (ii) $11x + 4 = 49 - 4x$,

(iii) $8t - 15 = 9 + 2t$, (iv) $7p - 8 = 20 + 3p$.

4. What must be done to get rid of the term not containing the unknown on the right-hand side of each of the following equations?

(i) $5x + 12 = 7x + 6$, (ii) $3y - 15 = 4y - 25$,

(iii) $12t + 18 = 17t - 12$, (iv) $13 - 3t = 2 + 8t$.

Solve the following equations. Explain every step of your work, as in Example 6 above, and check your answers.

5. $7 - 2x = 5 - x$. **6.** $2t + 5 = 11 - t$. **7.** $2t + 7 = 27 - 3t$.

8. $4x - 28 = 12$. **9.** $43 - W = 2W - 5$. **10.** $8x = x$.

11. $\frac{x}{3} - 7 = 11$. **12.** $\frac{s}{7} + 3 = 12$. **13.** $38 + c = 2 + 5c$.

14. $x + \frac{x}{3} = 8$. **15.** $\frac{y}{3} = 1{\cdot}6 - y$. **16.** $8x - 9 = 33 - 4x$.

EXERCISE 8. b

(*Nos.* 1–4 *may be discussed orally*)

1. What must be done to get rid of the term containing the unknown on the right-hand side of each of the following equations?

(i) $5x = 3x + 8$, (ii) $6x = 64 - 10x$,

(iii) $8x + 23 = 38 + 3x$, (iv) $2t = 119 - 5t$.

2. What must be done to get rid of the term containing the unknown on the left-hand side of each of the following equations?

(i) $125 - 7x = 18x$, (ii) $25 + 4x = 9x$,

(iii) $12 - 3y = 4y - 9$, (iv) $22 - 4t = 7t$.

3. What must be done to get rid of the term not containing the unknown on the left-hand side of each of the following equations?

(i) $12x - 11 = 5x + 24$, (ii) $3x + 7 = 28 - 4x$,
(iii) $12x + 11 = 60 + 5x$, (iv) $6z - 3 = z + 22$.

4. What must be done to get rid of the term not containing the unknown on the right-hand side of each of the following equations?

(i) $3x - 11 = 5x - 37$, (ii) $2x + 17 = 5x - 10$,
(iii) $14 - 4t = 1 + 9t$, (iv) $11x + 42 = 19x + 10$.

Solve the following equations. Explain every step of your work, as in Example 6 above, and check your answers.

5. $3y - 7 = 8 - 2y$. **6.** $4z - 5 = 2z - 1$. **7.** $x + 3 = 39 - 3x$.
8. $7y = 3y$. **9.** $3x - 18 = 24$. **10.** $13 - 3y = 2y - 7$
11. $\frac{x}{5} + 2 = 7$. **12.** $44 + 4c = 6 + 7c$. **13.** $\frac{x}{4} - 5 = 9$.
14. $11 - 4x = 11x - 19$. **15.** $2y + \frac{y}{5} = 33$. **16.** $\frac{z}{4} = 3{\cdot}5 - z$.

EXERCISE 8. c

Solve the following equations and check your solutions :

1. $8a = 2a + 18$. **2.** $24 - 5x = x$. **3.** $5r + 4 = 12 + 3r$.
4. $8c - 7 = 21 - 6c$. **5.** $11 - R = 29 - 7R$. **6.** $7X - 5 = 4X + 13$.
7. $3\frac{1}{2} + 3y = 11\frac{1}{2}$. **8.** $9a - 5 = 2a + 7 + 3a$.
9. $18 + 4y - 2y - 7 = 5y - 1$. **10.** $5x + 3 = 2 + 5x + 3x - 8$.
11. $z + 4 - 3z = 4z + 4$. **12.** $Z - 2 + 4Z - 4 = 8 - 2Z$.
13. $0 = 5L - 7 - 8 - 2L$. **14.** $33 = 2x + 7x + 2x$.
15. $x - 2{\cdot}3 = 1{\cdot}4$. **16.** $y + \frac{3}{4} = 2\frac{1}{2}$. **17.** $\frac{t}{3} = \frac{2}{5}$.
18. $\frac{6x}{7} = 3$. **19.** $7\frac{1}{2} = l + 2\frac{3}{4}$. **20.** $5x + 3 = 5\frac{1}{2}$.
21. $\frac{4b}{5} = \frac{2}{3}$. **22.** $\frac{3c}{7} = \frac{1}{3}$. **23.** $\frac{5x}{6} = 0$.
24. $7 = 5X$. **25.** $t - \frac{4t}{7} = 12$. **26.** $\frac{5x}{9} - 2 = 4$.
27. $1 + \frac{7x}{4} = x + 10$. **28.** $2{\cdot}4 = \frac{2n}{3}$. **29.** $8 - \frac{y}{3} = 0$.
30. $5t - 2 = 2$. **31.** $\frac{K}{3} - \frac{1}{4} = 2$. **32.** $\frac{x}{5} = 3 + \frac{5}{6}$.
33. $\frac{2}{3} + \frac{W}{2} = 4$. **34.** $\frac{R}{3} + R = 0$.
35. $\frac{5x}{8} - 3 = 7$. **36.** $1 + \frac{5a}{3} = a + 5$.

CHAPTER V

EASY PROBLEMS LEADING TO SIMPLE EQUATIONS

23. Many problems may be easily solved by the use of Algebra. In the general method we first represent the unknown quantity by a symbol, e.g. x or t, and then use the data so as to obtain two equal expressions containing the unknown. We thus obtain an equation which, if simple, can be solved by the methods already given in the previous chapter. It may happen that the equation so obtained is not a simple equation. The solution of such equations will be considered later.

24. The method of solving problems will best be learnt by considering the following worked examples. The chief difficulty is in translating the words of the question into the language of symbols ; the beginner should observe the following instructions :

1. **Read the question carefully.** Make sure that you understand what you are given and what you are asked to find out.

2. Choose a letter to stand for some unknown **number** which the problem involves. **Do not use letters to represent quantities.** The letter chosen is usually connected with what you are asked to find out, e.g., if you are asked to find a number satisfying certain conditions, it is natural to choose a letter to represent the number sought ; if you are asked to find the cost of an article, it is natural to choose a letter to represent the number of pence or shillings or pounds in the cost.

3. In any problem which involves quantities, **state your units clearly. Carelessness with units is one of the most frequent causes of failure to solve problems.**

4. Always state clearly in words what is represented by the unknown letter, e.g. " Let the cost be £x ", or " Let x pence be the cost of a dozen eggs ", or " Let the length of the room be l feet ", or " Let the required number be n ".

5. If you can find two different expressions for the same thing, you can form an equation by equating these two expressions. If you have difficulty in forming the necessary equation, read through

the question again and make sure that each statement in the question has been translated into symbols.

6. **Check the answer by using the actual data of the problem.** It is not sufficient to check by substituting in the equation, because you may have made a mistake in obtaining the equation. If your check shows that your answer is wrong, try to find your mistake. In particular, make sure that you have not made an error in units, and check carefully all signs.

Example 1. *Find two numbers whose sum is* 65, *and whose difference is* 15.

Let x be the smaller number. Since the difference between the numbers is 15, the larger number must be $x+15$. But their sum is 65. Hence,

$$x+x+15=65,$$
$$\therefore\ 2x+15=65,$$
$$\therefore\ 2x=50,$$
$$\therefore\ x=25;$$

so that the smaller number is 25. It follows that the larger number is 40.

The required numbers are therefore 40 and 25.

The solution should always be tested to see whether it satisfies the conditions of the problem or not. In this instance we have $40+25=65$, and $40-25=15$, so that the conditions are satisfied.

Example 2. *The weight of a box and its contents is* 25 *lb.*; *the box weighs* 7 *lb. more than the contents. Find the weight of the box.*

Let W lb. be the weight of the box. Since the box weighs 7 lb. more than the contents, the contents weigh $(W-7)$ lb.

But the total weight is 25 lb., $\therefore\ W$ lb. $+\,(W-7)$ lb. $=25$ lb.

This is a relation between quantities; from it we obtain the relation between numbers;

$$W+W-7=25;$$
$$\therefore\ 2W-7=25,$$
$$\therefore\ 2W=32,$$
$$\therefore\ W=16;$$

$\therefore$ the weight of the box is 16 lb.

Check. If the weight of the box is 16 lb., the weight of the contents must be 16 lb. − 7 lb., i.e. 9 lb. Thus the total weight is 25 lb. and the solution is correct.

Example 3. *A newsagent sells a certain number of magazines at 6d. each and three times as many newspapers at 1d. each. How many of each has he sold, if his receipts amount to £3?*

The student should at once notice that two different units for money have been used in stating the question. He must therefore express pence in pounds or vice-versa. In this instance it will be more convenient to replace £3 by its equivalent, 720 pence.

Let x be the number of magazines sold. Then the number of newspapers sold is $3x$.

The newsagent receives from the sale of magazines $6x$ pence and from the sale of newspapers $3x$ pence. Hence

$$6x \text{ pence} + 3x \text{ pence} = 720 \text{ pence},$$
$$\therefore\ 6x + 3x = 720,$$
$$\therefore\ 9x = 720,$$
$$\therefore\ x = 80, \quad \text{and} \quad 3x = 240;$$

∴ 80 magazines and 240 newspapers are sold.

Check. For 80 magazines at 6d. the newsagent receives 480 pence, or £2; for 240 newspapers at 1d. he receives 240 pence, or £1. His total receipts are £3. Thus the necessary conditions are satisfied.

In some problems the solver has to supplement the data by making use of his own knowledge. Thus, in the following question a knowledge of the elements of Geometry is required.

Example 4. *The largest angle of a triangle is 20° larger than the smallest. The third angle is 10° larger than the smallest. Find the angles.*

Let the smallest angle be $x°$. Then the largest is $(x+20)°$, and the other angle is $(x+10)°$. But the three angles of a triangle are together equal to 180°,

$$\therefore\ x° + (x+20)° + (x+10)° = 180°,$$
$$\therefore\ x + x + 20 + x + 10 = 180,$$
$$\therefore\ 3x + 30 = 180,$$
$$\therefore\ 3x = 150,$$
$$\therefore\ x = 50;$$

∴ the smallest angle is 50°, the largest is 70° and the other is 60°.

The pupil will have no difficulty in verifying mentally that the solution is correct.

Note. It will sometimes be found easier not to put the unknown equal to the number directly required, but to some other number involved in the question.

Example 5. *A woman spent 5s. 9d. in buying eggs, and finds that* 10 *of them cost as much over one shilling as* 14 *cost under two shillings; how many eggs did she buy?*

If we take x to be the number of eggs bought it will be found that a rather inconvenient equation is obtained. It is better to obtain the required number indirectly.

Let the price of one egg be x pence.

Then 10 eggs cost $10x$ pence, and 14 eggs cost $14x$ pence;

$$\therefore\ 10x - 12 = 24 - 14x,$$

$$\therefore\ 24x = 36,$$

$$\therefore\ x = 1\tfrac{1}{2}.$$

Thus the price of an egg is $1\frac{1}{2}$d., and the number of eggs

$$= 69 \div 1\tfrac{1}{2} = 46.$$

The pupil will have no difficulty in checking the result.

EXERCISE 9. a

It is suggested that the part of the question which is in brackets should be discussed orally before any written work is done.

1. One number exceeds another by 11, and their sum is 37; find them.

[Suppose that the smaller number is n. What is the larger? What is the sum of the two numbers in terms of n?]

2. A number is multiplied by 7 and then 4 is added; the result is 60. Find the number.

3. I think of a number, divide it by 6 and subtract 3; the result is 9. What is the number?

4. I think of a number and add to it one-quarter of itself; the result is 30. What is the number?

5. The result of adding 48 to a certain number is the same as multiplying that number by 5. What is the number?

6. Find two consecutive integers which add up to 85.

[If 24 is the larger of two consecutive integers, what is the smaller? If n is the larger of two consecutive integers, what is (1) the smaller, (2) the sum?]

7. Find two consecutive even integers which add up to 118.

[If 34 is the larger of two consecutive even integers, what is the smaller? If n is the larger of two consecutive even integers, what is the smaller?]

8. Three consecutive integers add up to 117; find the middle one.

[If 13 is the middle of three consecutive integers, what are the other two? If n is the middle of three consecutive integers, what are the other two?]

9. The sum of three consecutive even integers is 54; find them.

10. The sum of four consecutive integers is 42; find them.

EXERCISE 9. b

It is suggested that the part of each question which is in brackets should be discussed orally before any written work is done.

1. One number is smaller than another by 17, and their sum is 49; find them.

[Suppose that the smaller number is n. What is the larger? What is the sum of the two numbers in terms of n?]

2. A number is multiplied by 9 and then 5 is taken away; the result is 49; find the number.

3. A number is divided by 7 and then 5 is taken away; the result is 2. Find the number.

4. The result of subtracting 24 from a certain number is the same as dividing that number by 3. What is the number?

5. I think of a number and subtract from it one-fifth of itself; the result is 12. What is the number?

6. Find two consecutive integers which add up to 77.

[If 15 is the smaller of two consecutive integers, what is the larger? If n is the smaller of two consecutive integers, what is (1) the larger, (2) the sum?]

7. Find two consecutive even integers which add up to 106.

[If 28 is the smaller of two consecutive even integers, what is the larger? If n is the smaller of two consecutive even integers, what is the larger?]

8. Find two consecutive odd integers which add up to 144.

[If 33 is the larger of two consecutive odd integers, what is the smaller? If n is the larger of two consecutive odd integers, what is the smaller?]

9. The sum of three consecutive integers is 66; find them.

10. The sum of four consecutive odd integers is 96; find them.

EXERCISE 9. c

It is suggested that the part of each question which is in brackets should be discussed orally before any written work is done.

1. What number exceeds 25 by the same amount as it falls short of 49?

2. If I divide a certain number by 3 and add 2, the result is the same as subtracting four from the number; find the number.

3. One number is twice another, and five times the smaller added to three times the greater amounts to 143; find the numbers.

4. Divide £79 between A, B, and C, so that A may have £15 more than B, and B £8 more than C.
[Suppose C has £x. How much has B? How much has A? How much have all three together?]

5. A sum of £135 is divided between A, B, and C, so that B has £10 more than A, and C has three times as much as A; find the share of each.

6. A batsman scores 51 runs. He hits seven times as many singles as fours. If he scores 18 by his other hits, find how many fours he hit.

7. A has 15 shillings more than B. Together they have 25 guineas. How much did each have?

8. The perimeter of a rectangular table is 13 feet, and the length is 18 inches more than the width; find the width.
[Fill in the blank: Let the width of the table be ——.]

9. x return and $3x$ single tickets cost £9. A return ticket is 6s., a single 4s. Find x.

10. £1 is divided between two boys A, B so that twice A's share is sixpence more than B's share. How was the money divided?

11. The base angles of an isosceles triangle are $2x^\circ$ and $(x+26)^\circ$; find the angles of the triangle.

12. Two angles of an isosceles triangle are x° and $(x+15)^\circ$. Find the angles of the triangle. [Notice that there are two cases.]

13. In a triangle ABC, $\angle A$ is three times as large as $\angle C$ and 16° more than $\angle B$. Find the angles of the triangle.

14. Three straight lines meet at a point O. The three angles formed are $(6x+12)^\circ$, $(4x+20)^\circ$, $(5x-17)^\circ$. Find the angles.

15. A train starts from X to go to Y, 234 miles away; at the same time a second train starts from Y to go to X by the same route. The first train travels at 56 miles an hour, and the second at 61 miles an hour. After how many hours will the trains meet?

16. An express train starts at 8 a.m. and travels at the rate of 60 miles per hour. At 12 noon an aeroplane sets out from the same place to overtake it, travelling at the rate of 150 miles an hour. Where will it overtake the train?

17. Two cyclists starting at the same time from two towns 147 miles apart meet in 7 hours. Find their rates of riding, given that one rate is three-quarters of the other.

18. At a concert 450 people were present; n paid 2*s*. and the rest 1*s*. The total receipts were £25; find n.

19. Divide £94 between *A*, *B*, and *C*, so that *B* may have £37 more than *A*, and *C*'s share may be £13 more than twice *A*'s share.

20. In an isosceles triangle the base angles are each four times the vertical angle. Find the angles of the triangle.

TEST PAPERS I

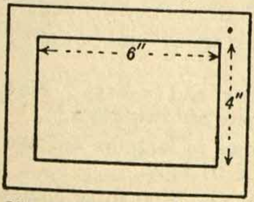
A

1. If $x=4$, find the values of (i) $\frac{5x}{2}$, (ii) $\frac{1}{2}x^2$, (iii) $2x^2-7x$, (iv) 2^x

2. Simplify, if possible:

(i) $3a\times 2$,	(ii) $5a\times 0$,	(iii) $3b+7$,
(iv) $3b+0$,	(v) $3c\div 1$,	(vi) $0\times 4d$,
(vii) $0\div 4d$,	(viii) $5x+0$,	(ix) $0+7t$.

3. Solve the equations (i) $7x-3x=42+x$, (ii) $3x-5=x+3$.



4. In the figure, which is not drawn to scale, the border is everywhere the same width. The perimeter of the outside rectangle is 36″. Find the width of the border.

5. There are $8a+3b+15c$ people at a meeting. $6a+2b$ are men, $2a+b+5c$ are women, and the rest are children. How many children are there?

6. (i) A camp has provisions for x men for y days. If z more men arrive how long will the provisions last?

(ii) How many shillings must I pay for y pens at x shillings a score?

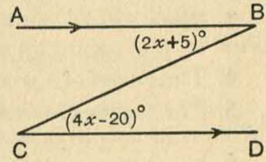
B

1. A boy is now a years old, and his father is $5a$ years old. How old will the father be when the boy is $3a$ years old? How old was the father when the boy was born?

2. Simplify (i) $5x \times 3xy$, (ii) $5a - a$, (iii) $7a - 3 - 2a + 6$.

3. Solve the equations (i) $17 - b = 4 + 2b$, (ii) $x + \frac{3x}{5} = 11\frac{1}{5}$.

4. Find the value of x in the figure, AB and CD being parallel.



5. The bus fare for a journey is 5 pence; what is the cost in shillings of p journeys? How much change would there be out of a ten-shilling note?

6. In Rugby football a goal counts 5 points and a try 3 points. In a match one side scored a certain number of goals, but the other side scored three times as many tries, and won by 12 points. What were the scores?

C

1. If $a=6$, $b=0$, $c=4$, find the values of

$$\text{(i)}\ \frac{2a+b}{c},\quad \text{(ii)}\ \frac{a^2+b^2}{3c},\quad \text{(iii)}\ \frac{3c^2-a^3b}{2a}.$$

2. Write in a shorter form, if possible:

(i) $\frac{2 \times x}{y}$, (ii) $c \times c \times c \times d$,

(iii) $s \times t + t \times s$, (iv) $u + v - 3uv$, (v) $z + z^3$,

(vi) $z \times z^3$, (vii) $6xy - 3x - 5y + 7y + 4x - 2xy$.

3. Two angles of a triangle are $23 - x$ and $67 + x$ degrees. What sort of a triangle is it?

4. Solve (i) $19 - \frac{2x}{3} = 6x - 1$, (ii) $12 - \frac{5a}{7} = 13\frac{1}{7} - \frac{19a}{7}$.

5. If the acute angles of a right-angled triangle are $5x^\circ$ and $(42 + x)^\circ$, find x.

6. A man left £1750 to be divided among his two daughters and four sons. Each daughter was to receive three times as much as a son. How much did each son and daughter receive?

D

1. (i) If I can type x words a minute, how many hours will it take me to type $7y$ words?

(ii) A train travels a miles in b hours, how many minutes does it take to travel one mile?

2. Simplify, and write in ascending powers of x:

$$5x^3 - 2x^4 + 2x - 12x^3 + 5x^4 + 7 + 14x^3 - 5.$$

What is (i) the coefficient of x^3, (ii) the term in x?

3. When the day is a hours long, the night is two-thirds as long. Find a.

4. The angles of a triangle are $2x°$, $5x°$, and $19°$. Find them.

5. The length of fence required for a square field is $8a$ yards. What is its area (i) in square yards, (ii) in square feet?

6. The adjacent angles of a parallelogram are $(2a+26)°$ and $(3a-6)°$. Prove that it must be a rectangle.

E

1. Simplify (i) $5a^2 - 3a + 2 + 4a - 2a^2$, (ii) $4 - 3b - 1$.

2. Solve the equations (i) $3x - \frac{5x}{2} = 0$, (ii) $3x = \frac{2x}{5}$.

3. (i) How many minutes are there between x minutes past 5 p.m. and 8 p.m. the same day? (ii) A clock loses n seconds a day. How many minutes will it lose in a week?

4. The angles of a quadrilateral taken in order are $x°$, $3x°$, $5x°$, $7x°$. Find x. Show also that two sides of the quadrilateral must be parallel.

5. What is the cost, in shillings, of $3a$ lb. of sugar at b pence a lb., and $5a$ lb. of tea at $2b$ pence a lb.?

6. I buy a house, and spend one-twentieth of the cost on repairs. I pay altogether £735. How much did I pay for the house?

F

1. (i) Square $3y$ and double the result. (ii) Add $7x - 4y + 2z$ to $3x + 5y - 2z$.

2. Prove that $A^3 + 3B^2 = A^2B^2 + 3$, if $A = 3$, and $B = 2$.

3. Two adjacent sides of a square have lengths $(3x - 5)$ in. and $(5x - 11)$ in. Find x.

4. How many packets of tea costing 1s. 6d. each can be bought for (i) one guinea, (ii) x guineas?

5. The expenses of a journey are made up of bus, train and taxi fares. The train fare is 5 times the bus fare, and the taxi fare is twice the bus fare. The total expense is £2 4s. What is the train fare?

6. For what value of P is $\frac{5P}{6} - 2$ equal to $5 - \frac{5P}{6}$?

G

1. (i) How many eggs at 5 for sixpence can I buy with N half-crowns? (ii) If x apples cost W pence, how much will z apples cost?

2. (i) Simplify, and arrange in descending powers of a,

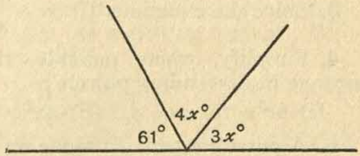
$$4a - 5a^3 + 2a^4 - 3a + 2 + 7a^3.$$

(ii) Subtract $5r$ feet from $6r$ yards and give the answer in inches.

3. If $p=5$ and $q=3$, find the values of

(i) $p+q$, (ii) $2pq$,
(iii) p^2+q^2, (iv) $3p^2-2q^3$.

4. Find the angles of the figure.

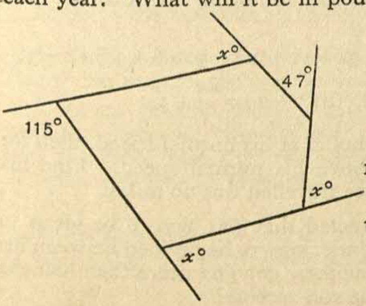


5. A boy gets as much pocket money in four weeks as his younger brother gets in eleven weeks. If the elder gets $3\frac{1}{2}$d. a week more than the younger, what does each get each week?

6. A boy starts the term with 10 shillings. He spends in the second week three times as much as he spends in the first week, and finds that he has then 2s. 4d. left. How much did he spend in the first week?

H

1. A man's salary is £x the first year, and increases by y shillings each year. What will it be in pounds the eighth year?



2. Simplify

(i) $3a^2b - 2a^2 + 2b + 5a^2$,
(ii) $4a - a^3 + 2a^2 - 3a + 4a^3$.

3. In the figure, which not accurately drawn, find x.

4. If $a=3$, $b=2$, $c=0$, find the values of

(i) $3a^2b - 6abc + 2b^2$,
(ii) $a^b - b^a$.

5. Solve the equations

(i) $5x=7x$, (ii) $8+10a-3a=18-8a$.

6. A bag contains some shillings and three times as many sixpences. The value of the money in the bag is £1 12s. 6d. How many shillings are there in the bag?

I

1. If $a=2$, $b=3$, $c=1$, $l=0$, $m=4$, $n=5$, $p=10$, find the values of (i) $(4n-5bc)^2$, (ii) $4c^b n^a$, (iii) $a^m+p^b+72b^2c^2l^2$.

2. I cycled $2a$ miles and walked $4b$ miles before lunch. After lunch I cycled three times as far and walked half as far as I did before lunch. What distance did I travel in the whole day?

3. Solve the equations (i) $7x-\frac{5x}{4}=42+x$, (ii) $\frac{3x}{2}-5=x+\frac{3}{2}$.

4. Simplify, where possible, the following expressions, and arrange in descending powers:

(i) $6c^3+7c-2c-4$, (ii) $4x^2+8+2x-3-3x^2-2+x^2+5x$.

5. A purse contains £1 made up of shillings and sixpences. The number of sixpences is six times the number of shillings. Find the number of coins.

6. Three sides of a rectangle taken in order are $(3x-6)$ cm., $(3x-15)$ cm., $(2x+1)$ cm. Find the numerical value of (i) its length, (ii) its breadth, (iii) its area.

J

1. The sides PQ, QR, RP of a triangle are respectively $3x$ in., $2x$ in., y in. What is (i) the perimeter, (ii) the excess of $PQ+RP$ over QR?

2. Simplify, where possible, the following expressions, and arrange in ascending powers:

(i) $3n^3+8+4n-7n^2$, (ii) $c^2+3c+11+2c^2+5c-6+3c^2-c-3$.

3. Solve (i) $18=4t+11t+18$, (ii) $7+2a=\frac{8a}{3}+3$.

4. A motorist travelled for 3 hours at his normal speed, then for 30 min. at 5 miles per hour above his normal speed. Find his normal speed, if the total distance travelled was 90 miles.

5. A man left £433. He directed that £21 was to be given to local charities and that the remainder was to be divided between his son and daughter, so that the daughter got £62 more than half the son's share. How much did the son receive?

6. Two angles of an isosceles triangle are $(x+10)^\circ$ and $(x-5)^\circ$. Find the angles of the triangle. [Notice that there are two cases.]

K

1. If $r=3$, $s=\frac{1}{4}$, $t=\frac{1}{2}$, find the values of the following:

$$\text{(i) } \frac{3}{st}, \quad \text{(ii) } 9+st-r^2, \quad \text{(iii) } \frac{rt}{s}.$$

2. Solve (i) $6-3z+z+8=0$, (ii) $\frac{4l}{7}=0$.

3. A parallelogram has adjacent sides $9a$ cm. and $5a$ cm. long. Find the area of a square whose perimeter is the same as that of the parallelogram.

4. In the first half of a football match Ireland scored a goals and Scotland scored b goals. In the second half Ireland did not score, but Scotland scored c goals and won the match by 2 goals. Write down an equation connecting a, b and c.

5. Simplify, if possible, the expressions (i) $3 \times 4uv - 5 \times 2vu$, (ii) $4lm+5l+7m+3ml$, (iii) $7yz-5xz-xy$.

6. Four straight lines meet at a point O. The four angles formed are 90°, $2x^\circ$, $(x+60)^\circ$, $(6x-105)^\circ$. Find the angles.

L

1. A taxi-driver is paid $12k$ shillings a week and receives in tips l shillings a day. Out of this he has to pay for oil etc. $(3k+4l)$ shillings per week. If he works each day except Sunday, how much does he make each week?

2. (i) Subtract $11x$ pence from $3x$ florins. Give the answer in pounds. (ii) What is the length of a fence formed by 16 hurdles, each a ft. long and 20 hurdles, each $2b$ ft. long?

3. Solve the equations (i) $2+\frac{5c}{3}=c+6$, (ii) $3s+\frac{s}{5}=0$.

4. The base angles of an isosceles triangle are $3x^\circ$ and $(2x+15)^\circ$. Prove that the triangle is right-angled.

5. In an election there were 3 candidates and the successful candidate received twice as many votes as the bottom candidate and 712 votes more than the second candidate. The total number of votes was 7968. How many votes did the successful candidate obtain?

6. Three sides of a rectangle taken in order are $(8x-56)$ ft., $(x-5)$ ft., $(3x-11)$ ft. Find the *numerical* value of its area.

CHAPTER VI

BRACKETS

Note. In the work of this chapter it is assumed that each operation is arithmetically possible. All other cases will be considered later in Chapter IX.

25. Meaning of Brackets. It was pointed out in Art. 3 that the contents of a bracket may be regarded as equivalent to a single number.

Thus, $(8+6)$ means the number obtained by adding 6 to 8, and $(x+y)$ means the number obtained by adding y to x.

Similarly $(8-5)$ means the number obtained by subtracting 5 from 8, and $(x-y)$ means the number obtained by subtracting y from x.

The product of 7 and $(8+6)$ is written $7(8+6)$.

The product of 7 and $(x+y)$ is written $7(x+y)$.

The product of $(p-q)$ and $(x+y)$ is written $(p-q)(x+y)$ or $(x+y)(p-q)$.

Similarly $(a-b)\div 5$ means "subtract b from a and divide the result by 5". It is usually written $\dfrac{a-b}{5}$ or $\frac{1}{5}(a-b)$.

The pupil will have no difficulty in attaching meanings to the expressions $\dfrac{p-q}{s+t}$, $(c+d)^2$, $(x-y)^3$ etc.

26. Brackets show the order in which operations must be performed. Thus $7+3(8-5)$ means "subtract 5 from 8, multiply the result by 3, add this result to 7", i.e.

$$7+3(8-5)=7+3\times 3=7+9=16.$$

But $(7+3)8-5$ means "add 3 to 7, multiply the result by 8 and from the answer subtract 5", i.e.

$$(7+3)8-5=10\times 8-5=80-5=75.$$

Also $(7+3)(8-5)$ means "add 3 to 7, subtract 5 from 8, multiply the two results together", i.e.

$$(7+3)(8-5)=10\times 3=30.$$

27. Use of brackets. The use of brackets is illustrated by the following example.

Example 1. *4 oz. of sweets are packed in a tin which weighs k oz. when empty. What is the weight of (a) 10 tins, (b) x tins of sweets?*

One tin when full of sweets weighs $(k+4)$ oz.,

$\therefore$ 10 full tins weigh $10(k+4)$ oz.

Similarly x full tins weigh $x(k+4)$ oz.

EXERCISE 10. a

(*Some of these may be taken orally*)

State in words the meaning of the following; and *afterwards* find their values, if $a=7$, $b=5$, $c=3$.

1. $3(b+2)$. **2.** $(c+3)a$. **3.** $8(a-4)$.
4. $2(b+3)$. **5.** $(2c-4)b$. **6.** $b(a-c)$.
7. $(a-b)c$. **8.** $(3a-4c)5$. **9.** $\dfrac{a+b}{c}$.
10. $\dfrac{3a-2c}{b}$. **11.** $\dfrac{a-2c}{b}$. **12.** $(a-b)(2b-c)$.

Find the values of the following, if $a=1$, $b=2$, $c=4$, $d=0$.

13. $a+c(b+d)$. **14.** $(a+c)(b+d)$. **15.** $(a+c)b+d$.
16. $a+b(c+d)$. **17.** $a+bc+d$. **18.** $(c+d)\div b$.
19. $c+(d\div b)$. **20.** $(c-b)\div(a-d)$. **21.** $c-(b\div a)-d$.
22. $c+b\div 2(a-d)$. **23.** $c-b\div 2(a+d)$. **24.** $a^2+(c-b)^2$.

Use brackets when answering the following questions; do not attempt to remove the brackets.

25. I have 12 coins; t of them are shillings and the rest are sixpences. What is their value in pence?

26. A grocer's store contains n jars of jam. One gross of them contain 2 lb. each and the rest contain 1 lb. each. What is the total weight of the jam in the store?

27. P is a point between A and B on the line AB. AB is 10 inches, and BP is c inches long. AP is divided into five equal parts. What is the length of each part?

28. A taxi-driver receives xs. yd. per day on six days of the week. If he pays out £z per week for petrol and rent of garage, how much has he left (in pence) for himself?

29. A man travels by car for 7 hours. For x hours his average speed is 40 miles per hour and for the remainder of the time it is 18 miles per hour. How far does he go?

30. A jar full of jam weighs P lb., and when empty weighs Q lb. What is the weight of the jam in x jars? What is the weight of a jar half-full of jam?

31. What is the result

(*a*) of multiplying $a-b+2c$ by 7?

(*b*) of subtracting $P+3T$ from $2R+S$?

(*c*) of dividing $x+2y$ by $a-b$?

(*d*) of subtracting $2a$ from x and dividing the result by 7?

(*e*) of squaring $r+s$?

(*f*) of reducing $(s-t)$ yards $(2r+s)$ feet k inches to inches?

EXERCISE 10. b

(*Some of these may be taken orally*)

State in words the meaning of the following, and *afterwards* find their values, if $x=8$, $y=3$, $z=2$.

1. $(x-5)z$. **2.** $7(y-2)$. **3.** $3(z+7)$.

4. $(5y-12)x$. **5.** $z(y-z)$. **6.** $5(2+x)$.

7. $(x-z)z$. **8.** $(2x-7z)y$. **9.** $\frac{3y-4z}{x}$.

10. $\frac{x+2z}{y}$. **11.** $\frac{3x-6z}{2y}$. **12.** $(x+z)(3y-z)$.

Find the values of the following, if $p=2$, $q=3$, $r=5$, $t=0$.

13. $q+p(r+t)$. **14.** $(q+p)r+t$. **15.** $(q+p)(r+t)$.

16. $3p+qt-r$. **17.** $p+r(q-p)$. **18.** $(r-q)\div p$.

19. $r-(q\div p)$. **20.** $(2r-p)\div(r-q)$. **21.** $2r-(p\div r)-q$.

22. $r-p\div(2q+t)$. **23.** $r+q\div(p-t)$. **24.** $(p+q+r-t)^2$.

Use brackets when answering the following questions; do not attempt to remove the brackets.

25. A store contains n packets of exercise books; three dozen of them have 68 pages each and the rest 136 pages each. How many pages are there altogether?

26. I have 30 coins; x of them are shillings, y are sixpences, and the rest are pennies. What is their value in pence?

27. A cask of butter when empty weighs c lb. and when full weighs d lb. What is the weight of the butter? How many casks must be bought to obtain a lb. of butter?

28. A man travels by train for 15 hours. For the first t hours the train goes at an average speed of 55 miles per hour. It then runs into fog and the average speed for the remainder of the journey is reduced to 15 miles per hour. What is the length of the journey?

29. A workman receives £r per week wages. He pays £x ys. zd. for his board and lodging and 1s. 4d. for his insurance. How much has he left (in pence) for himself?

30. A store contains n boxes of chocolate. Two dozen of them contain 1 lb. each, one gross contain $\frac{1}{2}$ lb. each and the rest contain $\frac{1}{4}$ lb. each. What is the weight (in ounces) of the chocolate in the store?

31. What is the result

(*a*) of multiplying $2a - b$ by $3c$?

(*b*) of taking three-fifths of the sum of X and $3Y$?

(*c*) of subtracting $3c$ from $4d$ and dividing the result by 5?

(*d*) of cubing the result of taking c from $2d$?

(*e*) of reducing $(a+b)$ quarts $(c-d)$ pints to pints?

(*f*) of subtracting $(x-3y)$ from $(5a-2c)$?

28. Removal of brackets. In evaluating an arithmetical expression containing a bracket, the first step is to obtain the contents of the bracket as a single number. In Algebra it is not usually possible to do this. Thus we can simplify the contents of the bracket

in $70+3(4+7)$ but not in $70+3(x+y)$,

in $360-(117+123)$ but not in $360-(x+y)$.

It is therefore necessary to obtain rules for removing brackets from algebraical expressions.

29. The expressions $(x+y)+z$, $(x+y)-z$, $(x-y)+z$, $(x-y)-z$. The expression $(25+12)+8$ means that we are to add 12 to 25 and then add 8 to the result. It is clear that this is also the meaning of $25+12+8$.

Thus $$(25+12)+8=25+12+8.$$

Similarly $$(x+y)+z=x+y+z.$$

It is easily seen that
$$(x+y)-z=x+y-z,$$
$$(x-y)+z=x-y+z,$$
$$(x-y)-z=x-y-z.$$

30. The expressions $x+(y+z)$ and $x+(y-z)$. The expression $25+(12+8)$ means that 12 and 8 are to be added and their sum added to 25. It is clear that we get the same result by adding 12 and 8 to 25 separately.

A man travels 25 miles in the morning and $(12+8)$ miles in the afternoon; he travels the same distance if he goes 25 miles in the morning, 12 miles in the afternoon and 8 miles in the evening.

Thus $$25+(12+8)=25+12+8.$$

Similarly $$x+(y+z)=x+y+z. \quad \text{......................(1)}$$

Again $25+(12-8)$ means that to 25 we are to add 12 diminished by 8. It is clear that, if we first add 12 to 25, we have added too much by 8, and must therefore take 8 from the result.

A boy has 25 pence in his pocket and sells an article for $(12-8)$ pence. He can receive $(12-8)$ pence or he can receive 12 pence and give 8 pence change. In either case he has the same amount of money when the transaction is completed.

Thus $$25+(12-8)=25+12-8.$$

Similarly $$x+(y-z)=x+y-z. \quad \text{......................(2)}$$

31. The expressions $x-(y+z)$ and $x-(y-z)$. The expression $25-(12+8)$ means that from 25 we are to subtract the sum of 12 and 8. It is clear that to take the sum of 12 and 8 from 25 is the same as to subtract them separately from 25. If we first subtract 12 from 25, we have subtracted too little by 8, and must therefore take 8 from the result.

A man has 25 shillings in his purse; he has the same amount of money left whether he spends $(12+8)$ shillings in one shop or 12 shillings in one shop and 8 shillings in another.

Thus $$25-(12+8)=25-12-8.$$

Similarly $$x-(y+z)=x-y-z. \quad \text{......................(3)}$$

Again, $25-(12-8)$ means that from 25 we are to subtract 12 diminished by 8. It is clear that, if we first subtract 12, we have taken away too much by 8, and must therefore add 8 to the result.

If a boy has 25 pence in his pocket and buys an article worth $(12-8)$ pence, he can pay $(12-8)$ pence or he can pay 12 pence and receive 8 pence change. In either case he has the same amount of money left.

Thus $25-(12-8)=25-12+8.$

Similarly $x-(y-z)=x-y+z.$ ....(4)

By considering the results numbered (1), (2), (3) and (4) we have the following rule :

In removing brackets, if the sign before the bracket is $+$, the $+$ and $-$ signs inside the brackets are unaltered ; if the sign before the bracket is $-$, the $+$ and $-$ signs inside the brackets are changed to $-$ and $+$ respectively.

Note 1. If no sign is placed before the bracket, as in the expressions in Art. 29, a $+$ sign is understood.

It will easily be seen that the results obtained in Art. 29 obey the rule stated at the end of Art. 31.

Note 2. The rules may be applied when the bracket contains more than two terms.

Thus
$$t+(x-y-z)=t+x-y-z,$$
$$t-(x-y-z)=t-x+y+z,$$
$$t-(x-y+z)=t-x+y-z.$$

Note 3. In the results of Arts. 30 and 31, put $y=5a$, $z=2a$.

Then $x+(5a+2a)=x+5a+2a,$

i.e. $x+7a$ is the same as $x+5a+2a$. ....(1)

Also $x+(5a-2a)=x+5a-2a,$

i.e. $x+3a$ is the same as $x+5a-2a$. ....(2)

Again $x-(5a+2a)=x-5a-2a,$

i.e. $x-7a$ is the same as $x-5a-2a$. ....(3)

Lastly, $x-(5a-2a)=x-5a+2a,$

i.e. $x-3a$ is the same as $x-5a+2a$. ....(4)

These results should be carefully noted. The first two have already been used in simplifying expressions containing unlike terms. The pupil should now master the third and fourth. He should convince himself of their truth by considering several concrete examples, similar to those discussed in Arts. 30 and 31.

Example 2. *Simplify* $4a-3b+2c+5a+b-4c.$

The expression equals $4a+5a-3b+b+2c-4c$

$=9a-2b-2c$, using (1) and (4).

Example 3. *Simplify* $7x - 3y - 2z + 3x - y - 5z$.

The expression equals $7x + 3x - 3y - y - 2z - 5z$

$= 10x - 4y - 7z$, using (1) and (3).

Example 4. *Simplify* $5p - 4q + 7r - 3q - 9r$.

The expression equals $5p - 4q - 3q + 7r - 9r$

$= 5p - 7q - 2r$, using (3) and (4).

EXERCISE 11. a

(*Many of these may be taken orally*)

Simplify :

1. $16 - (8 + 2)$. **2.** $16 - (8 - 2)$. **3.** $16 + (8 - 2)$.
4. $16 + (8 + 2)$. **5.** $a - (b + c)$. **6.** $a - (b - c)$.
7. $a + (2b - c)$. **8.** $a + (2b + 3c)$. **9.** $(9 + 7) - (5 + 3)$.
10. $(9 - 7) - (5 - 3)$. **11.** $(9 + 7) - (5 - 3)$. **12.** $(9 - 7) + (5 - 3)$.
13. $(l + m) - (r + s)$. **14.** $(l + m) - (r - s)$. **15.** $(l - m) + (r - s)$.
16. $(l - m) - (r - s)$. **17.** $(c + 3) - (c - 2)$. **18.** $k - (k - 5)$.
19. $5a + 3b - (2a + b)$. **20.** $(c + 2d) - (c - 2d)$. **21.** $x + (x + 3y)$.
22. $3p - (p + q)$. **23.** $3s + (t - 2s)$. **24.** $(h - k) + 2h$.
25. $5b + (11 - 3b)$. **26.** $(2m + 3n) - (m - 5n)$.
27. $a - (2b - 3a)$. **28.** $4x + (3y - x)$.
29. $4c - (c + 2d)$. **30.** $4p + 5q - (3q - p)$.
31. $2a + 3b - (a + 4b)$. **32.** $6a - (8a - 4b)$.
33. $5a - b + 2c - (2a + b + 3c)$. **34.** $5x - 2y - z + (2x - 3y - 7z)$.
35. $5x - 2y - z - (8x - 3y - 7z)$. **36.** $(c + 3) - (2c - 7)$.

EXERCISE 11. b

(*Many of these may be taken orally*)

Simplify :

1. $25 - (8 - 7)$. **2.** $25 + (8 + 7)$. **3.** $25 - (8 + 7)$.
4. $25 + (8 - 7)$. **5.** $a + (b - c)$. **6.** $a - (b + 2c)$.
7. $a + (b + 3c)$. **8.** $a - (3b - 2c)$. **9.** $(9 - 6) + (4 - 1)$.
10. $(9 + 6) - (4 - 1)$. **11.** $(9 - 6) - (4 - 1)$. **12.** $(9 + 6) - (4 + 1)$.
13. $(c - d) + (h - k)$. **14.** $(c - d) - (h - k)$. **15.** $(c + d) - (h + k)$.
16. $(c + d) - (h - k)$. **17.** $3t - (3t - 8)$. **18.** $7x + 5y - (4x + 3y)$
19. $(3h + 4k) - (3h - 4k)$. **20.** $(2l + 5) - (l + 2)$.
21. $(1 + y) - (1 - y)$. **22.** $k - (3l - 2m)$. **23.** $(W + 7) - (W + 3)$.
24. $(4x + 7y) - (x + y)$. **25.** $(8u + 9v) - (7v + u)$.

26. $3x-(5y-6x)$. **27.** $4s+(7b-3s)$. **28.** $6u-(12-3u)$.
29. $(2c-3d)+(3d-2c)$. **30.** $(13a-20z)+(40z-10a)$.
31. $5a-(9a-6c)$. **32.** $3a-2b-(a+3b)$.
33. $(R+5S)-(3R+S)$. **34.** $3x-2y+4z-(2x+3y+z)$.
35. $7x-3y-4z-(5x-5y+3z)$. **36.** $7x-3y-4z+(5x-5y+3z)$.

EXERCISE 11. c

Simplify :

1. $2k-(5k-25)$. **2.** $5a+b-(2a+3b)$.
3. $(3c+2d)-(c+7d)$. **4.** $3y+(2x-6y)$.
5. $4p-(2p-q)$. **6.** $2s+(2t-3s)$.
7. $(h-k)-(2h-9k)$. **8.** $3t+(17-7t)$.
9. $3t-(17-7t)$. **10.** $(2m+3n)-(m+5n)$.
11. $a+(2b-3a)$. **12.** $(4c+3d)-(c+5d)$.
13. $7a+(a-5b)$. **14.** $7c-(b-2c)$.
15. $(a+2b+3c)+(a-2b+3c)$. **16.** $(a+2b+3c)-(a-2b+7c)$.
17. $(a-2b+3c)+(a-2b-3c)$. **18.** $(a+2b+3c)-(a-2b-3c)$.
19. $(a-b)+(b-c)+(c-a)$. **20.** $(3x-y)+(2y-z)-(x-2z)$.
21. $(7a-4)-(2a-b)+(2b-a)$. **22.** $(4a-2)-(a-3)-(2a+1)$.
23. $(5a^2-3b^2)+(7b^2-2c^2)-(9c^2-a^2)$.
24. $(3x^2-2y^2)-(3y^2-4z^2)-(z^2+x^2-7y^2)$.
25. From $(3x-2y+4z)$ take $(x-4y+3z)$.
26. From $(3x-2y+4z)$ take $(x+4y-3z)$.
27. Take $(x-3z)$ from the sum of $(5x-2y-3z)$ and $(3y+5z-2x)$.
28. Take $(3p-q)$ from $(7p-3q+4r)-(2p+4q-r)$.
29. What must be added to $(c-2b)$ to make $(3a+2b+4c)$?
30. What must be added to $(x-y)$ to make $(2x-2y-3z)$?

32. The expressions $x(y+z)$ and $x(y-z)$. Consider the area of the figure, which represents a rectangle of length $(y+z)''$ and breadth x'', divided into two smaller rectangles, as shown in Fig. 1. The areas of the two smaller rectangles are xy sq. in. and xz sq. in. The area of the whole rectangle is $(y+z)x$ or $x(y+z)$ sq. in.,

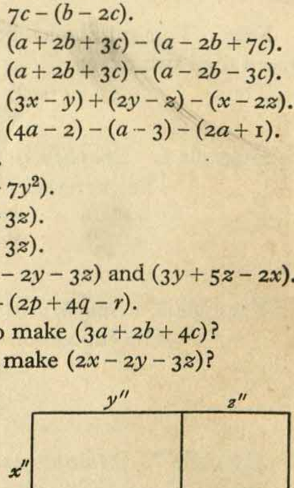


FIG. 1.

$$\therefore\ xy+xz=(y+z)x \quad\text{or}\quad x(y+z).$$

1. $14(t+u)+5(t-u)$.
2. $9x-4(x-y)$.
3. $3(a-b)-2(a+b)$.
4. $3(x-2y)-(x-y)$.
5. $3(a-b)+2(a+b)$.
6. $3(x-3y)+5(x+y)$.
7. $3a(4a+5)-5(a-2)$.
8. $5a(4a-3b)+3b(5a-2b)$.
9. $A(1-A+A^2)+A^2(2-A)$.
10. $5(8-3m)-27$.
11. $6(5-2x)-4(4+x)$.
12. $5(7-3a)-3(2-4a)$.
13. $2a(a-3b)-b(3a-b)$.
14. $2a(a-3b)+b(3a-b)$.
15. $\frac{1}{2}(8-M)-\frac{1}{2}(4-M)$.
16. $\frac{2}{3}(x-12)+\frac{2}{3}(18-x)$.
17. $2(3+z)-2(3-2t)-2(z-2t)$.
18. $a(1+a)-a(1-a)$.

33. It is sometimes convenient to use more than one set of brackets.

Example 8. *A book costs x shillings and the discount is y pence. A customer orders 5 copies to be sent to him by post (postage 9d.). Express, in pence, the change he will receive out of £2.*

We must express each sum of money in pence.

Each book with the discount taken off costs $(12x-y)$ pence,

$\therefore$ the parcel of 5 books with postage will cost $(5(12x-y)+9)$ pence,

$\therefore$ the change he receives is $(480-(5(12x-y)+9))$ pence.

To avoid confusion, when different sets of brackets are needed, it is better to use brackets of different shapes.

Thus it is better to write the above expression

$$\{480-[5(12x-y)+9]\}.$$

It is sometimes convenient to write $\overline{12x-y}$ instead of $(12x-y)$.

The line above $\overline{12x-y}$ is called a vinculum, which means a "bond", and serves to bind together everything underneath.

Similarly the line in a fraction, e.g. $\frac{12x-y}{7}$, serves as a bracket.

When simplifying an expression that has several sets of brackets, it is best to remove the innermost brackets first.

Example 9. *Simplify* $\{480-[5(12x-y)+9]\}$.

$$\begin{aligned}\text{The expression} &= \{480-[(60x-5y)+9]\}\\ &= \{480-[60x-5y+9]\}\\ &= \{480-60x+5y-9\}\\ &= 480-60x+5y-9=471-60x+5y.\end{aligned}$$

Example 10. *Simplify* $a+3b-(b-3a)-\{a+2b-(2a-b)\}$.

$$\begin{aligned}\text{The expression} &= a+3b-b+3a-\{a+2b-2a+b\}\\ &= a+3b-b+3a-a-2b+2a-b\\ &= 5a-b, \text{ collecting like terms.}\end{aligned}$$

EQUATIONS AND IDENTITIES

34. The equalities which we have written down in this chapter when removing brackets are true for all values of the letters which occur. Thus the statement $6(x+2)=6x+12$ is true for all values of x. Such statements are called **identities.** They must be carefully distinguished from equations, or statements which are true only for certain values of the letters. Thus the statement

$$6(x+2)=18$$

is true only when $x=1$, and is called an equation.

The correct link between the two sides of an identity is $\equiv$, which may be read **"is identically equal to".**

Thus $$4x-y(a-b)\equiv 4x-ay+by.$$

If two expressions are identically equal, either can be transformed into the other by application of the laws of algebra.

It is recommended that a small selection only of the examples in Exercises 13 *a, b, c* should be done at this stage. The remainder may be postponed.

EXERCISE 13. a

Simplify by removing brackets and collecting like terms :

1. $3(2x-3)+4[x+8(x+1)]$. **2.** $8a-[4+\frac{1}{3}(a-6)]$.

3. $15x-2[x-5(2-x)]$. **4.** $c+3+2(c+4)+3(2c+3)$.

5. $c+3-2(c+4)+3(2c+3)$. **6.** $c+3+2(c-4)-3(c-3)$.

7. $2x-\{3x-(1+2x)\}$. **8.** $2x-[3x+(1-2x)]$.

9. $2x+[3x-\overline{1+2x}]$. **10.** $2x+[3x-\overline{1-2x}]$.

11. $7a+3\{4-(a-1)\}$. **12.** $7a-3[4-2(1-a)]$.

13. $3(2x-3)+4[3x-8(x-1)]$. **14.** $8c-[4d+\frac{1}{4}(c-8d)]$.

15. $4y+3[x-5(y-x)]$. **16.** $5+d-2(2+3d)+3(3-d)$.

17. $7u+6v-2(u+2v)+3(u-4v)$.

18. $7u+6v+2(u+2v)+5(v-3u)$.

Some of the following statements are equations and some identities. Find which are which.

19. $3(x+2)-3(x-2)=12$. **20.** $3(x+2)+3(x-2)=12$.

21. $x(y+z)+3x=xy+xz+5$. **22.** $c(c-2)+2(c-2)=0$.

23. $x^2 - 3(x-2) = x^2 + 3(x+2) - 6x$.

24. $y(y+2) + 5(y+3) = y^2 + 7y + 15$.

EXERCISE 13. b

Simplify by removing brackets and collecting like terms :

1. $7c - [5 + \frac{1}{2}(c-4)]$.
2. $x + 5 + 3(x+2) + 5(3x+1)$.
3. $4(3z-8) + 3[2z + 6(z+2)]$.
4. $x + 5 + 3(x-2) - 2(x-5)$.
5. $12t - 2[t - 3(3-t)]$.
6. $x + 5 - 3(x+2) + 5(3x+1)$.
7. $2a - [5a + (3-4a)]$.
8. $5a + [7a - \overline{1 + 4a}]$.
9. $3a - \{8a - (2+7a)\}$.
10. $5a + 4\{7 - (a-2)\}$.
11. $23a - 2[5 - 3(1-2a)]$.
12. $8a + [5a - \overline{4 - 3a}]$.
13. $6x - [5y + \frac{1}{3}(x - 9y)]$.
14. $5(4t-7) + 3[2t - 9(t-3)]$.
15. $8 + 2x - 3(3+4x) + 4(7-2x)$.
16. $4c + 7[d - 3(c-5d)]$.
17. $2a + 3b + 2(2a+b) + 5(a-3b)$.
18. $2a + 3b - 2(2a+b) + 5(a-3b)$.

Some of the following statements are equations and some identities. Find which are which.

19. $4(x+3) + 3(x-4) = x$.
20. $4(x+3) - 3(x+4) = x$.
21. $a^2 + 4(a-3) = a^2 - 4(a+3) + 8a$.
22. $3r(s-t) + 7r = 3rs - 3rt + 21$.
23. $x(2x+3) + 4(2x+3) = 2x^2 + 11x + 12$.
24. $c^2(c-3) + 3c(c-3) + 9(c-3) = 0$.

EXERCISE 13. c

Simplify by removing brackets and collecting like terms :

1. $3a - \{5a - (3+a)\}$.
2. $3c - [7c - (2c+5b)]$.
3. $2t + [5s - 3(s+2t)]$.
4. $2t - [5s - 3(s+2t)]$.
5. $9c + 5[4d - \overline{3c + d}]$.
6. $9c - 5[4d - \overline{3c + d}]$.
7. $4k + l - \{5k - 4(k+l)\}$.
8. $5\{a - \overline{b - a}\} - 7(a+b)$.
9. $c^2 + 7c + \{5c^2 - (3c - 4c^2)\}$.
10. $l + 2m - (l - 2n) - \{2m + l - \overline{2n - l}\}$.
11. $a + 3b + (b - 3a) - [2a - b - (a - 2b)]$.
12. $3 + z - \{z + (z - \overline{z - 1})\}$.
13. $c[c(c+d) - d(c-d)]$.
14. $x[x(x-y) + y(2x - \overline{x + y})]$.
15. $2x - \{3x - x(3-x)\}$.
16. $x(2+y) - [4x - x(2-y)]$.
17. $7a + a\{4b - \overline{8 - a}\}$.
18. $8x^2 - x[4y - 3(x+2)]$.

CHAPTER VII

EQUATIONS AND PROBLEMS INVOLVING BRACKETS

35. Example 1. *Solve the equation* $7(3x-2)-3(5x-3)=7.$

We have $(21x-14)-(15x-9)=7.$

Removing brackets, $21x-14-15x+9=7,$

$\therefore\ 6x-5=7$ (this step should be carefully noted),

$\therefore\ 6x=12$ (adding 5 to each side),

$\therefore\ x=2$ (dividing each side by 6).

Check. When $x=2$,

L.H.S. $=7(6-2)-3(10-3)=28-21=7=$ R.H.S.

$\therefore$ the solution $x=2$ is correct.

EXERCISE 14. a

Solve the following equations :

1. $5(x-3)=3(x-1).$

2. $7(x+4)=4(x+10).$

3. $6(14x-3)-13(7x-2)=1.$

4. $5(5-2x)=3(8-3x).$

5. $4(2x-3)=7(x-1).$

6. $5-x=4(x-3)-2(x-1).$

7. $10-(3+2y)=1.$

8. $9t-5(3t-8)=4.$

9. $1-\{2x-(3x-5)-1\}=0.$

10. $5(x-1)=3(x+3).$

11. $4x-[7-2(x-1)]=3(x+1)-2(x-4).$

12. $3[3(4-x)+1]=7[4(3-x)-1].$

13. $20-(5+4z)=3.$

14. $5\{3(3x-16)-2\}=2\{3(x+1)-(x+5)\}.$

15. $28-\{5x-(3x+2)+24\}=0.$

16. $2(x+1)-[3-(x+1)]=10x+4-3(4x-2).$

17. $36-3[2(x-1)-8]=4(x-10)-4.$

18. $2(x-1)-[5-(x+2)]=x+1-3(x-3).$

EXERCISE 14. b

Solve the following equations :

1. $3(x-7)+5(x-4)=15.$

2. $9(x+2)=2(x+16).$

3. $18-5(x+1)=3(x-1).$

4. $6(4x-7)=10(2x-1).$

5. $8(7-3x)=4x$.
6. $32=5(x+4)-3(x-2)$.
7. $7t-3(t+5)=17$.
8. $18-(5+3y)=4$.
9. $2x-[5-(x+3)]=x+2-3(x-2)$.
10. $25-\{7x-(5x-11)+8\}=0$.
11. $9t-7(t-2)=60$.
12. $6[2x-(x+8)]=4x-3(x-9)$.
13. $26-(3+5y)=8$.
14. $72-3[2(x-4)-2]=4(x-2)$.
15. $5(9-x)=4x$.
16. $5x-[3-2(x-3)]=7(x+3)-5(x+2)$.
17. $3[4(3-x)+1+x]=7[2(6-x)-(1+2x)]$.
18. $5[3(3x-8)-1]=[3(x+1)-(5-x)]$.

EXERCISE 14. c

Solve the following equations:

1. $5(7-x)=3(10-x)$.
2. $16(2x+3)=5(5x+10)$.
3. $9=2\{10-2(3-2x)\}$.
4. $7(5-3x)=5(7-2x)$.
5. $2(4-t)+3(7-t)-1=16t$.
6. $21-7\{2x-3(3-x)\}=0$.
7. $\frac{x}{4}+3(x+1)=5$.
8. $\frac{1}{2}c=3(c-3)-(2c-5)$.
9. $\frac{1}{3}(5-2x)=2-3x$.
10. $8[3(x-1)+1]-8x=7(9-x)-(x-1)$.
11. $t^2+3(3t-5)=t(2+t)$.
12. $2t(3-2t)=4t(5-t)-7$.
13. $\frac{s}{3}+2(s+4)=11$.
14. $3a+\frac{a}{5}-2(a+3)=2$.
15. $3(n-7)-\frac{2n}{3}=\frac{4}{3}$.
16. $4(l-5)-\frac{3l}{5}=\frac{2}{5}$.
17. $\frac{t}{3}=5-4\left(t-\frac{1}{2}\right)$.
18. $3\frac{1}{2}-6\left(\frac{1}{3}+m\right)=\frac{m}{2}$.
19. $12(x+2)=7(x+5)$.
20. $3(8-3x)=5(3-x)$.
21. $4(6+3x)=6(4+3x)$.
22. $12-4(2n-1)=2n+1$.
23. $6(y-1)+7=3(y+3)+2$.
24. $3x+4[1+2(4-x)]=2(5x+8)$.
25. $\frac{1}{7}z=1-(z-1)$.
26. $2(2x+3)+\frac{x}{5}=7$.
27. $\frac{1}{5}(14x+4)=4x-1$.
28. $3(t-2)=5(t+4)-30\frac{2}{5}$.
29. $x^2+5(2x-7)=x(4+x)$.
30. $y(y+7)=3-y(1-y)$.
31. $\frac{u}{2}+4(u+3)=17$.
32. $6k+\frac{k}{3}-5(k-2)=14$.

33. $5(t-3)-\dfrac{3t}{4}=\dfrac{7}{4}$.

34. $2(r-7)-\dfrac{5r}{6}=\dfrac{1}{6}$.

35. $\dfrac{w}{2}=9-8\left(w-\dfrac{3}{4}\right)$.

36. $2\dfrac{1}{3}-5\left(h-\dfrac{2}{5}\right)=\dfrac{2h}{3}$.

PROBLEMS

36. No new principles are introduced, but the equations obtained contain brackets. The pupil should revise Art. 24.

Example 2. *A father is now three times as old as his son. Five years ago he was four times as old as his son. Find their present ages.*

Let x years be the present age of the son.

Then $3x$ years is the present age of the father.

Five years ago the age of the son was $(x-5)$ years, and the age of the father was $(3x-5)$ years.

Hence we have $\quad 4(x-5)$ years $=(3x-5)$ years,

$$\begin{aligned}
\therefore\ 4(x-5)&=(3x-5),\\
\therefore\ 4x-20&=3x-5,\\
\therefore\ 4x&=3x+15,\\
\therefore\ x&=15.
\end{aligned}$$

Thus the present age of the son is 15 years, and of the father is 45 years.

The pupil will have no difficulty in checking the result.

Example 3. *A bag contains shillings and half-crowns. There are* 18 *coins and their value is* 30s. *How many shillings are there?*

[In this question the choice of unit is important. Sixpence is the most convenient unit. Why?]

Let x be the number of shillings; then the number of half-crowns is $(18-x)$. We have

$$2x \text{ sixpences} + 5(18-x) \text{ sixpences} = 2\times 30 \text{ sixpences},$$

$$\begin{aligned}
\therefore\ 2x+5(18-x)&=60,\\
\therefore\ 2x+90-5x&=60,\\
\therefore\ 90-3x&=60,\\
\therefore\ 30&=3x, \quad \text{(adding } 3x-60 \text{ to each side)}\\
\therefore\ 10&=x, \text{ i.e. } x=10.
\end{aligned}$$

The number of shillings is therefore 10.

Check. If there are 10 shillings, there are 8 half-crowns, i.e. the value of the coins in the bag is 10s. + 20s. = 30s.

Example 4. *A man rode a distance of 61 miles in 7 hours; for part of the time he rode at 8 miles an hour and for the rest he rode at 10 miles an hour. For how long did he ride at 8 miles an hour?*

Let x hours be the required time at 8 miles an hour. Then he rode at 10 miles an hour for $(7-x)$ hours,

$\therefore$ the distance he rode is $8x$ miles $+ 10(7-x)$ miles,

$\therefore$ $8x$ miles $+ 10(7-x)$ miles $= 61$ miles,

$$\therefore 8x + 10(7-x) = 61,$$

$$\therefore 8x + 70 - 10x = 61,$$

$$\therefore 70 - 2x = 61,$$

$\therefore$ $9 = 2x$, (adding $2x - 61$ to each side)

$\therefore$ $4\frac{1}{2} = x$, i.e. $x = 4\frac{1}{2}$,

$\therefore$ he rode at 8 miles an hour for $4\frac{1}{2}$ hours.

Check In $4\frac{1}{2}$ hours at 8 miles an hour he rode 36 miles.
In $2\frac{1}{2}$ hours at 10 miles an hour he rode 25 miles,
$\therefore$ the total distance he rode is 61 miles and the solution is correct.

EXERCISE 15. a

1. If 4 be taken from a number, and the result multiplied by 7, the product is 49. Find the number.

2. I thought of a number, trebled it, then added 4. The result multiplied by 5 came to 200. Find the number.

3. Find three consecutive numbers such that four times the greatest added to five times the least amounts to 98.

4. Divide 40 into two parts so that twice the greater equals three times the less.

5. Divide 96 sheep into two flocks, so that one-fifth of one flock may be equal in number to three times the other flock.

6. A father is now three times as old as his son; in 15 years' time he will be twice as old as his son. How old is he now?

7. Three years ago a man was 8 times as old as his son. Their united ages at present total 51 years. How old is the son now?

8. In 17 years a man's age will be twice his age 13 years ago. What is his present age?

9. A man is 30 years older than his son; in 4 years' time he will be four times as old as his son. Find his present age.

10. A man is 36 years older than his son. In 10 years' time his son's age plus half his age will be 42 years. How old is he now?

EXERCISE 15. b

1. If 7 be added to a number, and the sum multiplied by 3, the result is 54. Find the number.

2. Find three consecutive numbers, such that five times the middle one added to three times the greatest amounts to 43.

3. I thought of a number, trebled it, then subtracted 5. The result multiplied by 6 came to 186. Find the number.

4. Divide 72 into two parts, so that seven times the greater equals eleven times the less.

5. Divide 92 into two parts, so that one-seventh of one part exceeds the other part by 4.

6. A is 16 years older than B; 16 years ago he was twice as old as B then was. How old is B now?

7. In 15 years a woman's age will be three times her age 15 years ago. What is her present age?

8. In 6 years' time a boy's age will be twice his age 4 years ago. How old is he now?

9. A man is 28 years older than his son; in 5 years' time he will be five times as old as his son. Find his present age.

10. A man is 32 years older than his son. In six years' time his son's age plus one-third of his age will be 40 years. How old is he now?

EXERCISE 15. c

1. The cost of 24 books is £3 7s. Some cost 3s. 6d. each; the rest cost 2s. 6d. each. How many of the latter are there?

2. A man spent £16 14s. in buying turkeys at 12s. each and hens at 5s. each. He bought 50 birds; how many of each did he buy?

3. In a shop there are 128 bags of flour, of total weight 3 cwt. Some hold 2 lb. and the rest 7 lb. How many 2 lb. bags are there?

4. A man bought 20 tickets for £4 16s. Some cost 4s. each; the rest cost 5s. 4d. each. How many were 4s. tickets?

5. A man cycled 75 miles in 7 hours, partly at the rate of 12 miles an hour, and partly at the rate of 10 miles an hour. For how long did he ride at the former rate?

6. A man spent £1 in buying 36 stamps. Some were insurance stamps costing 1s. 2d. each; the rest were 2d. receipt stamps. How many of each kind were bought?

7. A tramp set out to walk 71 miles. After he had walked a certain distance at the rate of 4 miles an hour he was given a lift by a lorry travelling at 24 miles an hour. His time for the whole journey was 4 hours. How far did he walk?

8. A man rode a distance of 57 miles in $5\frac{1}{2}$ hours; for part of the time he rode at 9 miles an hour, for the rest at 12 miles an hour. For how long did he ride at 12 miles an hour?

9. A has £72 and B has £12; how much must A give B so that B may have three times as much as A?

10. Two boys, A and B, start on a day's outing together, A with 5s., B with 4s. During the day A spends 10d. more than B and A then has twice as much money as B. How much has each spent?

11. In a subscription library the members are divided into three divisions, the second division paying 16s. 6d. less than the first, and the third 10s. 6d. less than the second. There are 20 members in the first division, 50 in the second, and 400 in the third, and the total subscriptions are £300. What is the subscription for each division?

12. The $\angle$s A, B, C of a pentagon are equal and so are the $\angle$s D and E. The $\angle A$ is 20° greater than the $\angle E$. Find the angles.

13. Twenty books cost £3 13s. Some cost 7s. 6d. each, and the rest cost 2s. each. How many are there of the latter price?

14. A man covers 130 miles in 5 hours. For part of the way he travels at 40 miles an hour, for the rest at 20 miles an hour. For how long does he travel at the former rate?

15. The $\angle$s A, C, E, F of a hexagon are equal, and so are the $\angle$s B and D. The $\angle B$ is 12° greater than the $\angle A$. Find the angles.

16. The price of oranges having risen $\frac{1}{4}$d. each, it costs 4d. more to buy 32 oranges than it cost to buy 36 oranges before the increase. What was the original price of an orange?

17. A has £45 and B has £18; how much must A give B so that B may have six times as much as A?

18. A covers 182 miles in 6 hours. For part of the way he travels at 35 miles an hour, for the rest at 25 miles an hour. For how long does he travel at the latter rate?

19. A man is 23 years older than his son. In four years' time his son's age plus two-fifths of his age will be 19 years. How old is he now?

20. Thirty books cost £5 19s. Some cost 4s. 6d. each, and the rest cost 2s. 6d. each. How many are there of the former price?

21. Red, green and white tickets are sold for an entertainment. A red ticket costs 3s. more than a green ticket, and a green ticket costs twice as much as a white ticket. The total receipts from the sale of 50 red, 100 green and 400 white tickets are £42 10s. Find the price of a green ticket.

22. One hundred oranges cost 8s. 11d. Some were bought at four for 5d., and the rest were bought at five for 4d. How many were bought at the latter price?

23. A man covers 237 miles in $5\frac{1}{2}$ hours. For part of the way he travels at 48 miles an hour, for the rest at 36 miles an hour. For how long does he travel at 48 miles an hour?

24. The $\angle$s A, C, E, F of a hexagon are equal. The $\angle B$ is $24°$ greater than the $\angle A$, and the $\angle D$ is $54°$ less than the $\angle E$. Find the angles.

25. A man is 35 years older than his son. In five years' time his son's age plus three-tenths of his age will be 30 years. How old is he now?

26. A scout set out to walk $4\frac{1}{4}$ miles. After he had walked a certain distance at the rate of $3\frac{3}{4}$ miles an hour he was given a lift in a car travelling at 35 miles an hour. His time for the whole journey was 18 minutes. How far did he walk?

27. A has £342 and B has £186; how much must A give B so that B may have three times as much as A?

28. Two men, X and Y, start on a holiday together, X with £13, Y with £13 15s. During the holiday X spends £1 15s. more than Y and Y returns from the holiday with three times as much money as X. How much has each at the end of the holiday?

29. The $\angle$s A, C of a pentagon are equal and so are the $\angle$s B, E. The $\angle A$ is $67°$ greater than the $\angle D$, and the $\angle D$ is half the $\angle E$. Find the angles.

30. Tickets for an international football match were sold as follows: tickets for reserved seats in the grand stand cost twice as much as tickets for reserved seats in the enclosure, and four times as much as tickets for unreserved seats in the enclosure; tickets for unreserved seats in the enclosure cost 1s. 6d. more than tickets for admission to the ground. The total receipts from the sale of 5000 grand stand tickets, 4000 reserved enclosure tickets, 2000 unreserved enclosure tickets and 40,000 tickets of admission to the ground were £5750. Find the cost of an unreserved enclosure ticket.

CHAPTER VIII

THE CONSTRUCTION AND USE OF GRAPHS

(*The examples in the text are for class discussion*)

37. The following table gives the average maximum temperature, to the nearest degree (in degrees Fahrenheit), at a certain town during the first six months of 1935 :

Month - - -	Jan.	Feb.	Mar.	Apr.	May	June
Average maximum temperature -	44	48	52	56	64	72

It is usually easier to grasp the meaning of such a set of figures, if they are shown in the form of a picture, or graph. The figure (Fig. 3) shows them as a column graph.

AVERAGE MAXIMUM TEMPERATURE JAN. TO JUNE 1935

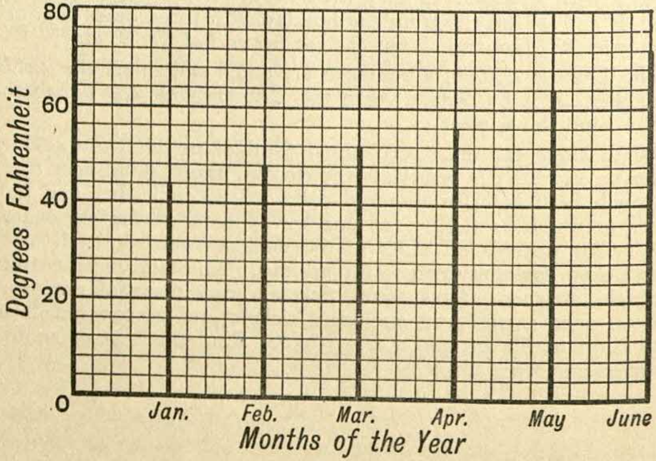


FIG. 3.

The graph is drawn as follows :

(1) Two lines called **axes** are taken, along which measurements may be made. The intersection of the axes is called the **origin,** and is usually denoted by the letter *O*. It is usual to take lines at right angles as axes.

(2) The axes are labelled, to show what quantities are measured along them.

(3) The axes are graduated, in one case to show the various months, and in the other to show the temperature.

(4) Along the first axis the months are represented at equal intervals.

(5) Along the second axis the temperature is represented by a certain scale ; in this case 1 small division is taken to represent 4°.

(6) On the first axis, at each point which represents one of the months, a line is drawn parallel to the second axis, the length of this line representing the temperature.

(7) The graph is given a title.

Note 1. It is usual to draw graphs on " squared paper " ruled in inches and tenths, or centimetres and millimetres. Such " squared paper " is not essential and, at a later stage, the pupil should be encouraged to draw rough sketches on ordinary paper.

Note 2. It is not essential to show the zero mark on the axes. In the above table all the temperatures lie between 44° and 72°, and there was no need to graduate the scale to show all the numbers from 0° to 80°. The alterations in temperature would have been as clearly shown, if along the second axis we had shown only the range from 40° to 80°. Had we done this, we could have taken a larger scale, say 1 small division, to represent 2°.

When the range of values to be shown is small, it is very important to choose a large scale. Consider a graph which shows that a man's temperature during a certain day moves between the limits 98·5° and 101·3° ; if we were to show all values between 0° and 102°, the scale would be so small that it would be difficult to derive any accurate information from the graph.

38. Sometimes the tops of consecutive uprights are joined by straight lines as an aid to the eye.

When two sets of measurements are drawn on the same diagram it is very desirable to draw these connecting lines.

Example 1. *The number of persons killed in road accidents in Great Britain during each week of the last quarter of the years* 1934 *and* 1935 *is given in the following table :*

Number of week	1	2	3	4	5	6	7	8	9	10	11	12	13
1934 -	143	131	153	169	178	143	140	146	140	155	167	160	187
1935 -	127	128	141	123	156	134	164	164	145	127	123	108	144

Draw the graph and use it to find

(i) *during which weeks the deaths in* 1934 *were less than in* 1935,

(ii) *in which consecutive weeks the increase or decrease in the number of deaths was the same.*

(iii) *in which weeks the number of deaths in* 1934 *differed* (*a*) *most,* (*b*) *least from the number of deaths in* 1935.

DEATHS FROM ROAD ACCIDENTS IN GREAT BRITAIN

1935 is represented by a thick line ———

1934 is represented by a dotted line ....

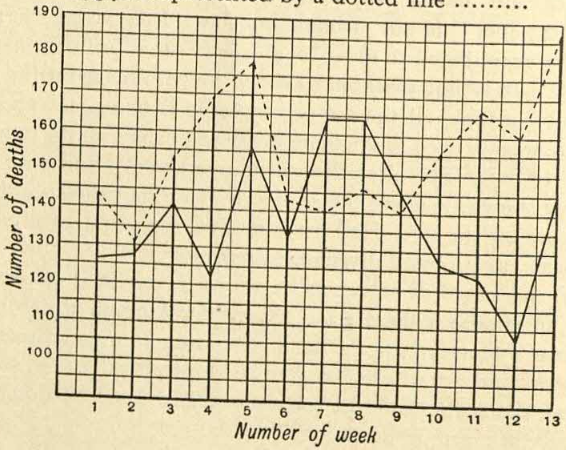


Fig. 4.

The help given by the joining lines will be readily appreciated. Note that only the range 100 to 190 has been shown along the second axis.

39. There is no need to draw the uprights. It is sufficient to mark with a cross the points where their ends would be, as in the following example.

Example 2. *The following table gives a boy's temperature (in degrees Fahrenheit) at intervals of 2 hours from 7 a.m. to 7 p.m. Draw the graph.*

Time - -	7 a.m.	9 a.m.	11 a.m.	1 p.m.	3 p.m.	5 p.m.	7 p.m.
Temperature	100·2	99	98·3	99·5	100·2	100·8	101

Describe in general terms the changes in the boy's temperature.

TEMPERATURE CHART

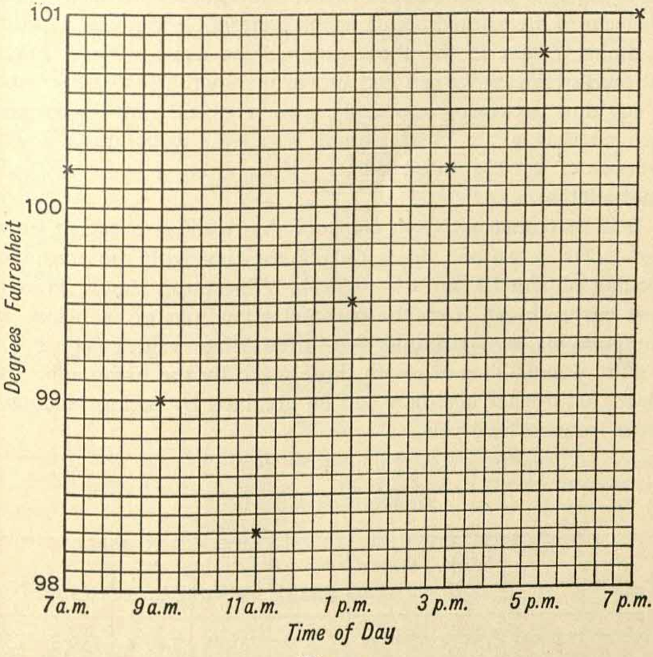


Fig. 5.

(Nos. 1 and 2 in Exs. 18 *a* and 18 *b* may now be done.)

LOCUS GRAPHS

40. The last graph differs in one important respect from the first two. If the boy's temperature had been taken at intermediate times, we should have intermediate points on the graph. If we took the boy's temperature at very small intervals of time, the points on the graph would be very close together. If it were possible to keep a continuous record we should get a curve which passed through all the points plotted in the above figure.

Whenever the intermediate points have a meaning, it is usual to draw a smooth curve through the marked points in the figure. This curve is not as a rule quite accurate, but it represents fairly closely the locus of the marked points which correspond to the intermediate readings. Thus, in the above graph, if we draw a curve through the marked points, we can find approximately the boy's temperature at any time between 7 a.m. and 7 p.m., e.g. at 8 a.m. the temperature is nearly 99·5°. Such a graph is called a **locus graph**, and the process of getting intermediate values from a graph is called **interpolation.**

It is most instructive to compare the readings obtained from a barometer at various times during the day with the continuous reading obtained from a barograph. The pupil should draw the best curve he can from the barometer readings given below, and compare his own readings for intermediate values (say 1 a.m., 5 a.m., 9 a.m., 11 a.m.) with those given by the barograph. It is clear that greater accuracy can be obtained by taking readings at more frequent intervals.

BAROMETER READINGS

Time - -	Midnight	2 a.m.	4 a.m.	6 a.m.	8 a.m.	10 a.m.	Noon
Height in mm.	752·0	749·0	746·6	746·6	749·4	750·5	750·5

The barograph readings are shown by the curve on p. 69.

BAROGRAPH

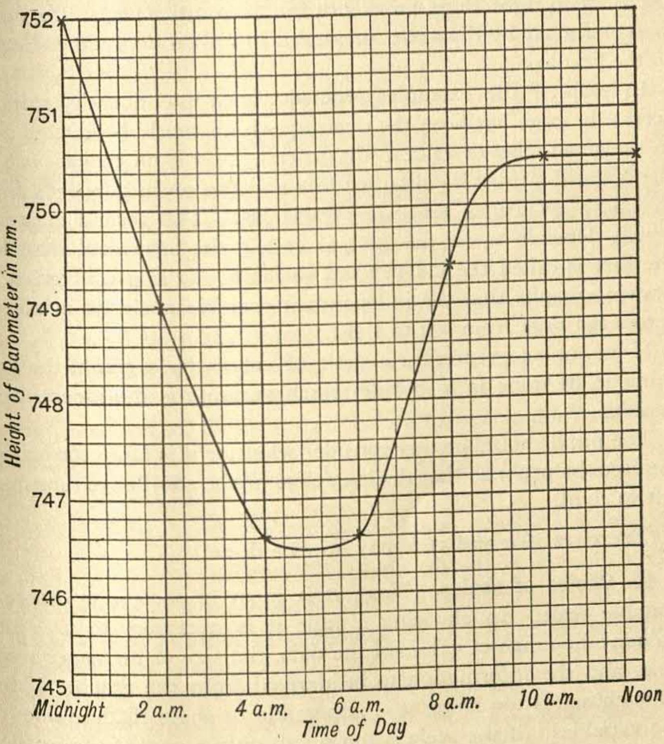


FIG. 6.

Note. The pupil should consider what the graph would look like on the same sheet of paper, if readings were shown from 0 to 752 instead of from 745 to 752.

(Nos. 3 and 4 in Exs. 18 *a* and 18 *b* may now be done.)

CHOICE OF AXES AND SCALES

41. Choice of axes. It will be clear from the examples studied above that most graphs represent in picture form the alterations which changes in one quantity produce in the values of a second quantity related to the first.

One of these quantities is represented by distances measured from left to right along an axis drawn across the page; the other is represented by distances measured in a direction perpendicular to the first axis.

In each of the examples studied above the quantity marked across the page has been the quantity which would be entered first in the given table of values. In general there is no choice—one of the quantities must be obtained first and the second depends upon it. Both quantities are called **variables.** The second quantity, which depends upon the first, is called the **dependent variable**; the first is called the **independent variable.** It is a convention in drawing graphs that the independent variable is always measured across the page from left to right.

If we wish to represent a table of values by a graph, the first thing to be done is to decide which variable is the independent variable.

The pupil should always consider whether it is more convenient to place the paper so that the long edge or the short edge runs from left to right.

(Exercises 16 *a* and 16 *b* may now be done.)

42. Choice of scales. Next, it is most important to choose a suitable scale. In choosing a scale it is necessary to take into account the range of values of the data, the size of the paper available, and the information to be derived from the graph. If the scale chosen is too large, it is impossible to show all the values of the variables; if the scale is too small, only a part of the paper is used, it is difficult to plot the points accurately, and the results obtained from the graph are not sufficiently reliable.

It is also important to choose a scale which makes plotting and reading easy. The most convenient scales for this purpose are those in which 1 inch represents 1, 10, 100 ... or 0·1, 0·01, ... units.

If such a scale is inconvenient, 1 inch may be taken to represent 2, 20, 200 ... or 0·2, 0·02, ... units, or 5, 50, 500, ... or 0·5, 0·05, ... units.

It is advisable to avoid scales in which 1 inch represents 3, 30, ... 7, 70, ..., 11, 110, ... units, unless all the readings given or required are multiples of those numbers.

It has already been pointed out that it is not necessary to show the zero graduation. Thus, if we wish to represent along an axis numbers ranging from 117 to 198, we should show the range from 110 to 200 only.

(Exercises 17 *a* and 17 *b* may now be done.)

EXERCISE 16. a (*Oral*)

Graphs are to be drawn to show the connection between the following pairs of quantities. State (i) which is the independent variable (which must be measured across the page), (ii) whether the intermediate points have any meaning.

1. Date and population of Wales.

2. Number of metres equivalent to 10, 20, 30, ... yards.

3. Time of day and height of tide.

4. Day of the month and lighting-up time on that day.

5. Amount of tax of a motor-car of 8, 10, 12, ... H.P.

6. Amount of premium for £100 Life Assurance for males aged 30, 35, 40, ... years.

7. The distance d yards in which a train running at v miles per hour can be stopped.

8. Day of the month and number of hours' sunshine on that day.

EXERCISE 16. b (*Oral*)

See instruction at the head of Exercise 16 *a*.

1. Date and number of candidates for the Cambridge School Certificate.

2. Time of day and number of hours' work done.

3. Number of francs equivalent to 10, 20, 30, ... shillings.

4. Price of a motor lorry of 12, 15, 18, ... H.P.

5. Compound Interest on £100 at 5 per cent. for 1, 2, 3, ... years.

6. Day of the month and time of sunrise.

7. Date and National annual expenditure.

8. The time of a complete oscillation for pendulums of lengths 2, 4, 6, ... ft.

EXERCISE 17. a (*Oral*)

With a sheet of paper $6\frac{1}{2}$ in. broad, choose scales for readings ranging from :

1. 0 to 11.	**2.** 15 to 78.	**3.** 42 to 167.
4. 336 to 948.	**5.** 18·35 to 18·97.	**6.** 0·678 to 0·74.
7. 4350 to 10,550.	**8.** 284 to 400.	

EXERCISE 17 b. (*Oral*)

With a sheet of paper $6\frac{1}{2}$ in. broad, choose scales for readings ranging from :

1. 0 to 58. **2.** 417 to 470. **3.** 80 to 320.
4. 168 to 474. **5.** 17·02 to 18·23. **6.** 0·313 to 0·434.
7. 2170 to 5300. **8.** 1801 to 1921.

EXERCISE 18. a

1. The following table shows the average rainfall at Cherrapunji :

Month - - -	Jan.	Feb.	Mar.	Apr.	May	June
Rainfall in inches -	0·5	2·8	9·6	31·4	50·5	92·9

Month - - -	July	Aug.	Sept.	Oct.	Nov.	Dec.
Rainfall in inches -	99·4	82·6	35·4	21·7	2·8	0·4

Draw the graph. By looking at the graph, find

(i) which are generally the wettest months of the year,

(ii) which months have an average rainfall of less than 3″,

(iii) which months have nearly the same average rainfall.

Have the intermediate points any meaning?

2. The following table gives the value of the diamonds produced in the Union of South Africa in millions of £'s. Draw the graph.

Year - -	1926	1927	1928	1929	1930	1931	1932	1933
Value - -	10·7	12·4	16·7	10·6	8·3	4·2	1·7	1·6

By looking at the graph, find

(i) in how many years the value of the diamonds produced exceeded £10,000,000,

(ii) which was the most prosperous year for the diamond industry,

(iii) the two years in which unemployment in the diamond industry was most severe.

3. The following table gives the population of India (in millions).

Year - -	1881	1891	1901	1911	1921	1931
Population -	254	287	294	315	319	353

Draw the graph. Estimate the population of India in the years 1897, 1920, 1925.

4. The cost of fuel (C) per week of 54 hours, for an engine of brake-horse power (P), is given in a certain price list as follows:

P -	10	20	50	80	100
C -	5s.	9s.	21s. 6d.	31s. 6d.	39s. 6d.

Draw the graph. What is the probable cost for engines of 15, 30, 70, 85 horse-power?

5. The following table shows the average rainfall at Winnipeg:

Month - - -	Jan.	Feb.	Mar.	Apr.	May	June
Rainfall in inches -	0·9	0·8	1·3	1·6	2·3	3·3

Month - - -	July	Aug.	Sept.	Oct.	Nov.	Dec.
Rainfall in inches -	3·2	2·3	1·9	1·4	1·0	0·9

Draw the graph. By looking at the graph, find

(i) which is generally the wettest month of the year,

(ii) which is likely to be the best month for a holiday.

6. The following table gives the annual premium £P payable during 20 years by a man aged A years next birthday for a whole life assurance of £1000 (without profits):

A - -	30	33	36	39	42	45	48	50
P - -	25	26·5	26·7	30	32·5	35·5	38·5	41

Draw the graph. In the table from which the figures were taken there was a misprint, which is reproduced above. Find from your curve which entry is wrong, and estimate the correct entry. Have the intermediate points any meaning? Estimate the annual premium for men aged 32, 40, 46 years next birthday.

7. The following table shows the average temperature at Milan:

Month - - -	Jan.	Feb.	Mar.	Apr.	May	June
Temp. in degrees F.	32·2	37·9	46·1	55·1	62·4	70·2

Month - - -	July	Aug.	Sept.	Oct.	Nov.	Dec.
Temp. in degrees F.	75·1	72·9	65·8	55·4	44·0	35·3

Draw the graph. From the graph find

(i) the months in which the average temperature exceeds 70° F.,

(ii) the three coldest months,

(iii) the months in which the average temperatures are most nearly equal.

8. In an experiment with a certain pulley the effort P lb., required to raise a load W lb., was found to be as follows. The efficiency e is also given.

W	10	20	30	40	50	60	70	80	90	100
P	$3\frac{1}{4}$	$4\frac{7}{8}$	$6\frac{1}{4}$	$7\frac{1}{2}$	9	$10\frac{1}{2}$	$12\frac{1}{4}$	$13\frac{3}{4}$	15	$16\frac{1}{2}$
e	12·8	17·1	20·0	22·2	23·1	23·8	23·8	24·2	25·0	25·3

Draw graphs on the same sheet, showing P and e as W alters.
Find P and e when $W=25$, 53, 88.
Find W (i) when $P=10$, (ii) when $e=22$.

EXERCISE 18. b

1. The following table shows the average temperature at Vladivostok in a certain year:

Month - - -	Jan.	Feb.	Mar.	Apr.	May	June
Temp. in degrees F.	8	13·8	27·2	39·6	49·1	56·8

Month - - -	July	Aug.	Sept.	Oct.	Nov.	Dec.
Temp. in degrees F.	64·9	69·1	61·7	49·1	30·7	13·8

Draw the graph. From the graph find

(i) the three hottest months,

(ii) the months in which the average temperature is between 30° F. and 50° F.

What do you notice about the temperatures of the three coldest months? Have the intermediate points any meaning?

2. The following table shows the average rainfall at Durban:

Month	Jan.	Feb.	Mar.	Apr.	May	June
Rainfall in inches	4·5	4·4	4·5	2·9	2·0	0·8

Month	July	Aug.	Sept.	Oct.	Nov.	Dec.
Rainfall in inches	0·9	1·9	3·6	3·8	4·3	4·4

Draw the graph. By looking at the graph find

(i) which month has the highest and which the lowest average rainfall,

(ii) which months have an average rainfall of less than 2 inches.

3. The following table gives the population of U.S.A. in millions.

Year	1830	1840	1850	1860	1870	1880
Population	12·9	17·1	23·2	31·4	39·8	50·2

Year	1890	1900	1910	1920	1930	
Population	62·9	76·0	92·0	105·7	122·8	

Draw the graph. Estimate the population of U.S.A. in the years 1845, 1875, 1925.

4. The price £P, of certain engines of brake horse-power H is given as follows. Draw the graph.

H	3	$4\frac{1}{2}$	$6\frac{1}{2}$	8	10	13	$14\frac{1}{2}$
P	103	129	160	183	211	247	259

Estimate the price of engines of 5, 9, $12\frac{1}{2}$ horse-power. If the price of an engine is £225, what is its probable horse-power?

5. The following table gives approximately the amount £P of £1 at compound interest for n years at $2\frac{1}{2}$ per cent.

n -	5	10	15	20	25	30	35	40
P -	1·13	1·28	1·45	1·64	1·85	2·10	2·37	2·69

Draw the graph. Estimate (i) the amount of £100 in 18 years, (ii) in how many years £1 will amount to £2, reckoning $2\frac{1}{2}$ per cent. Compound Interest in each case.

6. The following table gives the annual premium £P payable during life by a man aged A years next birthday for a whole life assurance of £1000 (without profits). Draw the graph.

A -	20	25	30	35	40	45	50	55
P -	11·6	13·1	13·3	17·5	21	25·5	31·5	40

In the table from which the figures were taken there was a misprint, which is reproduced above. Find from your graph which entry is wrong and estimate the correct entry. Estimate the annual premium for men aged 24, 42, 53 years next birthday.

7. The following table shows the average rainfall at Seathwaite:

Month - - -	Jan.	Feb.	Mar.	Apr.	May	June
Rainfall in inches -	13·5	11·1	10·5	6·8	7·6	7·0

Month - - -	July	Aug.	Sept.	Oct.	Nov.	Dec.
Rainfall in inches -	8·8	11·6	11·4	12·8	13·5	15·1

Draw the graph. By looking at the graph, find

(i) which is generally the wettest month of the year,

(ii) which is likely to be the best month for a holiday.

8. The following table gives the immediate annuity £I which can be purchased for £1000 by a man aged A years next birthday:

A -	45	50	55	60	65	70	75	80
I -	59·6	65·6	73·8	84·8	99·8	121	152·6	192·4

Draw the graph. Estimate the annuity which can be purchased for £1000 by men aged 52, 63, 77 years next birthday.

TEST PAPERS II

A

1. If $a=2$, $b=4$, $c=6$, find the values of

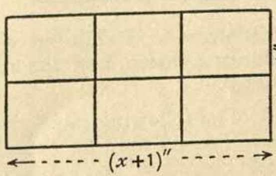
(i) $a+b(a+c)$, (ii) $(a+b)a+c$, (iii) $a+ba+c$.

2. Solve the equations

(i) $3(5x+2)-2(3x-1)=4(8-x)$, (ii) $\frac{1}{5}(x-3)=1$.

3. Two base angles of an isosceles triangle are $2x^\circ$ and $(x+36)^\circ$. Find all the angles of the triangle.

4. A has 3 times as much money as B. If A gives B 5 shillings, he will then have twice as much as B. What had each originally?



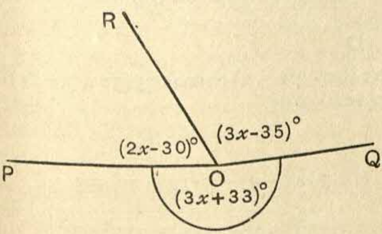
5. The length of wire required to make the grid in the diagram is 26". Find the length and breadth of the grid.

6. (i) Add $3a-4b+6c$ to $2a+3b-3c$, and take the result from $6a-6c$. (ii) Simplify $5x-3[2y+3\{4z-6x\}-5z]+x$.

B

1. If $x=4$, $y=0$, $z=3$, find the values of

(i) $x(2y+z)$, (ii) $(3x-2z)^2$, (iii) $3x^2-2z^2$.



2. Solve the equations :

(i) $\frac{1}{2}(3a-7)=4$,

(ii) $\frac{11}{2}(1-x)=4(2-3x)+x$.

3. Find x in the diagram, and hence prove that POQ is a straight line.

4. Simplify

(i) $x-\frac{1}{2}(x-y)$, (ii) $15x-4(x-2y)-3(z+x)$.

5. A cup and saucer cost 3s. 7 cups cost 3d. less than 10 saucers. Find the price of each.

6. In a cricket match one side scored x runs and $(x+71)$ runs: the other side scored $(3x-7)$ and $(2x-50)$ runs, and lost by 2 runs. What were the scores?

C

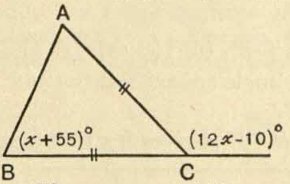
1. Simplify, and arrange in descending powers of a:

$$(a^2-4a-2)3a^2-(2a-1+a^2)3a.$$

What is (i) the coefficient of a^4, (ii) the term in a.

2. If $x=5$, $y=2$, find the values of

(i) $5x-2(x-y)^2$, (ii) $4(x+y)^2-3(x-y)^2$.



3. What value of t makes $\frac{3-2t}{7}$ equal to $\frac{3}{7}-2t$?

4. In the figure $AC=BC$; find $\angle ACB$.

5. If a greengrocer buys 15 dozen cauliflowers at x shillings a dozen, and sells 11 dozen at $(x-\frac{1}{4})$ shillings a dozen, and the rest at $(x+3\frac{3}{4})$ pence each, what is his profit?

6. The following table gives the number of years E that a male aged A years may be expected to live (i.e., "the expectation of life"):

A -	20	25	30	35	40	45	50
E -	44·2	40	35·8	31·7	27·7	23·9	20·3

Draw the graph. Have the intermediate points any meaning? Estimate the expectation of life of males aged 28, 36, 49 years.

D

1. (i) What must be added to $3(4x-2y+z)$ to make $5(3x+y-2z)$? (ii) Simplify $3b-[5a-\{6a+2(10a-b)\}]$.

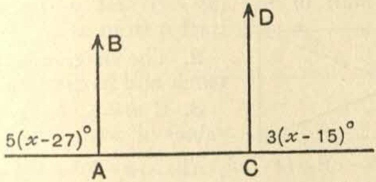
2. Solve the equations:

(i) $5-(3-2z)=\dfrac{z+6}{2}$, (ii) $3(2x-3)-5(12-2x)=5$.

3. If $x=7$, $y=3$, verify that $(x-y)^3=x^3-3x^2y+3xy^2-y^3$.

4. By riding a part of a journey at 32 miles per hour and walking the rest at 3 miles per hour, I travel 60 miles in $5\frac{1}{2}$ hours. How far do I walk?

5. In the figure, AB, CD are parallel. Find x, and show that $\angle BAC = 90°$.



6. (i) A boy was a years old b years ago. How old will he be in $(a-b)$ years?

(ii) Find the cost of $2a$ apples at $(3a-2)$ pence each, and 4 pears at $(2a-3)$ pence each.

E

1. If $x=15$, $y=4$, $z=3$, $t=1$, find the values of:

(i) $(x-y)\div(z-t)$, (ii) $x-y\div(z-t)$, (iii) $(x-y)\div z-t$.

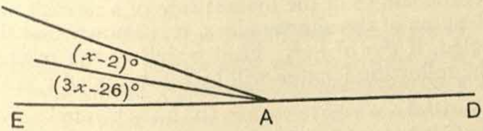
2. Simplify (i) $2(x-y+z)+3(x+2y-3z)-4(2y+z-3x)$,

(ii) $a(3a^2-7a+5)-a^3+2-a(7-3a)+7+2a$.

3. Solve the equations:

(i) $3x^2+5=x(3x+2)$, (ii) $5(3x-1)=4(2x+2)+\frac{x}{2}$.

4. In the diagram the marked angles are equal. Find the other angle.



5. A is 6 times as old as B, and A's age 32 years ago was the same as B's age will be 28 years hence. Find their ages at present.

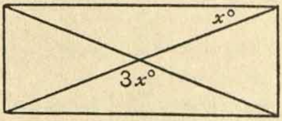
6. A man, aged A years next birthday, wishes to obtain £1000 (without profits) on attaining the age of 65 years or at death, if that should occur previously. The following table gives the annual premium £P payable:

A -	20	25	30	35	40	45	50	55
P -	14	16·5	19·5	24	31	41	58·5	92·5

Draw the graph. Estimate the premium payable by a man aged 28, 42, 53 years next birthday.

F

1. Find the sum of $6x-(2y-x)$ and $y-(3x-2y)$ and subtract it from $2(2x+y)$.



2. The diagram represents a rectangle and its diagonals. Find x.

3. If $a=4$, $b=2$, $c=1$, find the values of

(i) $(a-b)^2-(b-c)^2+(a-c)^2$, (ii) $4(a-b)^2+3(b-c)^2-2(a-c)^2$.

4. Solve the equations:

(i) $5a+(7-3a)+\{(6a-3)-(4-2a)\}=10$,

(ii) $\frac{2}{3}(x-10)=2(x-2\frac{1}{2})-65$.

5. If the area of the walls of a room is denoted by A sq. ft., its length by l ft., its breadth by b ft., and its height by h ft., it is known that $A=2h(l+b)$. What will be the height of a room whose length is 18 ft., breadth 12 ft., and the area of whose walls is 650 sq. ft.?

6. Some of the following statements are identities and the others are equations. Find which are the equations, and solve them.

(i) $4(x+3)-3(x+4)=x$, (ii) $4(x+3)-3(x-4)=30$,

(iii) $x^2-x(x-3)=5x-5(x-3)$, (iv) $x^2-x(x-y)=y^2-y(y-x)$.

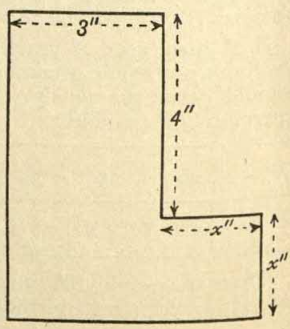
G

1. If c'' is the length of the longest side of a triangle, and a'' and b'' are the lengths of the shorter sides, it is known that the triangle is right-angled, if $c^2=a^2+b^2$. Find which of the triangles whose sides are the following lengths will be right-angled:

(i) $3''$, $4''$, $5''$; (ii) $6''$, $15''$, $17''$;

(iii) $25''$, $24''$, $7''$; (iv) $5''$, $6''$, $7''$.

2. Solve the equations (i) $\frac{3a-5}{7}=0$, (ii) $1-\frac{5}{4}(3-x)=\frac{1}{4}(x-2)$.



3. I buy a cwt. of sugar for £c. I sell 4 cwt. at c shillings a pound, and the rest at $9c$ pence a pound. What profit in pence do I make?

4. Find an expression for the perimeter of the diagram. If the perimeter is $22''$, what is x? What is then the area of the figure?

5. In an audience of 546 people, there are 52 more men than women, and 163 more women than children. How many are there of each?

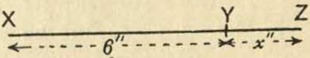
6. In catering for a dinner the following scale of charges was given.

Number of guests -	50	100	150	200	250	300	350
Charge per head -	7s. 6d.	6s.	5s. 6d.	5s. 3d.	5s.	4s. 9d.	4s. 8d.

Draw the graph, and estimate to the nearest penny, the charge per head for 130, 275, 315 guests.

H

1. If $XZ = 4YZ$, find x.



2. Simplify (i) $7x - 5[\{3 - 2(2x - 1)\} + 2x] - 2$,

(ii) $(3a^2)^3 + (2a^3)^2$.

3. Solve the equations:

(i) $11b + 3[2(2b + 5) - 3b] = 4(8 - b)$,

(ii) $25c - 19 - [3 + (5 - 4c)] = 3c - (6c - 5)$.

4. (i) If paraffin costs £x ys. for K pints, what is its price in shillings per gallon?

(ii) Find five consecutive odd numbers whose sum is 155.

5. If a number of half-crowns and one less than twice that number of florins make 3 guineas, how many florins are there?

6. In a factory the men get 10s. a day and the women 8s. a day. 200 people are employed and the wages are £92 a day. How many men are employed at the factory?

I

1. Write down the number represented by $3x^3 + 5x^2 + 2$ when $x = 10$; and the number represented by $5 + \frac{3}{y} + \frac{7}{y^2}$ when $y = 0{\cdot}1$.

2. Solve the equations:

(i) $3(3x - 1) - 6(x + 1) = 12(x + 1) - 2(5x + 3)$,

(ii) $4(4t + 0{\cdot}2) - 3(1 - 2t) = 1{\cdot}4(t + 8) + 0{\cdot}6t$.

3. A man buys $(5u + 3v)$ lb. of tea at 1s. 10d. per lb. and $(7u - v)$ lb. at 2s. 1d. per lb. He sells the whole for 2s. per lb. What is his gain in pence?

4. Twenty-two books cost £4 14s. Some cost 6s. 6d. each, and the rest cost 3s. each. How many are there of the latter price?

5. Simplify by removing brackets and collecting like terms:

(i) $8p+3[4q-\overline{5p+q}]$, (ii) $2B(3-B+2B^2)+5B^2(1+B)$.

6. Of three brothers, each was three years older than the next younger and the eldest was four times as old as the youngest. What were their ages?

J

1. A man walks at the rate of c miles an hour for x hours; he then rides for y hours at the rate of d miles an hour. How far has he travelled, and how long would it have taken to ride the same distance at z miles an hour?

2. Solve the equations:

(i) $15(3x-4)-4(7x-9)=4(x+17)$,
(ii) $-4(6-t)+3(t-4)=-4(5-t)+3(t-2)-(3t-2)$.

3. If $x=7$, $y=5$, find the values of

(i) $(x-y)^4$, (ii) $(x^2+y^2)^2-4xy(x^2-xy+y^2)$.

4. Two men, C and D, start on a holiday together, C with £34 and D with £30. During the holiday D spends £4 more than C and at the end C has twice as much as D. How much has each spent?

5. Simplify by removing brackets and collecting like terms:

(i) $8p-3[4q-\overline{5p+q}]$, (ii) $k(5+2l)-[8k-k(1+3l)]$.

6. The following table gives the weight of an infant during the first year of life:

Age in months -	0	2	4	6	8	10	12
Weight in pounds -	8	11	$14\frac{1}{2}$	$16\frac{1}{2}$	$18\frac{1}{4}$	$20\frac{1}{2}$	$23\frac{1}{2}$

Draw the curve, and estimate the infant's weight at 3, 7, 11 months.

K

1. (i) Simplify $(4c^3)^2-(2c^2)^3$.

(ii) What must be added to $5(3a-2b-7c)$ to make

$$8(a+3b-2c)?$$

2. Simplify by removing brackets and collecting like terms:

(i) $4a+\{7b-2(b+3a)\}$, (ii) $x[3x(2x+5y)-5y(3x-4y)]$.

3. Solve the equations:

(i) $9x-29-3(7-x)=5(2x-8)$, (ii) $0{\cdot}3x+4{\cdot}7(x-2)=2{\cdot}6$

4. A dealer buys p tons of coke at q shillings per ton. If he sells it at r pence per cwt. without gain or loss, prove that $3q=5r$.

5. Some of the following statements are identities and the others are equations. Find which are the equations and solve them.

(i) $x-3[x-2(3-x)]=2$,
(ii) $5(x-2)-2(x-5)=3x$,
(iii) $3a^2-a(3a-2b)=5b^2-b(5b-2a)$,
(iv) $3t^2-t(3t-5)=4t-2(2t-5)$.

6. In a subscription library the members are divided into three divisions, the second division paying half the subscription paid by the first, and the third 7s. 6d. less than the second. There are 30 members in the first division, 100 in the second and 240 in the third, and the total subscriptions are £210. What is the subscription for each division?

L

1. A cricketer finds that when he has played a completed innings his average is t runs, and that when he has played b completed innings more, his average is s runs. What was his average for the last b completed innings?

2. (i) From $(6a-5b+9c)$ take $(2a+9b+4c)$.
(ii) What must be added to $(4x+3y)$ to make $(7x+y-2z)$?

3. The angles A, C, E of a hexagon are equal, and so are the angles B, D and F. The angle B is 42° greater than the angle A Find the angles.

4. Solve the equations:

(i) $2(4+5t)-5(3t-2)=3t+2$,
(ii) $7[3(2-c)-4]=70-28(2-c)$.

5. The perimeter of a rectangular playing field is half a mile. Twice the width is 160 yards more than the length. Find the length of the field.

6. A rifle sighted at 1200 yards is fired from a point X. The height of the bullet above the horizontal through X is given below.

Horizontal distance from X in yards -	100	200	400	500	600	700	900	1000	1100
Height of bullet in feet - - -	10	19·7	35·1	40·3	43·5	44·2	37·3	28·7	16·5

Draw the graph, and estimate the height of the bullet at (i) 300 yards, (ii) 800 yards from X.

CHAPTER IX

DIRECTED NUMBERS

43. In the previous chapters the signs + and − have stood for orders to add or to subtract, as in Arithmetic. Thus, the expression, $8+5$, means "To 8 add 5". The expression, $12-7$, means "From 12 subtract 7". The numbers 8, 5, 12 and 7 are the ordinary numbers used in Arithmetic. For many purposes these numbers are all that we need. But there are occasions when it is convenient to use a different kind of number, called a directed number. Consider, for example, a town lying on a road running East and West. If a car passes through at a speed of 25 miles per hour, where will it be two hours later? We cannot answer this question unless we know whether the car is travelling East or West. The answer may be either 50 miles west of the town or 50 miles east of the town.

When quantities which involve the idea of direction occur, it is a convenience to extend the meaning of the symbols + and −. Thus, in the above example, we may agree to attach the + sign to all distances measured east of the town, and the − sign to all distances measured west of the town. If the car is travelling East, the answer will be $+50$ miles from the town; if the car is travelling West the answer will be -50 miles from the town.

The numbers $+50$, -50 are called **directed numbers.** The number $+50$ is called a **positive number,** the number -50 is called a **negative number.** The number 50 used in Arithmetic is called a **signless number.** To avoid confusion we shall, for the present, enclose directed numbers in brackets, e.g. $(+50)$, (-50).

The symbol 0 means that there is no change either way. Thus $(+0)$ is the same as (-0) and, as there is no possibility of confusion, each is denoted by 0.

Time may be reckoned forwards or backwards from a fixed date. Thus A.D. 1860 may be called $(+1860)$ and B.C. 65 may be called (-65).

Temperature may be measured by the amount above or below zero. In the Centigrade system freezing point is represented by 0°;

a temperature of 30° above freezing point is represented by $(+30)°$; a temperature of 10° below freezing point by $(-10)°$.

If a man has £20 to his credit at the bank, we may say that he has £$(+20)$, if his account is overdrawn and he owes the bank £15, we may say that he has £(-15).

It should be noticed that the choice of the signs + and − is quite arbitrary. Thus in the first example it would have been equally convenient to denote distances measured West by +, and distances measured East by −. The choice is merely a matter of convention.

EXERCISE 19. a (*Oral*)

1. What are the meanings of $(-x)$, when $(+x)$ has the following meanings :

(i) " put £x in the bank " ;
(ii) " a fall in the temperature of $x°$ " ;
(iii) " x minutes after midnight "?

2. What are the meanings of $(+p)$, when $(-p)$ has the following meanings :

(i) " a fall in the temperature of $p°$ " ;
(ii) " put the clock on p minutes " ;
(iii) " go back p yards " ?

3. If the hours of the day are reckoned positive from noon onwards, 2 p.m. would be written $(+2)$ and 9 a.m. would be written (-3). Express similarly :

(i) 6 p.m. ; (ii) 4 a.m. ; (iii) 10 p.m. ; (iv) 2 a.m.

4. In contour maps, if heights above sea-level are reckoned positive, depths below sea-level would be reckoned negative. Express in directed numbers :

(i) 3650 ft. above sea-level ; (ii) 1300 ft. below sea-level ;
(iii) 2840 ,, ,, ,, ; (iv) 2863 ,, ,, ,, .

5. What meaning is to be attached to the following statements?

(i) Bradford is (-10) miles East of Leeds ;
(ii) The clock is (-5) minutes fast ;
(iii) The barometer has risen (-2) mm.

EXERCISE 19. b (*Oral*)

1. What are the meanings of $(-y)$, when $(+y)$ has the following meanings :

(i) " put the clock back y minutes " ;
(ii) " go y miles West " ;
(iii) " an increase in salary of £y "?

2. What are the meanings of $(+t)$, when $(-t)$ has the following meanings:

(i) "take £t out of the bank";
(ii) "buy t lb. of sugar";
(iii) "I owe my employer £t"?

3. In using the Centigrade scale, temperatures above freezing point are written +, temperatures below freezing point are written −. Express in directed numbers:

(i) 50° above freezing point; (ii) 20° below freezing point;
(iii) 110° " " "; (iv) 42° " " ".

4. If velocity vertically upwards is reckoned positive, velocity vertically downwards would be reckoned negative.

Express in directed numbers velocities of:

(i) 40 ft. per sec. upwards; (ii) 8 ft. per sec. downwards;
(iii) 24 " " " ; (iv) 336 " " " ".

5. What meaning is to be attached to the following statements?

(i) The school is (-42) feet below sea-level.
(ii) The price of tea has gone down (-2) pence per lb.
(iii) The population of England has decreased by (-2) per cent.

ADDITION AND SUBTRACTION

44. Consider the figure on p. 87, which represents part of the scale of a Centigrade thermometer.

If the initial temperature is $(+2°)$ what is the result of rises of $(+5°)$, $(-3°)$, $(-8°)$, $(+4°)$ respectively?

A rise of $(+5°)$ takes us **up** 5 steps from $(+2°)$ and brings us to $(+7°)$. A rise of $(-3°)$ takes us **down** 3 steps from $(+2°)$ and brings us to $(-1°)$.

Similarly rises of $(-8°)$ and $(+4°)$ bring us to $(-6°)$ and $(6°)$ respectively.

Again, if the initial temperature is $(+1°)$ and the final temperatures are $(+6°)$, $(-7°)$ it is easily seen that the rises of temperature are $(+5°)$ and $(-8°)$ respectively.

More generally, consider the expressions:

$$(+3)+(+4), \quad (+3)+(-4), \quad (-3)+(+4), \quad (-3)+(-4).$$

(i) To add $(+4)$ to $(+3)$, start at $(+3)$ on the scale and move up 4 steps, reaching $(+7)$. Thus $(+3)+(+4)=(+7)$.

(ii) To add (-4) to $(+3)$, start at $(+3)$ on the scale and move up (-4) steps, i.e. move down 4 steps, reaching (-1). Thus $(+3)+(-4)=(-1)$.

(iii) To add $(+4)$ to (-3), start at (-3) on the scale and move up 4 steps, reaching $(+1)$. Thus $(-3)+(+4)=(+1)$.

(iv) To add (-4) to (-3), start at (-3) on the scale and move up (-4) steps, i.e. move down 4 steps, reaching (-7). Thus $(-3)+(-4)=(-7)$.

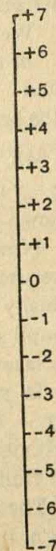


FIG. 7.

For subtraction, since

(i) $(+3)+(+4)=(+7)$, we have $(+7)-(+4)=(+3)$;
(ii) $(+3)+(-4)=(-1)$, we have $(-1)-(-4)=(+3)$;
(iii) $(-3)+(+4)=(+1)$, we have $(+1)-(+4)=(-3)$;
(iv) $(-3)+(-4)=(-7)$, we have $(-7)-(-4)=(-3)$.

Thus to subtract $(+4)$ from $(+7)$, start at $(+7)$ on the scale and move down 4 steps, reaching $(+3)$.

To subtract (-4) from (-1), start at (-1) on the scale and move down (-4) steps, i.e. move up 4 steps, reaching $(+3)$.

To subtract $(+4)$ from $(+1)$, start at $(+1)$ on the scale and move down 4 steps, reaching (-3).

To subtract (-4) from (-7), start at (-7) on the scale and move down (-4) steps, i.e. move up 4 steps, reaching (-3).

In general $+(+\mathbf{a})$ or $-(-\mathbf{a})$ **means move up a steps,**
and $+(-\mathbf{a})$ or $-(+\mathbf{a})$ **means move down a steps.**

We thus obtain the **rule of signs:**

Whenever $+(+a)$ or $-(-a)$ appears in an expression, we may write it as $+a$.

Whenever $+(-a)$ or $-(+a)$ appears in an expression, we may write it as $-a$.

45. In Arithmetic we should say that the expression $3-4$ has no meaning; but, if the numbers are understood to be directed numbers, i.e. $(+3)-(+4)$, it has been shown above that the expression is equal to (-1).

Once this is clearly understood the use of brackets to denote directed numbers may be dropped, and the symbols + and − may be used to denote **either** addition and subtraction **or** direction.

Thus we may write

$$3-4=-1, \qquad -15+7+6=-2,$$
$$3x-4x=-x, \qquad -15a+7a+6a=-2a.$$

In Art. 31 we stated certain rules for dealing with brackets, but these rules were restricted, because every expression had to be arithmetically intelligible. The above work shows that with directed number the same rules still hold, and the restriction is no longer necessary.

We therefore have as a general rule :

In removing brackets, if the sign before the bracket is +, the + and − signs inside the bracket are unaltered ; if the sign before the bracket is −, the + and − signs inside the bracket are changed.

46. Consider again the scale in Art. 44. Take the expression $4-7+5-3$, i.e. start at $+4$, go down 7 steps, go up 5 steps, go down 3 steps, finally arriving at -1. Thus $4-7+5-3=-1$.

Now take the same steps but in a different order, e.g. $4+5-7-3$, i.e. start at $+4$, go up 5 steps, go down 7 steps, go down 3 steps, finally arriving at -1. Thus the two results are the same. The pupil should verify that the same result is obtained whatever order is taken.

In general, in an expression consisting of numbers connected by + and − signs, the numbers may be re-arranged in any order without altering the value of the expression, provided that the signs in front of the numbers are moved with them.

The pupil is reminded that if no sign stands before a number, the sign + is understood.

Example 1. *Simplify* $(3a-7a-12a+15a-9a)$.

The expression equals $3a+15a-7a-12a-9a$
$$=18a-28a=-10a.$$

Example 2. *Simplify* $-5x+7x-11x+3x+4x$.

The expression equals $7x+3x+4x-5x-11x$
$$=14x-16x=-2x.$$

Example 3. *Simplify* $(2x-5y)-(4x-9y)$.

The expression equals $2x-5y-4x+9y$
$$=2x-4x+9y-5y=-2x+4y$$
$$=4y-2x.$$ (See Note 2, below.)

Note 1. In performing the calculations it is simpler to group together like terms having the same sign. This may be done mentally, if preferred.

Note 2. If positive and negative terms occur in the final result, it is customary to write the positive term first.

Note 3. In the expression $4y - 2x$, the coefficient of y is 4 and the coefficient of x is -2. The terms of the expression are $4y$ and $-2x$.

EXERCISE 20. a

(Many of these may be taken orally)

1. The temperature (Centigrade) is $(+12°)$. What does it become after (i) a fall of $(+18°)$, (ii) a rise of $(-8°)$, (iii) a fall of $(-8°)$?

2. The temperature (Centigrade) is $(-12°)$. What does it become after (i) a rise of $(+18°)$, (ii) a fall of $(+18°)$, (iii) a rise of $(-8°)$, (iv) a fall of $(-8°)$?

3. I have $(+6a)$ shillings at the beginning of the day. How much have I at the end of the day, if during the day:

(i) I earn $(+2a)$ shillings and spend $(+9a)$ shillings,

(ii) I earn $(+2a)$ shillings and borrow $(+4a)$ shillings?

4. A man has a bank balance of £21; he deposits £30, withdraws £10, withdraws £50. What is his balance now?

5. A man starts the year with a deficit of £40, and ends the year with a deficit of £10. What has he (*a*) gained, (*b*) lost during the year? Express each answer as a directed number.

6. A train is moving with a speed of 50 miles an hour. What are the speeds relative to this train of others moving in the same direction at 66 miles an hour, and at 30 miles an hour; also of one moving in the opposite direction at 28 miles an hour?

7. Find the values of:

(i) $(-11)+(-9)$, (ii) $(-11)+(+9)$, (iii) $(-2)+(+2)$,
(iv) $(-2)+(-2)$, (v) $(-8)+(+4)$, (vi) $(-11)+(-7)$.

8. Find the values of:

(i) $(+11a)+(-9a)$, (ii) $(-11a)+(-9a)$, (iii) $(-2b)+(-2b)$,
(iv) $(-2b)+(+2b)$, (v) $(+4c)+(-7c)$, (vi) $(-3c)+(-9c)$.

9. Fill in the blank spaces:

(i) $(+3)+(\quad)=(+5)$, (ii) $(+3)+(\quad)=(-2)$,
(iii) $(+7)+(\quad)=0$, (iv) $(+7)+(\quad)=(-7)$,
(v) $(-11)+(\quad)=(-11)$, (vi) $(+8)+(\quad)=(+4)$.

10. Find the values of :

(i) $(+7)-(+9)$, (ii) $(-3)-(+9)$, (iii) $(-3)-(-9)$,
(iv) $(-3)-(-1)$, (v) $(+7)-(-5)$, (vi) $(-5)-(-5)$.

11. Find the values of :

(i) $(-5x)-(+12x)$, (ii) $(-5x)-(-12x)$, (iii) $(-3t)-0$,
(iv) $(-3t)-(-t)$, (v) $(+8t)-(-6t)$, (vi) $(-8t)-(-7t)$.

12. Find the values of :

(i) $(+7)-(-3)$, (ii) $(+8)+(-8)$, (iii) $(-6)+(-15)$,
(iv) $(-6)-(-15)$, (v) $0-(-2)$, (vi) $(-7)+(+4)$.

EXERCISE 20. b

(Many of these may be taken orally)

1. The temperature (Centigrade) is $(-16°)$. What does it become after (i) a rise of $(+28°)$, (ii) a fall of $(+28°)$, (iii) a rise of $(-7°)$, (iv) a fall of $(-7°)$?

2. The temperature (Centigrade) is $(+16°)$. What does it become after (i) a fall of $(+28°)$, (ii) a rise of $(-7°)$, (iii) a fall of $(-7°)$?

3. I have £$(+7b)$ in the bank at the beginning of the week. How much have I at the end of the week, if during the week :

(i) I pay in £$(+3b)$ and take out £$(+15b)$,
(ii) I pay in £$(-2b)$ and take out £$(-3b)$?

4. A man has a bank balance of £10 ; he deposits £20, withdraws £15, withdraws £35. What is his balance now?

5. A man starts the year with a balance of £20, and ends the year with a deficit of £15. What has he (*a*) gained, (*b*) lost during the year? Express each answer as a directed number.

6. A train is moving with a speed of 45 miles an hour. What are the speeds relative to this train of others moving in the same direction at 62 miles an hour and at 27 miles an hour ; also of one at rest?

7. Find the values of :

(i) $(-13)+(+7)$, (ii) $(+13)+(-11)$, (iii) $(-13)+(-7)$,
(iv) $(-3)+(+7)$, (v) $(-3)+(+3)$, (vi) $(-17)+(-12)$.

8. Find the values of :

(i) $(-13x)+(+7x)$, (ii) $(+13t)+(-11t)$, (iii) $(-13t)+(-7t)$,
(iv) $(-3c)+(+3c)$, (v) $(+7d)+(-12d)$, (vi) $(-17d)+(+12d)$

9. Fill in the blank spaces :

(i) $(+11)+(\quad)=0$, (ii) $(-9)+(\quad)=(+6)$,
(iii) $(+5)+(\quad)=(-9)$, (iv) $(+8)+(\quad)=(-8)$,
(v) $(-7)+(\quad)=(-7)$, (vi) $(-9)+(\quad)=(-2)$.

10. Find the values of :

(i) $(+13)-(+8)$, (ii) $(-2)-(+14)$, (iii) $(-6)-0$,
(iv) $(+10)-(-8)$, (v) $(+13)-(+20)$, (vi) $(-7)-(-20)$.

11. Find the values of :

(i) $(-7u)-(-5u)$, (ii) $(+4l)-(-5l)$, (iii) $(-20z)-(-35z)$,
(iv) $(-7x)-(+8x)$, (v) $(-7m)-(-9m)$, (vi) $(+11k)-(+28k)$.

12. Find the values of :

(i) $(+41)-(-11)$, (ii) $(-33)-(-22)$, (iii) $(-1)+(+8)$,
(iv) $(-2)+(-33)$, (v) $(+6)+(-6)$, (vi) $0-(-7)$.

EXERCISE 20. c

1. Find the values of :

(i) $(-5x)-(-3x)$, (ii) $(+7y^2)-(-3y^2)$, (iii) $(+4xy)+(-7xy)$,
(iv) $(+4xy)-(-7xy)$, (v) $(+ab)+(-6ab)$, (vi) $(-st)-(-ts)$.

2. Find the values of :

(i) $-5-4$, (ii) $-11+7$, (iii) $5-7+3$,
(iv) $5+8-17$, (v) $-3+8-9$, (vi) $-4-3-5$.

3. Find the values of :

(i) $-8-7+3$, (ii) $-2-7-12$, (iii) $-4+23-27$,
(iv) $10-25+20+5$, (v) $-25+20-10+5$, (vi) $-17+11-5+11$.

4. Simplify :

(i) $2a-3b-5b-a$, (ii) $5x-3y-7x+9y$,
(iii) $0-3s-4t+5s$, (iv) $7c^2-4c-5c^2$,
(v) $4t^2-3t-7t+t^2$, (vi) $4xy-x^2-3yx-2x^2$.

5. Simplify, and arrange in descending powers of x :

(i) $3-2x^2-3x-7x+5x^2-7$, (ii) $-2-4x+7x^2+7+2x-9x^2$,
(iii) $4x-9x^2-5-3+2x-x^2$, (iv) $4x^2-3x-11+5x-2-7x^2$,
(v) $2-3x^3-4x-5x^2+11x$, (vi) $x^3-3x^2+11-4x^3$.

6. Simplify, and arrange in ascending powers of x :

(i) $5-2x^2-5x-9x-3x^2+7$, (ii) $-4-6x+9x^2+9+4x-3x^2$,
(iii) $3x-8x^2-7-4+4x-3x^2$, (iv) $7x^2-5x+13-5x-4-9x^2$,
(v) $11-4x^3-7x+x^2-5x$, (vi) $3x^3-7x^2+3+3x^2$.

7. Simplify :

(i) $(2a+7)-(3a+9)$, (ii) $(c^2-2d)-(c^2-d)$,
(iii) $(s+t+5)-(s+t+8)$, (iv) $(7-a-3b)+(2a+b+2)$,
(v) $0-(2z-3)$, (vi) $(t^2-2t+1)-(t^2+2t+1)$.

8. Simplify :

(i) $(x^2-3x)-(2x^2+x)$, (ii) $(3x^2+2x+1)-(7x^2+5x+3)$,
(iii) $(x^3-1)-(x^3-3x^2+3x-1)$, (iv) $(2a^2-3a-4)-(3a^2+3a+4)$,
(v) $(8-a^3)+(a^3-8)$, (vi) $(3a-4b-5c)-(8a-7b+2c)$.

MULTIPLICATION AND DIVISION

47. Multiplication. To illustrate the rule for multiplying two directed numbers it is necessary to find two quantities measured by directed numbers such that their product has a meaning.

Consider a man who saves x shillings a week. At the end of t weeks he will have saved xt shillings.

(i) If $x = +3$, $t = +8$, the man saves 3 shillings per week and will have saved 24 shillings at the end of 8 weeks.

Thus $$(+3) \times (+8) = +24.$$

(ii) If $x = +3$, $t = -8$, the man saves 3 shillings per week and at the end of (-8) weeks, i.e. 8 weeks ago, he was 24 shillings worse off than now, i.e. he had saved (-24) shillings.

Thus $$(+3) \times (-8) = -24.$$

(iii) If $x = -3$, $t = +8$, the man saves (-3) shillings per week, i.e. he loses 3 shillings per week, and at the end of 8 weeks he will have lost 24 shillings, i.e. he will have saved (-24) shillings.

Thus $$(-3) \times (+8) = -24.$$

(iv) If $x = -3$, $t = -8$, the man saves (-3) shillings per week, i.e. he loses 3 shillings per week, and at the end of (-8) weeks, i.e. 8 weeks ago, he was 24 shillings better off than now, i.e. he had saved 24 shillings.

Thus $$(-3) \times (-8) = +24.$$

Other illustrations may be taken, e.g. the letters may represent speed and time, or rate of increase of temperature and time. From all such illustrations it is clear that products of directed numbers are calculated according to the following rules:

$$(+\mathbf{a}) \times (+\mathbf{b}) = (+\mathbf{ab}) = \mathbf{ab}; \qquad (-\mathbf{a}) \times (-\mathbf{b}) = (+\mathbf{ab}) = \mathbf{ab}.$$
$$(+\mathbf{a}) \times (-\mathbf{b}) = (-\mathbf{ab}) = -\mathbf{ab}; \qquad (-\mathbf{a}) \times (+\mathbf{b}) = (-\mathbf{ab}) = -\mathbf{ab}.$$

Division.

Since $(+3) \times (+8) = (+24)$, $\therefore$ $(+24) \div (+8) = (+3)$.
Since $(+3) \times (-8) = (-24)$, $\therefore$ $(-24) \div (-8) = (+3)$.
Since $(-3) \times (+8) = (-24)$, $\therefore$ $(-24) \div (+8) = (-3)$.
Since $(-3) \times (-8) = (+24)$, $\therefore$ $(+24) \div (-8) = (-3)$.

In general, we have the following rules for division:

$$(+\mathbf{a}) \div (+\mathbf{b}) = \left(+\frac{\mathbf{a}}{\mathbf{b}}\right) = \frac{\mathbf{a}}{\mathbf{b}}; \qquad (-\mathbf{a}) \div (-\mathbf{b}) = \left(+\frac{\mathbf{a}}{\mathbf{b}}\right) = \frac{\mathbf{a}}{\mathbf{b}}.$$

$$(+\mathbf{a}) \div (-\mathbf{b}) = \left(-\frac{\mathbf{a}}{\mathbf{b}}\right) = -\frac{\mathbf{a}}{\mathbf{b}}; \qquad (-\mathbf{a}) \div (+\mathbf{b}) = \left(-\frac{\mathbf{a}}{\mathbf{b}}\right) = -\frac{\mathbf{a}}{\mathbf{b}}.$$

To sum up, in multiplication and division of one directed number by another, the final result is positive if the two numbers have the same sign, and negative if the two numbers have opposite signs.

48. Multiplication and division by zero.

0×3 means three times nothing, which is nothing;

$$\therefore\ 0 \times 3 = 0.$$

Also $$3 \times 0 = 0 \times 3 = 0,$$

and $$-3 \times 0 = -0 = 0.$$

Similarly, if there are more than two factors, the product is zero, if one factor is zero.

Also, if 0 is divided by any number which is not itself zero, the quotient is 0.

But division by 0 is quite meaningless. The operation $a \div b$, when $b=0$, has not been defined and has no meaning. The pupil must be very careful not to attempt to evaluate an expression containing a zero factor in the denominator.

As soon as the pupil has mastered these rules the use of brackets to denote directed numbers may be dropped.

EXERCISE 21. a

(Many of these may be taken orally)

Find the values of:

1. $(-3) \times (+7)$. **2.** $(-3) \times (-6)$. **3.** $(+3) \times (-5)$.
4. $(+3) \times (+4)$. **5.** $(+3) \times (-8)$. **6.** $(-3) \times (-10)$.
7. $(-3) \times (+10)$. **8.** $(-2) \times (-12)$. **9.** $(+2) \times (+17)$.
10. $(-2) \times (+20)$. **11.** $(+2) \times (-9)$. **12.** $(-5) \times (+6)$
13. $(-5) \times (-12)$. **14.** $(+5) \times (-1)$. **15.** $(-6) \times (0)$.
16. $(-12) \div (-2)$. **17.** $(-12) \div (+3)$. **18.** $(+12) \div (-4)$.
19. $(-12) \div (-1)$. **20.** $(+18) \div (-6)$. **21.** $(-18) \div (-9)$.
22. $(-18) \div (+3)$. **23.** $(-48) \div (+12)$. **24.** $(+48) \div (-3)$.
25. $(-48) \div (-8)$. **26.** $(-72) \div (+24)$. **27.** $(-72) \div (-12)$.
28. $(-72) \div (+4)$. **29.** $(+84) \div (-7)$. **30.** $(-84) \div (-21)$.

If $a=(+1)$, $b=(-2)$, $c=(+3)$, $p=(-1)$, $q=(+2)$, $r=0$, find the values of:

31. $2a-3b$. **32.** $abc+pqr$. **33.** $4bpq$. **34.** ar^2.
35. ap^2c. **36.** $2bc$. **37.** $3qr$. **38.** bcq^2.
39. a^3b^3. **40.** p^3q^3. **41.** $3cp^2$. **42.** $3cq^2$.

Example 2. *Subtract* $3p-2q+5r$ *from* $4p-9q+2r$

If we use brackets we write

$$(4p-9q+2r)-(3p-2q+5r)$$
$$=4p-9q+2r-3p+2q-5r.$$

We now simplify, by collecting like terms, obtaining $p-7q-3r$.

It is clear that the same result is obtained by changing the sign of every term in the expression $3p-2q+5r$ and adding the changed expression to $4p-9q+2r$.

Hence, if we arrange the working as in Arithmetic, we have the following rule :

Write the expression to be subtracted beneath the expression from which it is to be subtracted, placing like terms in the same column. Change the sign of each term in the bottom line (mentally) and add, e.g.

$$\begin{array}{r} 4p-9q+2r \\ \underline{3p-2q+5r} \\ p-7q-3r \end{array}$$

As with addition, it is very doubtful whether this method is better than the method previously adopted. It will, however, be found very useful in the solution of simple simultaneous equations, and also in long division and square root. For this reason it seems desirable to provide ample opportunity for practising the method.

EXERCISE 22. a

Add together the following expressions :

1. $3a+4b-2c$; $-2a+2b+3c$; $4a-2b+2c$.

2. $3x+5y-6z$; $5x-3y+3z$; $7x+3y-4z$.

3. $6a+b-2c$; $-5a-b+c$; $-a+3b-c$.

4. $l-3m-3n$; $5l-3m+2n$; $7l-5m-7n$.

5. $6xy+3yz-zx$; $2xy-4yz$; $-xy+zx$.

6. $p-2q+3r-s$; $2r+3s$; $2p-q+s$; $q-3r$.

7. $-K^2+3K-1$; $3K^2+K-9$; $-2K^2-5K+6$.

8. $5t^2+2t-3$; $2t^2+2t+1$; $3t^2-4t+8$.

9. $4l^2-3$; $4l+7$; $-3l^2-6$; l^2+3l.

10. $6x^4+3x^3$; x^4-x^3 ; $-2x^3+7x^2$; $-5x^2-8x$.

11. $-2x^3+2x^2y-2xy^2-y^3$; $-4x^2y+6xy^2$; $-2x^2y+y^3$.

12. $1-t^3$; t^4-t^2-3 ; t^5+3t^2-6 ; $-2t+4+t^3$.

Subtract :

	(i)	(ii)	(iii)	(iv)
13.	$\begin{array}{r} 3a+2b \\ \underline{a+7b} \end{array}$	$\begin{array}{r} 7a-5b \\ \underline{4a+b} \end{array}$	$\begin{array}{r} 7a+3b \\ \underline{2a-6b} \end{array}$	$\begin{array}{r} 8a-3b \\ \underline{7a-7b} \end{array}$
14.	$\begin{array}{r} 2r-5s \\ \underline{r-s} \end{array}$	$\begin{array}{r} 6r+3s \\ \underline{2r-3s} \end{array}$	$\begin{array}{r} 7r+2s \\ \underline{2r+7s} \end{array}$	$\begin{array}{r} 8r-5s \\ \underline{2r+3s} \end{array}$
15.	$\begin{array}{r} 2x+3y \\ \underline{x+3y} \end{array}$	$\begin{array}{r} 5x-7y \\ \underline{8x-9y} \end{array}$	$\begin{array}{r} 2x-8y \\ \underline{7x-8y} \end{array}$	$\begin{array}{r} 2x+8y \\ \underline{7x-8y} \end{array}$
16.	$\begin{array}{r} a \\ \underline{4a-3k} \end{array}$	$\begin{array}{r} c-d \\ \underline{3d} \end{array}$	$\begin{array}{r} 4x+4y \\ \underline{7x} \end{array}$	$\begin{array}{r} 2z \\ \underline{y+5z} \end{array}$

17. $2x+2y+z$ from $x+5y-z$ **18.** $3x-2y+5z$ from $4x-7y+2z$.

19. $-2p-5q$ from $p+2q-3r$. **20.** $-2a-6b$ from $4a-3b-5c$.

21. $-xy+2yz-3zx$ from $5xy-3yz+4zx$.

22. x^2+x+1 from x^3-1. **23.** x^2+x-1 from x^3+1.

24. $-x^4-x^3+5x$ from x^3+5x^2+x.

25. a^3-b^3 from $a^3-3a^2b+3ab^2+b^3$.

26. $4-a+a^2+2a^3$ from $7+a-a^2$

What must be added to

27. $x-y-z$ to give $x+z$? **28.** $2a^2+b^2-c^2$ to give $5a^2-3b^2+2c^2$?

EXERCISE 22. b

Add together the following expressions :

1. $3x-2y+4z$; $2x+5y-2z$; $2x+3y-2z$.

2. $5r-4s-t$; $-3r+3s+t$; $-2r+s+5t$.

3. $8a+5b-2c$; $-6a+4b+7c$; $-a-8b-c$.

4. $3c-4b+5a$; $c-4b+3a$; $-2c+8b-5a$.

5. $-ab+bc+ca$; $-5ab-2bc+4ca$; $ab+bc-ca$.

6. $4p-q+r+2s$; $q-r+2s$; $r-5s$; $3p+2s$.

7. $-2x^2+4x+2$; x^2-6x+7 ; $3x^2+3x-9$.

8. t^3-2t^2+t-5 ; $3t^2+5t+2$; t^3+2t^2-6t.

9. $a^4+3a^3+a^2-7a$; $-2a^3-3a^2+a$; $5a^4+2a^2+6a$.

10. $-c+3c^3$; $6c^2+3c$; c^4-8c^3 ; $7c^4-2c^2$.

11. $a^4-2a^2b^2+b^4$; $-3a^3b+2a^2b^2$; $5a^3b-b^4$; $2ab^3-a^4$.

12. $5-2x+3x^3$; $-3+x^3+x^4$; $-2+2x-x^3-x^4+x^5$.

Subtract :

	(i)	(ii)	(iii)	(iv)
13.	$4a+9b$ $a-5b$	$6a-5b$ $2a-8b$	$5a+4b$ $2a+5b$	$9a-7b$ $2a+2b$
14.	$3x-7y$ $5x-7y$	$3x+7y$ $5x-7y$	$6x+7y$ $2x+7y$	$3x-6y$ $6x-9y$
15.	$2a$ $7a-5h$	$2c-5d$ $8d$	$2x+7y$ $9x$	$5z$ $3x+8z$
16.	$5r-3s$ $2r+s$	$7r-2s$ $3r+4s$	$4r+7s$ $r-7s$	$5r+s$ $3r+8s$

17. $5a-4b+3c$ from $3a+b-c$. **18.** $-p-q-4r$ from $-p+q-4r$.

19. $-l+9m-3n$ from $7l-5m$. **20.** $ab-2bc-4ca$ from $5ab-2ca$.

21. $-ab+7bc-5ca$ from $-5ca$. **22.** $c^2-2cd+d^2+7$ from c^2+d^2.

23. a^3-3ab^2 from $a^3+3a^2b+3ab^2+b^3$

24. $2x^2y-3xy^2$ from $3x^3-5xy^2+2y^3$.

25. $4+4m+3m^2-m^4$ from $1-m+m^4$.

26. x^4+2x^3-5x from $4x^2-5x^3+3$.

What must be added to

27. $x-2y-3z$ to give $2x+5z$?

28. $3x^2-4y^2-5z^2$ to give $-x^2-2y^2+3z^2$?

SIMPLE EQUATIONS INVOLVING DIRECTED NUMBERS

50. In Chapter IV we solved the equation $7-2x=3$ by the successive steps :

(1) $7=3+2x$, (2) $7-3=2x$, (3) $4=2x$,

(4) $2=x$, (5) $x=2$.

The steps given were then necessary because we had not learnt to use directed numbers.

It is simpler to proceed as follows :

$$7-2x=3,$$
$$\therefore\ -2x=3-7 \text{ (subtracting 7 from each side)},$$
$$\therefore\ -2x=-4,$$
$$\therefore\ x=2 \text{ (dividing each side by } -2\text{)}.$$

The pupil is now in a position to realise that any term in an equation may be transferred to the other side of the equation,

provided that the sign is changed. In the above example the effect of subtracting 7 from each side is to transfer 7 to the opposite side of the equation with a changed sign.

In practice it is quicker to apply this rule than to apply the axioms, but the student should not use it, unless he is able to justify its use and give the argument in full.

The following example illustrates the usual procedure.

Example 3. *Solve the equation* $2x+12-x=6x-8-3x$.

Collect all terms containing x on the left-hand side of the equation, all other terms on the right-hand side. We then have,

$$2x-x-6x+3x=-8-12,$$
$$\therefore\ -2x=-20,$$
$$\therefore\ x=10.$$

Check. If $x=10$, L.H.S. $=20+12-10=22$,
R.H.S. $=60-8-30=22$.

EXERCISE 23. a (*Oral*)

Solve the equations : [Further practice in substitution may be obtained by checking the solutions.]

1. $7x=-7$. **2.** $-4x=8$. **3.** $-2x=-12$.
4. $-3x=27$. **5.** $-5x=-25$. **6.** $3t=18$.
7. $4t=-24$. **8.** $-10a=20$. **9.** $8b=32$.
10. $-11r=-77$. **11.** $3z=-33$. **12.** $-6s=0$.
13. $\frac{x}{5}=-3$. **14.** $\frac{x}{6}=-\frac{1}{2}$. **15.** $-\frac{x}{4}=2$.
16. $-\frac{x}{3}=-1$. **17.** $-\frac{x}{3}=-\frac{2}{3}$. **18.** $\frac{t}{5}=-\frac{3}{5}$.
19. $-\frac{s}{10}=-\frac{2}{5}$. **20.** $-\frac{a}{8}=\frac{1}{4}$. **21.** $\frac{b}{8}=-\frac{3}{4}$.
22. $-\frac{r}{5}=0$. **23.** $7x=-2$. **24.** $-\frac{z}{18}=\frac{2}{9}$.

EXERCISE 23. b (*Oral*)

Solve the equations : [Further practice in substitution may be obtained by checking the solutions.]

1. $-3x=6$. **2.** $-5x=-15$. **3.** $8x=-8$.
4. $3z=21$. **5.** $-7u=42$. **6.** $-9t=-63$.
7. $-12a=36$. **8.** $4c=20$. **9.** $6z=-60$.

10. $-11s=0.$ **11.** $-17x=-51.$ **12.** $13z=-65.$

13. $\frac{x}{2}=-1.$ **14.** $-\frac{x}{8}=3.$ **15.** $\frac{x}{6}=-2.$

16. $-\frac{x}{8}=-\frac{1}{2}.$ **17.** $\frac{u}{12}=-\frac{2}{3}.$ **18.** $-\frac{z}{5}=-5.$

19. $-\frac{a}{20}=\frac{1}{5}.$ **20.** $\frac{c}{36}=-\frac{2}{9}.$ **21.** $-\frac{s}{28}=-\frac{4}{7}.$

22. $-\frac{z}{42}=\frac{3}{7}.$ **23.** $-\frac{l}{11}=0.$ **24.** $-9x=-5.$

EXERCISE 23. c

Solve the equations :

1. $2x+x=24+7x.$ **2.** $9+x+6x=2x.$

3. $-4x-5=2+3x.$ **4.** $7x+27=3x+11.$

5. $5x+15=8x+24.$ **6.** $2x+15=-27-4x.$

7. $x-6x=3x.$ **8.** $19-2x-x=3-7x.$

9. $-3x+48=20+4x+7x.$ **10.** $-7x+5x+16=29+x-4.$

11. $11x-11+6x=5x-35.$ **12.** $x+11-3x=-15-15x.$

13. $7(x-2)-4(x-3)=-8.$ **14.** $7(x-2)+4(x-3)=-4.$

15. $4-2(2-x)=4(x-5)-6.$ **16.** $4(x-1)+8+5x=3(x-2).$

17. $2(x-1)-3(3-x)=8.$ **18.** $2(2z-1)-5(2+z)=10.$

19. $(t-14)-6(2t+5)=10(t+4).$

20. $2(26-t)-4(4t+1)+3(2-3t)=0.$

21. $6(1-z)=4(3+z)-3(z+9).$ **22.** $2(2s-1)-5(3s+1)=4.$

23. $2(x+1)-3(x-1)=6+4x.$

24. $0=6(K+1)-(3-K)-3(K-2).$

25. $6(4-c)-4(10+2c)-12=0.$

26. $5(x-4)-2(x-3)-13=12-3(x-5).$

27. $12(2l-1)-3(4l-3)=9-17(1-l).$

28. $3x-2(5x-9)+3=7(2x-3)-2(x-2).$

29. $5(d-2)-3(3d-4)=34-7(3-d).$

30. $2(x-14)-5(13+2x)=39.$

31. $0=3[15-2(10-x)]-5(x-4).$

32. $20+(x-6)-(x-4)+2(x-2)=22+(x-2)-(x-4).$

33. $4+(7t-1)-3(t+8)=2-7(1-t)+3t+14.$

34. $(3a+10)-4(1-a)-(a-5)=3(a+3)-7(a+4).$

35. $2(c-8)-7(3c+5)=4(c-2)-6(3c-1).$

36. $7(z-3)-3(2-z)=(z-6)-(3-7z)-18.$

CHAPTER XI

THE INDEX LAWS. SIMPLE MULTIPLICATION AND DIVISION. SIMPLE ROOTS. H.C.F. AND L.C.M.

51. In Chapter II the index notation was explained. The pupil should revise this chapter.

We now proceed to further work involving the use of indices.

Example 1. *Multiply* $2x^2 \times 3x^4$.

$$2x^2 \times 3x^4 = 2 \times x \times x \times 3 \times x \times x \times x \times x$$
$$= 2 \times 3 \times x \times x \times x \times x \times x \times x,$$

since the factors may be taken in any order,

$$= 6 \times x^6, \text{ by definition}$$
$$= 6x^6.$$

Example 2. *What is the shortest way of writing*

$$2 \times a \times b \times 3 \times a \times a \times b?$$

The expression $= 2 \times 3 \times a \times a \times a \times b \times b$, rearranging the factors,

$$= 6a^3b^2.$$

Example 3. *Multiply* $18t^4$ *by* $\frac{4}{9}st$.

The expression $= 18 \times t \times t \times t \times t \times \frac{4}{9} \times s \times t$

$$= 18 \times \tfrac{4}{9} \times s \times t \times t \times t \times t \times t,$$

rearranging the factors, $\qquad = 8st^5$.

52. As in Arithmetic, the value of a fraction is unaltered by multiplying or dividing both its numerator and denominator by the same number or expression. The number or expression must not, however, be zero. It must be borne in mind in the work which follows that there are values of the letters for which division is impossible.

Example 4. *Divide* $24x^5y^2z$ *by* $15x^3yz$.

$$24x^5y^2z \div 15x^3yz = \frac{24 \times x \times x \times x \times x \times x \times y \times y \times z}{15 \times x \times x \times x \times y \times z}$$

$$= \frac{8 \times x \times x \times y}{5}, \text{ cancelling as in Arithmetic,}$$

$$= \frac{8x^2y}{5}.$$

Note. If $x=0$, or if $y=0$, or if $z=0$, $24x^5y^2z \div 15x^3yz$ has no meaning

53. As in Arithmetic, the root of any expression is that quantity which will produce the given expression by being raised to the power denoted by the index of the root.

Thus $\sqrt[2]{x}$ (which is usually written $\sqrt{x}$), the **square root** of x, is such that $\sqrt{x} \times \sqrt{x} = x$.

Similarly $\sqrt[3]{x}$, the **cube root** of x, is such that

$$\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x} = x;$$

$\sqrt[10]{(a+b)}$, the tenth root of $(a+b)$ is such that

$$\{\sqrt[10]{(a+b)}\}^{10} = (a+b).$$

54. Since, by the rule of signs,

$$(+x) \times (+x) = (+x^2) = x^2 \quad \text{and} \quad (-x) \times (-x) = (+x^2) = x^2,$$

it follows that a positive number, x^2, has two square roots, $(+x)$ and $(-x)$ or $+x$ and $-x$. These are sometimes written in the form $\pm x$. **There is no square root of a negative number.**

Similarly $\quad \sqrt[4]{x^4} = \pm x, \quad \sqrt[6]{x^6} = \pm x$ etc.

But $\quad (+x) \times (+x) \times (+x) = (+x^3) = x^3$

and $\quad (-x) \times (-x) \times (-x) = (-x^3) = -x^3.$

The cube root of x^3 is therefore $(+x)$ or x, and the cube root of $-x^3$ is $(-x)$ or $-x$.

Similarly $\quad \sqrt[5]{x^5} = x, \quad \sqrt[7]{x^7} = x$, etc.

and $\quad \sqrt[5]{-x^5} = -x, \quad \sqrt[7]{-x^7} = -x$, etc.

Example 5. *Find the values of the square root of t^6.*

$$t^3 \times t^3 = t^6 \quad \text{and} \quad (-t^3) \times (-t^3) = t^6,$$

$$\therefore \text{ the square roots of } t^6 \text{ are } t^3 \text{ or } -t^3.$$

Similarly the square roots of $4t^6$ are $2t^3$ and $-2t^3$.

It is a convention to write $+\sqrt{x}$ or $\sqrt{x}$ when the positive value of the square root is to be taken, and $-\sqrt{x}$ when the negative value is to be taken. Similarly when the fourth, sixth or any even root is to be taken.

[Exercises 25 *a* and *b*, Nos. 1–7 may now be done.]

THE INDEX LAWS

55. By generalising the above work we obtain the index laws.

I. Just as $x^2 \times x^4 = x \times x \times x \times x \times x \times x = x^6$, so

$$a^m \times a^n = a \times a \times \dots \text{ to } m \text{ factors } \times a \times a \times \dots \text{ to } n \text{ factors},$$
$$= a \times a \times \dots \text{ to } (m+n) \text{ factors},$$
$$= a^{m+n}, \text{ provided that } m \text{ and } n \text{ are positive integers.}$$

Similarly $$a^m \times a^n \times a^p = a^{m+n+p},$$
$$a^m \times a^n \times a^p \times a^r \times \dots = a^{m+n+p+r} \dots,$$
provided that $m, n, p, r \dots$ are positive integers.

Or in words,

When a power of a variable is multiplied by other powers of the **same** variable, the index of that variable in the product is the **sum** of the indices in the terms of the product.

It should be particularly noted that this rule does not enable us to simplify the product of different variables. Thus $a^3 \times b^2 = a^3b^2$, and cannot be written in any simpler form.

II. Just as $x^5 \div x^3 = \dfrac{x \times x \times x \times x \times x}{x \times x \times x} = x^2$, so

$$a^m \div a^n = \frac{a \times a \times a \times \dots \text{ to } m \text{ factors}}{a \times a \times a \times \dots \text{ to } n \text{ factors}}$$
$$= a \times a \times a \times \dots \text{ to } (m-n) \text{ factors},$$
$$= a^{m-n}, \text{ provided that}$$

(i) m and n are positive integers,

(ii) $m > n$ (if $m = n$ the quotient is 1),

(iii) a is not zero.

It should be particularly noted that $a^m \div a^n$ is meaningless if $a = 0$; if $m < n$, we cannot at this stage attach a meaning to a^{m-n}, but we can divide the numerator into the denominator, getting $\dfrac{1}{a^{n-m}}$.

We may express this result in words:

When a power of a variable is divided by a **smaller** power of the **same** variable, the index of that variable in the quotient is the greater index **minus** the smaller index.

It should be particularly noted that this rule does not enable us to simplify the quotient of different variables. Thus $a^3 \div b^2 = \dfrac{a^3}{b^2}$, and cannot be written in any simpler form.

III. As a particular case of I, we have

$$(a^m)^n = a^m \times a^m \times a^m \times \dots \text{ to } n \text{ factors},$$
$$= a^{m+m+m+\dots \text{ to } n \text{ terms}},$$
$$= a^{mn}.$$

There is no rule for simplifying

$$a^m + a^n \quad \text{and} \quad a^m - a^n.$$

EXERCISE 24. a

(Many of these may be taken orally)

1. Write in the shortest way :

(i) $a \times 3 \times a$, (ii) $b \times b \times b$, (iii) $c \times c \times 3 \times c$,
(iv) $x \times x^2 \times x$, (vi) $x^2 \times x^3$, (vi) $x^5 \times x$,
(vii) $zzzz$, (viii) $a + a + a + a$, (ix) $x \times 5$,
(x) $1 \times t \times t \times t$, (xi) $dddd - d$, (xii) $x \times y \times y \times x \times x$.

2. Simplify :

(i) $x^3 \times x^4$, (ii) $a^5 \times a^4$, (iii) $a^6 \times a$,
(iv) $2x \times 3y$, (v) $3x \times 4xy$, (vi) $ab \times bc$,
(vii) $x^7 \div x^2$, (viii) $x^7 \div x^3$, (ix) $a^5 \div a$,
(x) $4x^7 \div x^4$, (xi) $4x^7 \div 2x^4$, (xii) $6x^8 \div 4x^3$.

3. Simplify, *if possible* :

(i) $6c^2 \times 2c$, (ii) $6c^2 \div 2c$, (iii) $2de^2 \times 2e^2d$,
(iv) $p + p^2 + p^3$, (v) $p \times p^2 \times p^3$, (vi) $p \times 2p \times 3p$,
(vii) $6a \times \frac{a}{3}$, (viii) $6a \div \frac{a}{3}$, (ix) $\frac{3x}{4} \times \frac{4x}{3}$,
(x) $x^3 + 2x$, (xi) $2b \times 2b^2$, (xii) $2b \times (2b)^2$.

4. Find the squares of :

$$\frac{2}{3}x, \quad -\frac{1}{3}x, \quad \frac{z^2}{5}, \quad -\frac{2}{t}, \quad 2st, \quad -a^3, \quad -\frac{d^4}{4}, \quad x^3y^2z.$$

5. Find the square roots of :

$$\frac{x^2}{9}, \quad 4b^2, \quad \frac{a^2}{b^2}, \quad \frac{a^2}{b^2c^2}, \quad \frac{(-x)^4}{y^2z^2}, \quad K^8$$

6. Find the cubes of :

$$4x, \quad -3y, \quad \frac{2z}{5}, \quad -\frac{2}{t}, \quad -a^3, \quad 2pqr.$$

7. Find the cube roots of :

$$8a^3, \quad \frac{b^3}{64}, \quad -27z^3, \quad -\frac{y^3}{27}, \quad -K^6, \quad K^9.$$

8. Simplify :

(i) $8a^5b^3 \times 3bc$, (ii) $3a^2lm \times 5alm^4$, (iii) $-3 \times -4a^2$,

(iv) $-3\times(-4a)^2$, (v) $-3axy\times 6xyz$, (vi) $a^4x^3\times -abxy$,
(vii) $7a^7b^2\times 3bc^4$, (viii) $2a^2bc\times 5ab^2c^3$, (ix) $-3a^3b^5\times -2a^7b^4$,
(x) $-4a^2t^6\times 6at^2$, (xi) $a^2\times a^6\times a^5$, (xii) $x^3\times x^4\times x^8$.

EXERCISE 24. b

(*Many of these may be taken orally*)

1. Write in the shortest way:

(i) $c\times c\times 5$, (ii) $k\times 3\times k\times 2$, (iii) $t\times t\times t\times t$,
(iv) $x^2\times x^4$, (v) $x\times x^3\times 1$, (vi) $d+d+2d+1$,
(vii) $x^7\times x$, (viii) ccc, (ix) $1\times v\times v\times 1$,
(x) $a\times b\times c\times b$, (xi) $u\times u\times 5$, (xii) $aaa-aa$.

2. Simplify:

(i) $t^2\times t^4$, (ii) $t^6\times t^3$, (iii) $5c\times 4d^2$,
(iv) $a\times a^4$, (v) $ac\times ca$, (vi) $2t\times 5s^2t^2$,
(vii) $x^6\div x^3$, (viii) $t^8\div t^5$, (ix) $l^7\div l$,
(x) $3c^8\div c^3$, (xi) $9a^6\div 3a^2$, (xii) $10c^9\div 6c^4$.

3. Simplify, *if possible*:

(i) $5c^2\times 2c^2$, (ii) $8t^3\div 4t$, (iii) $3x^2y^2+5y^2x^2$,
(iv) $1\times r\times r^2$, (v) $1\times 2r\times 3r$, (vi) $1+r+r^2$,
(vii) u^4+5u^2, (viii) $\frac{2}{5}u\times\frac{5}{4}u^2$, (ix) $5t^2\div\frac{t}{2}$,
(x) $12k\times\frac{k^2}{3}$, (xi) $3s\times(3s)^3$, (xii) $3s\times 3s^3$.

4. Find the squares of:

$$\frac{1}{4}c^2,\quad -\frac{1}{2}a^4,\quad -\frac{z}{3},\quad \frac{3}{x^3},\quad 3st^2,\quad (3st)^2,\quad -l^5,\quad a^2b^3c^4.$$

5. Find the square roots of:

$$\frac{a^2}{16},\quad 25c^2,\quad \frac{9x^2}{4y^2z^2},\quad \frac{(-x)^2}{9t^2},\quad R^{10},\quad \frac{C^2D^4}{E^4F^2}.$$

6. Find the cubes of:

$$5t,\quad -2k,\quad -\frac{3s}{4},\quad \frac{2}{a^3},\quad -b^4,\quad 3rs^2t.$$

7. Find the cube roots of:

$$27m^3,\quad -\frac{a^3}{125},\quad -8z^6.\quad \frac{U^3V^3}{8},\quad -64M^{12},\quad M^{15}.$$

8. Simplify:

(i) $4a^3rs\times 7ar^3s$, (ii) $-5\times -3c^3$, (iii) $-5\times(-3c)^3$,
(iv) $6a^6b^2\times 4b^2c^2$, (v) $c^3d^4\times -abcd$, (vi) $9a^6b^3\times 2b^2c$,
(vii) $3a^4bc\times 7ab^5c^3$, (viii) $-2abl\times 5a^2bl^4$, (ix) $-3a^3t^5\times 5at^4$,
(x) $-6a^4b^6\times -7a^6b^3$, (xi) $z^4\times z^5\times z^9$, (xii) $c^3\times c^5\times c^2$.

EXERCISE 24. c

1. Simplify :

(i) $12t^6 \div 3t^3$, (ii) $x^{15} \div x^5$, (iii) $-6t^4 \div (-2t)$,
(iv) $-6t^5 \div 3t^3$, (v) $x^4y^3z^2 \div x^2z$, (vi) $x^4y^3z^2 \div (-xy^2)$,
(vii) $x^8y^6z^5 \div x^4y^3z^4$, (viii) $-x^7y^5z^3 \div (-x^5yz^2)$,
(ix) $6l^2m^2n^2 \div 3lmn$, (x) $6x^2y \div 9xy^2$, (xi) $20t^4 \div 35t^6$,
(xii) $24a^3b \div 15b^2c^2$, (xiii) $\dfrac{x^3 \times x^6}{x^5}$, (xiv) $3z^3 \div (3z)^3$,
(xv) $2t^4 \div (-2t)^4$, (xvi) $\dfrac{lm \times (-ml}{mn}$
(xvii) $3x^3 \times 4ax^2 \div 2x$, (xviii) $-54a^9b^{10}c^{12} \div (-6a^6b^5c^4)$.

2. Simplify :

(i) $\sqrt{25a^6b^4c^2}$, (ii) $\sqrt{9a^4x^{10}}$, (iii) $-\sqrt{t^{20}}$,
(iv) $\sqrt[3]{8K^{18}}$, (v) $\sqrt[3]{-x^9y^{12}}$, (vi) $-\sqrt{\dfrac{36t^{12}}{25s^6}}$,
(vii) $\sqrt{\dfrac{64t^4}{81c^{10}}}$, (viii) $\sqrt[5]{x^{15}}$.

3. Simplify, *if possible* :

(i) $3x^4 \times 4x^3$, (ii) $3x^4 + 4x^3$, (iii) $3x^4 \div 4x^3$,
(iv) $3x^4 - 4x^3$, (v) $(t^3)^4 + (t^4)^3$, (vi) $(t^3)^4 \div (t^4)^3$,
(vii) $(t^3)^4 \times (t^4)^3$, (viii) $(t^3)^4 - (t^4)^3$, (ix) $8x^7 + 5x^7$,
(x) $8x^7 \times 5x^7$, (xi) $8x^7 \div 5x^7$, (xii) $8x^7 - 5x^7$,
(xiii) $(2a)^5$, (xiv) $2(-a)^5$, (xv) $(-2a)^5$,
(xvi) $-2(-a)^5$, (xvii) $\dfrac{3x}{2y} \div \dfrac{y}{x}$, (xviii) $\dfrac{10x^3}{y} \div \dfrac{2x}{5y^3}$,
(xix) $\dfrac{5x}{y} \div 2x$, (xx) $\dfrac{12x}{y^2} \div \dfrac{3x^2}{y}$, (xxi) $\dfrac{x^3 \times x^6 \times x^9}{x^{20}}$.

4. (i) By what must $3a^3b^4$ be multiplied to produce $21a^8b^6$?

(ii) By what must $-2a^6b^2$ be multiplied to produce $10a^6b^5c^2$?

(iii) By what must $-5a^2b^2c^2$ be multiplied to produce $-30a^3b^7c^6$?

(iv) To what power must c^3 be raised to give

(*a*) c^6, (*b*) c^9, (*c*) c^{21}, (*d*) c^{300}?

(v) Write down the square root of $64a^{18}b^{12}$ and the cube root of the result.

(vi) Write down the cube root of $64a^{18}b^{12}$ and the square root of the result.

MULTIPLICATION AND DIVISION OF A COMPOUND EXPRESSION BY A SIMPLE EXPRESSION

56. It has already been shown (in Chapter VI) that

$$x(y+z)=xy+xz, \quad x(y-z)=xy-xz,$$

and $a(p-q+r)=ap-aq+ar$, etc.; i.e. in multiplying a polynomial by a monomial, each term of the polynomial must be multiplied by the monomial.

Example 6. $(5a^2-3ab-2b^2)\times(-2ab^2)$
$$=5a^2\times(-2ab^2)+(-3ab)\times(-2ab^2)+(-2b^2)\times(-2ab^2)$$
$$=-10a^3b^2+6a^2b^3+4ab^4.$$

After a little practice, the intermediate step may be left out, and the product written down at once.

Similarly, to divide a polynomial by a monomial, each term of the polynomial must be divided by the monomial.

Example 7. $(12a-15b+9c)\div(-3)$
$$=12a\div(-3)-15b\div(-3)+9c\div(-3)$$
$$=-4a+5b-3c.$$

Example 8. $(24a^4b^2-20a^3b^6-16a^5b^3)\div 4a^3b.$
$$=24a^4b^2\div 4a^3b-20a^3b^6\div 4a^3b-16a^5b^3\div 4a^3b$$
$$=6ab-5b^5-4a^2b^2.$$

Note. The expression has no meaning if $a=0$ or if $b=0$.

After a little practice the intermediate step may be left out, and the quotient written down at once.

EXERCISE 25. a

Multiply:

1. $a+4b-3c$ by 4.

2. $3a-7b-2c$ by -5.

3. $xy+x^2y^2-x^3y^3$ by $2x$.

4. $2a^2-3ab-5b^2$ by $-2b$.

5. $5a^3-3a^2-6a+2$ by $6a^2$.

6. $4lm-3lm^2-7lm^3$ by $-3m^2$.

7. $s^3t^2-s^2t^4+2s$ by $-st^2$.

8. $a^3-3a^2b+3ab^2-b^3$ by $-3ab$.

Divide:

9. $5a-15b$ by 5.

10. $5a-25b$ by -5.

11. $5x^2-7x$ by x.

12. $2x^2-9x$ by $-x$.

13. $4a^2-12ab$ by $2a$.

14. $6a^2b-18ab^2$ by $3ab$.

15. $15x^3y-35xy^3$ by $-5xy$.

16. $ax-bx+cx$ by $-x$.

17. $7x^3-11x^2$ by x^2.

18. $-11x^4+13x^3$ by $-x^3$.

19. $14c^4 - 28c^2d^2$ by $7c^2$. **20.** $-5a^3b + 25a^2b^2$ by $-5ab$.

21. $-a^5b^2 - a^2b^5 + a^3b^4 - a^4b^3$ by ab. **22.** $36a - 54b - 81c$ by -9.

EXERCISE 25. b

Multiply:

1. $a + 7b - 5c$ by 6. **2.** $2a - 5b + 3c$ by -3.

3. $a^2 + 2ab + b^2$ by $3a$. **4.** $2xy + 3x^2y^2 - 4x^3y^3$ by $3y$.

5. $2ab + 3a^2b - 5a^3b$ by a^3. **6.** $8x^3 - 5x^2 + 4x - 7$ by $5x^2$.

7. $a^3 - 6a^2b + 12ab^2 - 8b^3$ by $-2ab$.

8. $-ab^3 + a^2b^2 - a^3b$ by a^4b^3.

Divide:

9. $8a - 24b$ by 4. **10.** $9a - 36b$ by -3.

11. $9x^2 - 5x$ by $-x$. **12.** $7x^2 - 8x$ by x.

13. $12a^2b - 48ab^2$ by $4ab$. **14.** $14a^2 - 49ab$ by $-7a$.

15. $15x^3 - 9x^2$ by x^2. **16.** $4ac - 3bc + c^2$ by $-c$.

17. $-17x^4 + 23x^3$ by $-x^3$. **18.** $18x^3y - 63xy^3$ by $-9xy$.

19. $-8x^4y + 28x^3y^2$ by $-4x^2y$. **20.** $26d^5 - 39d^3c^2$ by $-13d^2$.

21. $21a - 14b - 28c$ by -7. **22.** $-x^3y - xy^3 + x^2y^2$ by xy.

HIGHEST COMMON FACTOR

57. When an integral expression (see Art. 12) exactly divides two or more integral expressions it is said to be a **common factor** of those expressions.

Thus, 7 is a common factor of 14, 21 and 35.

x is a common factor of x^3, $3x^2$ and $8x$.

ab is a common factor of $3a^3b$, $4a^2b^2$ and $5ab^3$.

58. The **Highest Common Factor** of two or more integral algebraical expressions is the integral expression of highest degree which divides each of them without remainder. The abbreviation H.C.F. is used for the words Highest Common Factor.

For example, 1, a, x, ax, a^2, a^2x are common factors of a^2x and a^2x^3; the factor of highest degree is a^2x, $\therefore$ the H.C.F. is a^2x.

If H is the H.C.F. of a number of integral expressions A, B, C, …, then $\frac{A}{H}$, $\frac{B}{H}$, $\frac{C}{H}$, … are integral expressions, i.e. H exactly divides each of A, B, C, … .

In the case of simple expressions the H.C.F. can be written down by inspection.

Example 9. *Find the H.C.F. of* a^4b^4, $a^2b^3c^3$, a^4b^3c.

Consider in turn each of the letters a, b and c.

The highest power of a which is a common factor is a^2;

" " " b " " " " b^3.

The first term does not contain c as a factor, so that no power of c is a common factor.

Hence the H.C.F. is a^2b^3.

If the expressions have numerical coefficients, the H.C.F. of the expressions must have a numerical coefficient which is the H.C.F. of the numerical coefficients.

Example 10. *Find the H.C.F. of* $15a^3x^2$, $35a^2x^3$, $20a^3x$.

Consider in turn the numerical coefficients and each of the letters a, x.

The H.C.F. of 15, 35, 20 is 5.

The highest power of a which is a common factor is a^2,

" " " x " " " " x.

Hence the H.C.F. is $5a^2x$.

59. This method for finding the H.C.F. is analogous to the method used in Arithmetic for finding the H.C.F. by prime factors. But care must be taken not to confuse H.C.F. in Algebra with H.C.F. in Arithmetic.

In Arithmetic the H.C.F. of two or more numbers is the greatest number which will exactly divide them, e.g. 5 is the H.C.F. of 15, 35 and 20. In Algebra the ideas of greatest and least are not valid. We cannot say whether x^3, x^2, x are in ascending or descending order of magnitude unless we know the value of x.

If $x > 1$, they are in descending order of magnitude; if $x < 1$ in ascending order; if $x = 1$ they are all equal. But although we cannot arrange them in the order of their magnitude, we can arrange them in the order of their degree. When we find the H.C.F. in Algebra we are concerned with the expression of *highest* degree which is a factor of the given algebraical expressions. This may or may not be the *greatest* factor; for some numerical values of the letters it is the greatest, for others it is not. Thus, $3x$ is the H.C.F. of $3x$ and $6x^2$; but, if $x = \frac{1}{3}$, $3x = 1$, so that $3x$ although it is the algebraical H.C.F. is, in this instance, numerically less than the common factor 3.

LOWEST COMMON MULTIPLE

60. Similarly we distinguish between the Least Common Multiple (L.C.M.) in Arithmetic and the Lowest Common Multiple (L.C.M.) in Algebra. In Arithmetic the L.C.M. of a number of integers is the lowest integer which is exactly divisible by each of them. In Algebra the L.C.M. of a number of integral expressions is the integral expression of lowest degree which is exactly divisible by each of them.

If L is the L.C.M. of a number of integral expressions $A, B, C, \cdots$, then $\frac{L}{A}, \frac{L}{B}, \frac{L}{C}, \cdots$ are integral expressions, i.e. A, B, C, each exactly divides L.

In the case of simple expressions, the L.C.M. can be written down by inspection.

Example 11. *Find the L.C.M. of x^3y, xy^4, xy^2z.*

Consider in turn each of the letters x, y, z.

The lowest power of x divisible by x^3, x, x is x^3,

" " " y " " y, y^4, y^2 is y^4,

" " " z " " z is z;

$\therefore$ the L.C.M. is x^3y^4z.

Example 12. *Find the L.C.M. of $4a^2bc$, $8a^3b^2$, $12bc^3$.*

The L.C.M. of 4, 8, and 12 is 24.

The lowest power of a divisible by a^2, a^3 is a^3,

" " " b " " b, b^2, b is b^2,

" " " c " " c, c^3 is c^3;

$\therefore$ the L.C.M. is $24a^3b^2c^3$.

This method for finding the L.C.M. is analogous to the method used in Arithmetic for finding the L.C.M. by prime factors.

EXERCISE 26. a

(Many of these may be taken orally)

Find the H.C.F. of:

1. ab, a^3b^2.	**2.** $3a^3, 12a^2b$.	**3.** $5xy^3, 7x^2y$.
4. $2lm^3, 8l^4m^2$.	**5.** $8, 16a^4c$.	**6.** $2a^3b, 8abc$.
7. a^3b, ab^3, a^2b^2.	**8.** x^3y, xy^3, xyz^2.	**9.** $3a^2b^2, 6a^3, 9a^5b$.
10. $x^2, 3x^2y^2, 5x^3y$.	**11.** $3lm, 9l^2, 6lmn$.	**12.** $abc^2, 2bcd^2, a^2c^2$.

Find the L.C.M. of:

13. x^2, xy.
14. x^2, xyz.
15. $2x^2, x^3y$.
16. $3a^2b, 2ab^2$.
17. $15a^3b, 5ab^4$.
18. $5lmn, 2l^2n$.
19. $3x, 4y, 5z$.
20. $2a^2, 3ab, 4b^2$.
21. $2r^2, 4rs^2, 8r^2s^2$.
22. $4a^2b, 5a^3bc^2, 6b^3c^3$.
23. $5z^3, 6t^3, 8t^4$.
24. $4u^3, 5v^3, 10u^4v^2$.

EXERCISE 26. b

(Many of these may be taken orally)

Find the H.C.F. of:

1. xy, x^4y^3.
2. $6ab^4, 11a^2b$.
3. $5x^4, 20x^3y$.
4. $4xy^3, 6x^3y^2$.
5. $3xy, 33x^2y^2$.
6. $11, 33x^2y^2$.
7. a^4b, a^2b^3, a^3b^2.
8. $2l^2m, 8l^4, 14l^3m^3$.
9. $8c^2d^3, 16c^4, 24c^7d^2$.
10. $xy^2z^3, 5yzt^6, x^3z^2$.
11. $x^4, 2x^2y, 9x^5y^3$.
12. $4abc^2, 20b^4, 12b^2c^4$.

Find the L.C.M. of:

13. a^3, ab^2.
14. $3t^3, tu^2$.
15. c, cd^2.
16. $7abc, 3a^2$.
17. $4x^3y, 5x^2y^2$.
18. $21x^2y^5, 7xy^6$.
19. $2l, 3m, 5n$.
20. $4a^2, 6b^2, 8c^2$.
21. $3x^2, 2xy, 8y^2$.
22. $4a^2, 6a^3, 18a^4$.
23. $3a^3, 4b^3, 6a^2b^2$.
24. $6x^3y, 8x^2y^2z^2, 12xz^4$.

EXERCISE 26. c

Find the:

1. L.C.M. of al^2, am^2, a^2l, a^2m.
2. H.C.F. of $7l^3mn^5, 21l^2m^7n^3$.
3. H.C.F. of $15ab^2c, 10a^2b^4, 5a^3bc^2$.
4. L.C.M. of $x^4y^4, x^3y^3, x^2y^2, xy$.
5. H.C.F. of $21x^3, 63x^2y, 35x^4$.
6. L.C.M. of $2k, 3k, 4k, 5k$.
7. L.C.M. of $3x^2yz, 4y^2zt, 5xz^2t$.
8. H.C.F. of $18l^3m^3n, 6lm^3n^2, 2l^2m^4n^4$.
9. L.C.M. of $2k^2, 3k^3, 4k^4, 5k^5$.
10. L.C.M. of $5a^3bc, 6ab^2c^3, 60a^3b^3c$.
11. H.C.F. of $33x^4y^4z^3, 22x^2z^5, 11x^3z^4$.
12. H.C.F. of $35x^7y^4z^6, 49x^3yz^5, 14xy^3z^2$.

Find the H.C.F. and L.C.M. of:

13. $5b^3c, 15b^2c^2, 25abc$.
14. $4yz, 7x^2y, 14xy^3z^2$.
15. $ay^2z, a^2xz^2, a^3y^2z^3$.
16. $4a^2b^3cd^2, 8a^2d^3, 10a^4c^2d^4, 12a^5b^2d^5$.

CHAPTER XII

FRACTIONS WITH VERY SIMPLE DENOMINATORS. EQUATIONS AND PROBLEMS INVOLVING FRACTIONS

61. The rules for dealing with fractions in Algebra are essentially the same as the arithmetical rules with which the pupil is already familiar. Several easy cases were considered in the previous chapter in dealing with Multiplication and Division. For convenience, the fundamental principle is repeated here :

The value of a fraction is unaltered by multiplying or dividing both its numerator and denominator by the same number or expression. The number or expression must not, however, be zero.

It must be borne in mind that there are values of the letters for which division is impossible. For brevity, this will be understood in future work.

The pupil will best learn how to deal with fractions in Algebra by considering worked examples illustrating the close resemblance to the work in Arithmetic.

Example 1. *Simplify* (i) $\frac{2}{5}\times\frac{10}{21}\times\frac{7}{8}$, (ii) $\frac{a}{b}\times\frac{ab}{xy}\times\frac{x}{a^3}$.

(i)

$$\frac{\overset{1}{\cancel{2}}}{\cancel{5}}\times\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{10}}}{\underset{3}{\cancel{21}}}\times\frac{\overset{1}{\cancel{7}}}{\underset{\underset{2}{\cancel{4}}}{\cancel{8}}}$$

$$=\frac{1}{6},$$

cancelling in the usual way, i.e. dividing numerator and denominator by the common factors 5, 2 2, 7 in succession.

(ii)

$$\frac{\overset{1}{\cancel{a}}}{\cancel{b}}\times\frac{\overset{1}{\cancel{a}}\overset{1}{\cancel{b}}}{\cancel{x}y}\times\frac{\overset{1}{\cancel{x}}}{\underset{\underset{a}{\cancel{a^2}}}{\cancel{a^3}}}$$

$$=\frac{1}{ay},$$

cancelling in the usual way, i.e. dividing numerator and denominator by the common factors b, a, a, x in succession.

Note. The case in Algebra corresponding to a mixed fraction, $1\frac{2}{5}$, will be considered later in the chapter.

62. The **reciprocal** of a number x is the number $\frac{1}{x}$.

Thus the reciprocal of 5 is $\frac{1}{5}$;

" " " " $\frac{3}{4}$ is $1 \div \frac{3}{4}$, i.e. $\frac{4}{3}$;

" " " " $\frac{a}{b}$ is $1 \div \frac{a}{b}$, i.e. $\frac{b}{a}$.

As in Arithmetic, to divide by a fraction multiply by its reciprocal.

Example 2. *Simplify* (i) $\frac{2}{5} \times \frac{4}{7} \div \frac{10}{21}$, (ii) $\frac{a}{b} \times \frac{a^2}{c} \div \frac{ab}{cd}$.

(i)

$$\frac{2}{5} \times \frac{4}{7} \div \frac{10}{21}$$

$$= \frac{\overset{1}{\cancel{2}}}{5} \times \frac{4}{\cancel{7}} \times \frac{\overset{3}{\cancel{21}}}{\underset{5}{\cancel{10}}}$$

$$= \frac{12}{25}.$$

(ii)

$$\frac{a}{b} \times \frac{a^2}{c} \div \frac{ab}{cd}$$

$$= \frac{\overset{1}{\cancel{a}}}{b} \times \frac{a^2}{\cancel{c}} \times \frac{\cancel{c}d}{\underset{1}{\cancel{a}}b}$$

$$= \frac{a^2d}{b^2}.$$

63. The pupil should remember that the order of operations is from left to right, unless expressions in brackets have to be worked first.

Thus $$\frac{a}{b} \div \frac{c}{d} \times \frac{e}{f} = \frac{a}{b} \times \frac{d}{c} \times \frac{e}{f} = \frac{ade}{bcf},$$

but $$\frac{a}{b} \div \left(\frac{c}{d} \times \frac{e}{f}\right) = \frac{a}{b} \div \frac{ce}{df} = \frac{a}{b} \times \frac{df}{ce} = \frac{adf}{bce}.$$

Also $$\frac{a}{b} \div \frac{c}{d} \text{ of } \frac{e}{f} = \frac{a}{b} \div \left(\frac{c}{d} \times \frac{e}{f}\right).$$

"Of" means the same as "$\times$", the terms connected by "of" being enclosed in a bracket. It is, however, more usual to avoid ambiguity by the insertion of brackets. Thus $\frac{a}{b} \div \frac{c}{d} \times \frac{e}{f}$ is usually written $\left(\frac{a}{b} \div \frac{c}{d}\right) \times \frac{e}{f}$, and this practice is to be encouraged.

EXERCISE 27.a

(*Some of these may be done orally*)

Simplify the following expressions :

1. $\frac{3k}{k}$.
2. $\frac{3xy}{4xz}$.
3. $\frac{3ab^2}{b^3}$.
4. $\frac{c^3}{3c}$.
5. $\frac{4l^2}{4l}$.
6. $\frac{5a^2}{15ab}$.
7. $\frac{9lm}{6l^2m}$.
8. $\frac{abc}{abc}$.
9. $\frac{5x}{10xy^2}$.
10. $\frac{4a^3}{6a^7}$.
11. $\frac{ab^3c}{abc^2}$.
12. $\frac{15c^2d^2}{5c^4d^3}$.
13. $\frac{15l^2m^3}{20lm^2n}$.
14. $\frac{x^2y^5}{3xy}$.
15. $a\div\frac{b}{c}$.
16. $3x\times\frac{y^2}{xz}$.
17. $\frac{a}{2b}\times\frac{4c}{3d}$.
18. $\frac{a}{b}\div\frac{c}{2d}$.
19. $\frac{a}{2b}\times\frac{2b}{a}$.
20. $\frac{l^2}{m}\times\frac{m^2}{l}$.
21. $\frac{c}{d}\times\frac{d}{x}\div\frac{2c}{3x}$.
22. $2x\div\frac{1}{y}$.
23. $z\div\frac{1}{3x}$.
24. $\frac{a^2b}{c^3}\div\frac{b}{ac^2}$.
25. $\frac{1}{a^5}\div a^2$.
26. $\frac{1}{6x^2y}\div\frac{1}{8xy^2}$.
27. $a^7\div\frac{2}{a^3}$.
28. $\frac{x^5}{y^3}\times\frac{y^4}{x^2}$.

EXERCISE 27.b

(*Some of these may be done orally*)

Simplify the following expressions :

1. $\frac{3k}{3}$.
2. $\frac{2s^4}{9s^2}$.
3. $\frac{7c^3}{7c^3}$.
4. $\frac{x^5}{4x}$.
5. $\frac{2ab}{4bc}$.
6. $\frac{4l^2}{8lm}$.
7. $\frac{7l}{21lm^3}$.
8. $\frac{6a^3b^3}{a^2b^2}$.
9. $\frac{z}{2xyz}$.
10. $\frac{6cd}{4cd^3}$.
11. $\frac{10c^4}{12c^6}$.
12. $\frac{lm^5n^2}{l^2mn^2}$.
13. $\frac{18a^3b^2}{6a^2b^4}$.
14. $y\div\frac{x}{z}$.
15. $\frac{r^3s^4}{5rs}$.
16. $\frac{r}{3s}\div\frac{2u}{v}$.
17. $5u\times\frac{t^3}{uv}$.
18. $\frac{x}{5t}\times\frac{15y}{4z}$.
19. $\frac{a^3}{b}\times\frac{b^3}{a}$.
20. $\frac{s}{t}\times\frac{2t}{u}\div\frac{4s}{5t}$.
21. $\frac{x}{5z}\times\frac{5z}{x}$.
22. $c\div\frac{1}{7b}$.
23. $\frac{x^3y}{z^4}\div\frac{y^2}{xz^3}$.
24. $3y\div\frac{1}{z}$.
25. $\frac{a^6}{b^2}\times\frac{b^5}{a^5}$.
26. $x^5\div\frac{2}{x^3}$.
27. $\frac{1}{x^3}\div 3x^4$.
28. $\frac{1}{12a^3c}\div\frac{1}{10ac^4}$.

EXERCISE 27. c

Simplify the following expressions :

1. $\dfrac{abc}{lmn} \times \dfrac{l^2mn}{abc^2}$.

2. $\dfrac{2lx}{3my^2} \times \dfrac{6m^2y}{5l}$.

3. $\dfrac{ac^2}{9x^2z^2} \times \dfrac{3xz^3}{ac^3}$.

4. $\dfrac{2u^2v}{5a^3b^2} \times \dfrac{3bv}{8u^2}$.

5. $\dfrac{35c^2d^5}{9l^2m^2} \div \dfrac{5c^6d}{27lm^3}$.

6. $\dfrac{2rst}{7x^3yz^2} \div \dfrac{6r^3t}{21x^2yz^3}$.

7. $\dfrac{7a^3bc^4}{9l^4} \div \dfrac{14a^2c^3}{27l^2}$.

8. $\dfrac{3r^3s^3}{5P^3Q^5} \div \dfrac{12r^2s^5}{15P^2Q^6}$.

9. $\dfrac{lm}{xy} \times \dfrac{x^2y^2}{l^2m}$.

10. $\dfrac{4ax^2}{5by} \times \dfrac{5b^3y}{6a^2x}$.

11. $\dfrac{xy^3}{10z^2t} \times \dfrac{5zt^3}{xy}$.

12. $\dfrac{7b^3c}{3b^4d} \times \dfrac{2cd}{14b}$.

13. $\dfrac{21a^3b^4}{8c^2d^3} \div \dfrac{7a^5b^2}{12c^2d^2}$.

14. $\dfrac{3abc}{10x^2y^2z} \div \dfrac{9bc^2}{5xy^2z}$.

15. $\dfrac{2x^2yz^3}{3t^3} \div \dfrac{14x^2y^2}{15t^2}$.

16. $\dfrac{4l^2m^2}{9c^2d^4} \div \dfrac{10l^3m}{7c^2d^3}$.

ADDITION AND SUBTRACTION OF FRACTIONS

64. If two or more fractions have the same denominator, we may add or subtract directly.

Thus, in Arithmetic, $\dfrac{1}{7}+\dfrac{2}{7}+\dfrac{3}{7}=\dfrac{1+2+3}{7}=\dfrac{6}{7}$,

$$\frac{5}{9}-\frac{2}{9}+\frac{4}{9}-\frac{3}{9}=\frac{5-2+4-3}{9}=\frac{4}{9}.$$

So, in Algebra, $\dfrac{a}{7}+\dfrac{2a}{7}+\dfrac{3a}{7}=\dfrac{a+2a+3a}{7}=\dfrac{6a}{7}$,

$$\frac{5x}{9y}-\frac{2x}{9y}+\frac{4x}{9y}-\frac{3x}{9y}=\frac{5x-2x+4x-3x}{9y}=\frac{4x}{9y}.$$

65. If the numerator of a fraction consists of two or more terms, these terms must be considered to be in a bracket. The beginner is strongly advised to insert this bracket and proceed as in the next example.

Example 3. *Simplify* $\frac{3}{4}-\frac{c-2}{4}$.

The expression $=\frac{3}{4}-\frac{(c-2)}{4}=\frac{3-(c-2)}{4}$

$=\frac{3-c+2}{4}=\frac{5-c}{4}$.

If the brackets are not inserted, there is considerable danger that an error in sign will be made.

66. If the denominators of the fractions to be added or subtracted are not the same, the fractions must first be brought to a common denominator as in Arithmetic.

Example 4. *Simplify* (i) $\frac{5}{6}-\frac{2}{3}+\frac{1}{2}$,

(ii) $\frac{5x}{6a}-\frac{2x}{3a}+\frac{x}{2a}$.

(i)

The L.C.M. of 6, 3, 2 is 6,

$\therefore\ \frac{5}{6}-\frac{2}{3}+\frac{1}{2}$

may be written

$\frac{5}{6}-\frac{4}{6}+\frac{3}{6}$

$=\frac{5-4+3}{6}=\frac{4}{6}=\frac{2}{3}$.

(ii)

The L.C.M. of $6a$, $3a$, $2a$ is $6a$,

$\therefore\ \frac{5x}{6a}-\frac{2x}{3a}+\frac{x}{2a}$

may be written

$\frac{5x}{6a}-\frac{4x}{6a}+\frac{3x}{6a}$

$=\frac{5x-4x+3x}{6a}=\frac{4x}{6a}=\frac{2x}{3a}$.

Example 5. *Simplify* (i) $2-\frac{3}{4}+\frac{2}{3}$,

(ii) $2-\frac{3t}{4x}+\frac{2t}{3x}$.

(i)

The L.C.M. of 1, 4, 3 (for $2=\frac{2}{1}$) is 12,

$\therefore$ expression $=\frac{24}{12}-\frac{9}{12}+\frac{8}{12}$

$=\frac{24-9+8}{12}$

$=\frac{23}{12}=1\frac{11}{12}$.

(ii)

The L.C.M. of 1, $4x$, $3x$ (for $2=\frac{2}{1}$) is $12x$,

$\therefore$ expression $=\frac{24x}{12x}-\frac{9t}{12x}+\frac{8t}{12x}$

$=\frac{24x-9t+8t}{12x}$

$=\frac{24x-t}{12x}$.

Note. The pupil should notice that there is no step in Algebra corresponding to the last step in the Arithmetic. We could, of course, write $2 - \frac{t}{12x}$ instead of $\frac{24x - t}{12x}$, but it is usual to leave the answer as above.

As soon as the method is thoroughly understood, the intermediate step, e.g. $\frac{24x}{12x} - \frac{9t}{12x} + \frac{8t}{12x}$ in Example 5, may be left out.

EXERCISE 28. a

Express as fractions with a single denominator:

1. (i) $\frac{1}{2}+\frac{1}{4}$, (ii) $\frac{a}{2}+\frac{a}{4}$, (iii) $\frac{1}{2x}+\frac{1}{4x}$, (iv) $\frac{a}{2x}+\frac{a}{4x}$.

2. (i) $\frac{2}{3}-\frac{1}{2}$, (ii) $\frac{2a}{3}-\frac{a}{2}$, (iii) $\frac{2}{3x}-\frac{1}{2x}$, (iv) $\frac{2a}{3x}-\frac{a}{2x}$.

3. (i) $\frac{1}{2}+\frac{3}{8}-\frac{1}{4}$, (ii) $\frac{k}{2}+\frac{3k}{8}-\frac{k}{4}$, (iii) $\frac{k}{2l}+\frac{3k}{8l}-\frac{k}{4l}$.

4. (i) $1+\frac{5}{8}$, (ii) $1+\frac{5k}{8}$, (iii) $1+\frac{5}{8l}$, (iv) $1+\frac{5k}{8l}$.

5. (i) $2-\frac{3}{7}$, (ii) $2-\frac{3t}{7}$, (iii) $2-\frac{3}{7t}$, (iv) $2-\frac{3x}{7y}$.

6. (i) $2-\frac{3}{7}-\frac{6}{7}$, (ii) $2-\frac{a}{7}-\frac{2a}{7}$, (iii) $2-\frac{1}{7b}-\frac{2}{7b}$, (iv) $2-\frac{a}{7b}-\frac{2a}{7b}$.

7. (i) $\frac{1}{c}+\frac{1}{d}$, (ii) $\frac{1}{c}-\frac{1}{d}$, (iii) $c+\frac{1}{c}+1$, (iv) $1+\frac{1}{c}+\frac{1}{c^2}$.

8. (i) $\frac{a}{3}-\frac{a-1}{5}$, (ii) $\frac{a}{3}-\frac{a-1}{2}$, (iii) $\frac{2a}{3}-\frac{(3a-1)}{2}$, (iv) $\frac{2a}{3}-\frac{(5a-2b)}{2}$.

9. (i) $3+\frac{4}{c}+\frac{5}{c^2}$, (ii) $x-2+\frac{1}{x}$, (iii) $\frac{x}{y}-2+\frac{y}{x}$, (iv) $3x+6+\frac{1}{3x}$.

10. (i) $2-\frac{a+1}{3}$, (ii) $5+\frac{x-2}{3}$, (iii) $8-\frac{2z-3}{2}$, (iv) $\frac{2}{3}-\frac{2z-3}{5}$.

11. (i) $t-\frac{t+2}{3}$, (ii) $\frac{t-1}{4}+\frac{t-3}{3}$, (iii) $\frac{t-3}{3}-\frac{2t-5}{2}$, (iv) $\frac{t}{4}-\frac{t+6}{5}$.

12. (i) $1-\frac{3(x-1)}{4}$, (ii) $1-\frac{3(2x-5)}{4}$,

(iii) $\frac{x-2}{3}-\frac{3(x-1)}{2}$, (iv) $\frac{x-2}{3}-\frac{3(2x-5)}{4}$.

EXERCISE 28. b

Express as fractions with a single denominator :

1. (i) $\frac{4}{5}-\frac{2}{3}$, (ii) $\frac{4c}{5}-\frac{2c}{3}$, (iii) $\frac{4}{5d}-\frac{2}{3d}$, (iv) $\frac{4c}{5d}-\frac{2c}{3d}$.

2. (i) $\frac{3}{7}+\frac{1}{5}$, (ii) $\frac{3k}{7}+\frac{k}{5}$, (iii) $\frac{3}{7l}+\frac{1}{5l}$, (iv) $\frac{3k}{7l}+\frac{k}{5l}$.

3. (i) $\frac{2}{3}+\frac{5}{6}-\frac{1}{2}$, (ii) $\frac{2a}{3}+\frac{5a}{6}-\frac{a}{2}$, (iii) $\frac{2a}{3X}+\frac{5a}{6X}-\frac{a}{2X}$.

4. (i) $3-\frac{2}{5}$, (ii) $3-\frac{2u}{5}$, (iii) $3-\frac{2}{5v}$, (iv) $3-\frac{2u}{5v}$.

5. (i) $2+\frac{4}{9}$, (ii) $2+\frac{4h}{9}$, (iii) $2+\frac{4}{9k}$, (iv) $2+\frac{4h}{9k}$.

6. (i) $4-\frac{2}{9}-\frac{5}{9}$, (ii) $4-\frac{2x}{9}-\frac{5x}{9}$, (iii) $4-\frac{2}{9y}-\frac{5}{9y}$, (iv) $4-\frac{2x}{9y}-\frac{5x}{9y}$.

7. (i) $4+\frac{7}{c}-\frac{2}{c^2}$, (ii) $3-t-\frac{4}{t}$, (iii) $\frac{2a}{3b}-5-\frac{3b}{2a}$, (iv) $2l-5-\frac{1}{4l}$.

8. (i) $\frac{2}{m}-\frac{3}{n}$, (ii) $\frac{5}{m}+\frac{4}{n}$, (iii) $\frac{d}{5}-\frac{5}{d}+3$, (iv) $3+\frac{1}{2w}+\frac{4}{w^2}$.

9. (i) $\frac{x}{5}-\frac{x-2}{8}$, (ii) $\frac{x}{8}-\frac{x-2}{5}$, (iii) $\frac{3x}{5}-\frac{(4x-3)}{3}$, (iv) $\frac{3x}{5}-\frac{(7x-3y)}{3}$.

10. (i) $1-\frac{4(2a-3)}{5}$, (ii) $1-\frac{3(a-3)}{5}$,

(iii) $\frac{2a-7}{4}-\frac{5(a-2)}{6}$, (iv) $\frac{2a-7}{4}-\frac{5(2a-3)}{6}$.

11. (i) $3+\frac{4-3a}{5}$, (ii) $7-\frac{t-9}{2}$, (iii) $4-\frac{5K+1}{7}$, (iv) $\frac{3}{5}-\frac{5K+1}{6}$.

12. (i) $m-\frac{m-6}{4}$, (ii) $\frac{m+2}{3}-\frac{m-6}{5}$,

(iii) $\frac{m-6}{5}-\frac{2m-13}{10}$, (iv) $\frac{m}{7}-\frac{m+8}{9}$.

EQUATIONS INVOLVING FRACTIONS

67. Example 6. *Solve* $\frac{2x-1}{3}-\frac{3x-2}{4}=\frac{5x-4}{6}-\frac{7x+6}{12}$.

Write the fractions with the numerators in brackets.

$$\frac{(2x-1)}{3}-\frac{(3x-2)}{4}=\frac{(5x-4)}{6}-\frac{(7x+6)}{12}.$$

Multiply each side by 12 (i.e. the L.C.M. of 3, 4, 6, 12),

$$\therefore\ 4(2x-1)-3(3x-2)=2(5x-4)-(7x+6),$$

N.B. Beginners should not attempt to multiply by 12 and remove brackets in one step.

$$\therefore\ 8x-4-9x+6=10x-8-7x-6,$$
$$\therefore\ 8x-9x-10x+7x=-8-6+4-6,$$
$$\therefore\ -4x=-16,$$
$$\therefore\ x=4.$$

Check. When $x=4$,

$$\text{L.H.S.}=\frac{7}{3}-\frac{10}{4}=\frac{28-30}{12}=-\frac{2}{12}=-\frac{1}{6},$$
$$\text{R.H.S.}=\frac{16}{6}-\frac{34}{12}=\frac{32-34}{12}=-\frac{2}{12}=-\frac{1}{6}.$$

68. When the coefficients involve decimals, we may express the decimals as common fractions and proceed as before. It is, however, sometimes simpler to work entirely in decimals. It is occasionally possible to get rid of decimals by multiplying each term of the equation by a suitable power of 10.

Example 7. *Solve* $\dfrac{x+0{\cdot}25}{0{\cdot}15}-\dfrac{x-0{\cdot}35}{0{\cdot}45}-18=0.$

Write the fractions with the numerators in brackets, and with the decimals replaced by common fractions.

$$\frac{(x+\frac{1}{4})}{\frac{3}{20}}-\frac{(x-\frac{7}{20})}{\frac{9}{20}}-18=0,$$
$$\therefore\ \tfrac{20}{3}(x+\tfrac{1}{4})-\tfrac{20}{9}(x-\tfrac{7}{20})-18=0,$$
$$\therefore\ \frac{20x}{3}+\frac{5}{3}-\frac{20x}{9}+\frac{7}{9}-18=0.$$

N.B. The brackets should be removed before clearing of fractions.

Multiply each side by 9 (i.e. the L.C.M. of 3, 3, 9, 9),

$$\therefore\ 60x+15-20x+7-162=0,$$

(*N.B.*—After multiplying by 9, we still have 0 on the right-hand side, for $0\times 9=0$.)

$$\therefore\ 60x-20x=162-15-7,$$
$$\therefore\ 40x=140,$$
$$\therefore\ x=3{\cdot}5 \text{ or } 3\tfrac{1}{2}.$$

Check When $x = 3{\cdot}5$,

$$\text{L.H.S.} = \frac{3{\cdot}75}{0{\cdot}15} - \frac{3{\cdot}15}{0{\cdot}45} - 18 = 25 - 7 - 18 = 0 = \text{R.H.S.}$$

Example 8. *Solve* $0{\cdot}6x - 3 = 0{\cdot}25x + 0{\cdot}3x$.

The work here is simple, and we may work entirely in decimals.

We have $\quad 0{\cdot}6x - 0{\cdot}25x - 0{\cdot}3x = 3$,

$$\therefore \quad 0{\cdot}05x = 3, \quad \therefore\ x = \frac{3}{0{\cdot}05} = 60.$$

Check. When $x = 60$

$$\text{L.H.S.} = 0{\cdot}6 \times 60 - 3 = 36 - 3 = 33,$$

$$\text{R.H.S.} = 0{\cdot}25 \times 60 + 0{\cdot}3 \times 60 = 15 + 18 = 33.$$

Alternatively, we may multiply each term by 100, getting

$$60x - 300 = 25x + 30x, \text{ etc.}$$

NOTE ON CHECKS

69. 1. To check the solution of an equation it is essential to find the value of each side of the equation **as given**. If a simplified form of the equation is checked, the check would not expose a mistake made in obtaining the simplified form from the original equation. It would merely show that no mistake had been made in the subsequent work.

2. The work in the check should not merely follow the procedure of the solution, for any mistake made (such as errors in sign or in clearing fractions) may be repeated. In Exs. 6, 7, 8 above, the work of the check is entirely different from that of the solution.

3. If the answer involves awkward numbers, e.g. if $x = 13\frac{107}{113}$, the checking is complicated and errors may be made in the working. In such cases it is better to check by going over the work carefully to make sure that no errors have been made.

4. If, after substitution, the L.H.S. is not equal to the R.H.S., an error has been made. It is then necessary to look through the working carefully and trace the error. If the error is not quickly found, it is a good plan to check each line of the solution separately.

Thus, if in the third line L.H.S. $\neq$ R.H.S., but in the fourth line L.H.S. = R.H.S., the error has been made in transforming the third line into the fourth line. It should then be easy to find the error.

EXERCISE 29. a

Solve the following equations and verify the solutions :

1. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=7\frac{5}{6}$.

2. $\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=4\frac{1}{4}$.

3. $\frac{x}{3}+\frac{5}{18}=\frac{x}{9}+\frac{x}{7}$.

4. $\frac{x}{3}+5-\frac{x}{6}+\frac{x}{4}=0$.

5. $\frac{x-1}{3}+\frac{5}{12}=\frac{x}{12}+\frac{2x+3}{15}$.

6. $6+\frac{x+1}{2}+\frac{x+2}{3}=\frac{3+x}{4}$.

7. $\frac{6x-3}{2}-(2x-6)=\frac{x-3}{4}$.

8. $\frac{5x-1}{9}-\frac{x-5}{3}-\frac{7x-1}{19}=0$.

9. $\frac{1}{12}(3t-2)-\frac{1}{45}(t-3)=4$.

10. $\frac{t+24}{21}-\frac{t+5}{9}-\frac{t+6}{30}=0$.

11. $\frac{3x+5}{4}+\frac{1}{4}=x+2$.

12. $x+\frac{2x+1}{7}=2(x-1)$.

13. $8\frac{3}{4}-y=\frac{3(y-1)}{4}-\frac{2y-1}{2}$.

14. $\frac{2(4y+2)}{5}-(y-2)=\frac{4(4y+5)}{13}$.

15. $\frac{4x-9}{11}=\frac{12x-11}{7}-6$.

16. $\frac{10x+1}{9}=\frac{14x-4}{15}-\frac{8x+7}{5}$.

17. $\frac{x-2}{9}=\frac{x+2}{7}$.

18. $\frac{2(x+2)}{3}=\frac{2x+1}{4}+\frac{3}{4}$.

EXERCISE 29. b

Solve the following equations and verify the solutions :

1. $\frac{x}{3}-\frac{x}{9}=\frac{5}{18}+\frac{x}{7}$.

2. $\frac{x}{3}-\frac{x}{6}+\frac{x}{4}-20=0$.

3. $\frac{x}{3}-\frac{7}{10}=\frac{x}{4}+\frac{x}{5}$.

4. $\frac{7x}{10}+1\frac{4}{5}=\frac{x}{5}-\frac{7}{10}$.

5. $\frac{x+3}{4}=\frac{x+4}{3}-\frac{1}{4}$.

6. $\frac{x}{4}+\frac{x+2}{14}=\frac{x}{2}-2$.

7. $\frac{2x-7}{11}-\frac{5x-3}{7}=\frac{x-2}{7}-6$.

8. $\frac{3+4x}{5}-\frac{4-5x}{9}-\frac{7x-11}{15}=0$.

9. $\frac{5x-3}{8}=\frac{7x+\text{[illegible]}}{6}$.

10. $\frac{x}{6}-\frac{x}{14}=\frac{x-5}{18}$.

11. $\frac{2x+3}{2}-\frac{2x-2}{3}=\frac{x-1}{12}$.

12. $\frac{x-3}{3}+\frac{5}{4}=\frac{x}{12}+\frac{2x+9}{15}$.

13. $\frac{4s+22}{7}-\frac{4s+3}{3}=\frac{2s+2}{5}$.

14. $\frac{x+10}{2}=\frac{x}{4}-\frac{x-1}{3}$.

15. $\frac{t}{3}-\frac{7t+69}{72}-\frac{33+8t}{90}=0.$ **16.** $-\frac{2(t+5)}{4}=1-\frac{t+4}{5}.$

17. $\frac{5z+1}{12}-\frac{3z+1}{5}=\frac{1-7z}{40}.$ **18.** $\frac{2(z-1)}{10}-\frac{z-7}{4}=1-\frac{3z+22}{5}.$

Further and harder examples are given in Ch. XXI, Exs. 63, *a* and *b*.

PROBLEMS INVOLVING FRACTIONS

70. No new principles are introduced, but the equations obtained contain fractional expressions. The pupil should revise Art. 24.

Example 9. *A boy cycles home to dinner at 12 miles an hour, takes half an hour for his dinner, and cycles back to school at 10 miles an hour. He is absent from school for $57\frac{1}{2}$ minutes altogether. How far from his home is the school?*

The pupil should at once notice that two different units for time have been used in stating the question.

He must therefore decide whether to take 1 hour or 1 minute as the unit of time. It is more convenient to take 1 hour.

Let the required distance from home to school be x miles.

The time taken to cycle home is $\frac{x}{12}$ hours,

the time spent at home is $\frac{1}{2}$ hour,

the time taken to cycle back to school is $\frac{x}{10}$ hours,

the total time is $\frac{23}{24}$ hours,

$$\therefore \frac{x}{12}+\frac{1}{2}+\frac{x}{10}=\frac{23}{24}.$$

Multiply each side by 120 (i.e. the L.C.M. of 12, 2, 10, 24),

$$\therefore 10x+60+12x=115,$$

$$\therefore 10x+12x=115-60,$$

$$\therefore 22x=55,$$

$$\therefore x=\frac{55}{22}=2\frac{1}{2}.$$

The required distance is therefore $2\frac{1}{2}$ miles.

Check. To cycle $2\frac{1}{2}$ miles at 12 miles an hour takes

$$\frac{2\frac{1}{2}}{12} \times 60 = 12\frac{1}{2} \text{ minutes.}$$

To cycle $2\frac{1}{2}$ miles at 10 miles an hour takes

$$\frac{2\frac{1}{2}}{10} \times 60 = 15 \text{ minutes.}$$

But $12\frac{1}{2} + 30 + 15 = 57\frac{1}{2}$, $\therefore$ the solution is correct.

EXERCISE 30. a

1. A number is 7 less than the sum of its one-half, its one-fifth and its five-twelfths. Find it.

2. Find a number such that, if you subtract 1 and divide the difference by 3, the result will exceed by 2 the number obtained by adding 7 and dividing the sum by 5.

3. Two-fifths of the coins in a box are sovereigns, one-third are half-sovereigns and the rest half-crowns. The total value of the coins is £27. Find the total number of coins in the box.

4. I have to catch a train at a station $5\frac{1}{4}$ miles away. I motor to a garage at the rate of 25 miles per hour, and complete the journey on foot at the rate of 3 miles per hour. I take 17 minutes in all and just catch the train. How far is the garage from my house?

5. What number must be added to the numerator and also to the denominator of $\frac{4}{9}$ so that the result may equal $\frac{2}{3}$?

6. A man walks up a mountain at an average rate of 2 miles an hour, and back by a way 9 miles longer at an average rate of $3\frac{1}{2}$ miles an hour. His whole journey takes 12 hours. How far does he walk altogether?

7. If I walk to the station at $3\frac{1}{2}$ miles per hour, I shall have 7 minutes to spare; but if I walk at 3 miles per hour, I shall miss the train by 3 minutes. How far off is the station?

8. A man can walk from A to B and back in a certain time at 4 miles an hour. If he walks at 3 miles an hour from A to B, and returns at 5 miles an hour, he takes 15 minutes longer for the double journey. Find the distance from A to B.

9. Divide £60 into two parts such that, if they are invested at $3\frac{1}{2}$ per cent. and 5 per cent. respectively, they may together yield the same annual income as if the whole were invested at $4\frac{1}{2}$ per cent.

10. A cyclist, whose average speed is 10 miles an hour, sets out to ride from X to Y; at the same time another cyclist, whose average speed is 8 miles an hour, sets out to ride from Y to X. If they meet 2 miles from half-way, how far is it from X to Y?

11. Divide £185 so that by investing part of it at 6 per cent. and the remainder at 5 per cent. the total income produced may be £10.

12. A man buys oranges at 6d. a dozen, and three times as many at $10\frac{1}{2}$d. a score; he sells the whole of them at 7d. a dozen, and makes a profit of 5s. 2d. How many oranges does he buy?

EXERCISE 30. b

1. One-quarter of the coins in a box are florins, two-ninths are shillings and the rest sixpences. The total value of the coins is £3 11s. Find the total number of coins in the box.

2. Think of a number. Take away one-third of it. Take away 44. One-seventh of the number remains. Find the number.

3. Find a number such that, if you add 5 and divide the sum by 6, the result will exceed by 5 the number obtained by subtracting 5 and dividing the difference by 10.

4. A man climbs a mountain at an average rate of 2 miles an hour, and takes the same time to return by a way 9 miles longer at an average rate of $3\frac{1}{2}$ miles an hour. What is the length of the total journey?

5. What number must be added to the numerator and also to the denominator of $\frac{31}{35}$, so that the result may equal $\frac{10}{11}$?

6. I have to catch a train at a station 12 miles away. I motor to a garage at the rate of 20 miles per hour, and complete the journey by tram and on foot at the rate of 8 miles per hour. I take 45 minutes in all and just catch the train. How far is the garage from my house?

7. I invest £100, partly at 3 per cent. and partly at 5 per cent., thereby obtaining the same income as if I had invested the whole at $3\frac{4}{5}$ per cent. How much do I invest at each rate?

8. A boy walks to school at the rate of $3\frac{1}{2}$ miles an hour, and is 6 minutes late; the next day he increases his pace by a quarter of a mile an hour, and is 3 minutes late: find the distance to the school.

9. A cyclist can ride from A to B and back in a certain time at an average rate of 10 miles an hour. If he were to ride from A to B at 9 miles an hour, and return at 12 miles an hour, he would save 15 minutes on the double journey. Find the distance from A to B.

10. A man buys apples at 6d. a dozen, and twice as many at 9d. a score; he sells them at 9d. a dozen, and makes a profit of 8s. 6d. How many apples does he buy?

11. A man bought a number of eggs at four for threepence, and four times as many at five for fourpence; he sells them at a penny each, and makes a profit of 7s. How many eggs does he buy?

12. Divide £340 so that by investing part of it at $2\frac{1}{2}$ per cent. and the remainder at $4\frac{1}{2}$ per cent., the total income produced may be £12 2s.

EXERCISE 30. c

1. I obtain 14 lb. of tea by mixing Grade A tea worth 2s. per lb. with Grade B tea worth 1s. 7d. per lb. I sell the mixture at 1s. 11d. per lb. and make a profit of 2s. 2d. How many lb. of Grade A tea are taken?

2. A man goes to a concert, paying 2s. for admission. He then spends one-quarter of what he has left, and afterwards pays 6d. for a train ticket. On reaching home he has seven-twelfths of what he started with. How much had he at first?

3. In an examination paper one boy M got eight marks less than 80 per cent. of the full marks, and another boy N got 5 marks more than 70 per cent. of the full marks. M beat N by 2 marks. What were N's marks?

4. I have a certain number of nuts to divide equally amongst 28 children; if the number of nuts were increased by 12 and the number of children decreased by 2, each child would receive two more nuts. How many nuts have I to distribute?

5. A man has 96 coins, some of them crowns and the rest shillings; if he exchanged each crown for a florin and each shilling for a half-crown he would neither gain nor lose. How many crowns has he?

6. A man buys one lot of eggs at 1s. 5d. a dozen and a secon lot, which is 96 more than the first lot, at 2s. 11d. a score; he sells them at 2s. 4d. a dozen and makes a profit of 15s. 2d. How many eggs does he buy altogether?

7. 7 lb. of tea at a certain price is mixed with 21 lb. of tea costing 8d. per lb. more. The average price of the mixture is 2s. 8d. per lb. Find the price of the dearer kind.

8. In an examination paper one boy X got 4 marks less than 75 per cent. of the full marks, and another boy Y got 6 marks more than 70 per cent. of the full marks. Y beat X by 1 mark. What were Y's marks?

9. I have a certain number of apples to divide equally amongst 48 children; if the number of apples were decreased by 20 and the number of children increased by 7, each child would receive one apple less. How many apples have I to distribute?

10. A man buys one lot of eggs at 1s. 4d. a dozen, and a second lot, which is 5 dozen more than the first lot, at 1s. 9d. a dozen; he sells them all at 2s. 2d. a dozen and makes a profit of 12s. 1d. How many eggs does he buy altogether?

11. A man has 84 coins, some of them half-crowns and the rest florins; if he exchanged each half-crown for a crown and each florin for a sixpence, he would gain 18s. How many florins has he?

12. A man goes to a theatre and pays 5s. 9d. for admission. He then spends one-fifth of what he has left, and afterwards pays 2s. 3d. for a cab. On reaching home he has 1s. 3d. more than half of what he started with. How much had he at first?

TEST PAPERS III

A

1. Simplify (i) $a^2-(-a)^2$, (ii) $27x^3\div(-3x)^2$, (iii) $\left(-\frac{x}{y}\right)(-3xy)$.

2. If $a=2$, $b=-6$ find the values of (i) a^2b, (ii) ab^2, (iii) a^2-b^2.

3. Find the H.C.F. and L.C.M. of $15a^3b^2$, $35a^2b^3c$ and $25ab^2c^2$.

4. Simplify $p-[p-\{2p-3(p-q)-q\}-q]-(p-q)$.

5. Solve the equations (i) $6(2-x)-8(x+2)=0$,

$$\text{(ii) } 1-\frac{1}{6}(2t+5)=\frac{4t+7}{3}-\frac{t-5}{2}.$$

6. A boy counts 2 marks for each sum he gets right and -1 mark for each he gets wrong. He does 21 sums and obtains 27 marks. How many does he get right?

B

1. If $x=2$, $y=-1$, $z=0$ and $u=-3$, find the values of

(i) $4u-3(x-y)$, (ii) $72x^3yz-u^2$, (iii) $(-xy)^2+(2u)^3$

2. Simplify

(i) $27m^2n^3\div(3mn)^2$, (ii) $(4xy)^2\div 4xy^2$, (iii) $(-xy)^2\div(-2xy)$.

3. (i) Add together $4-x+2x^2$, $-3x+7-11x^2$ and $6x^2-4$.

(ii) Subtract $z+3y-5x$ from $2y+7x$.

4. The following table gives the number of years E that a female aged A years may be expected to live (i.e. "the expectation of life").

A -	20	25	30	35	40	45	50
E -	47·1	42·8	38·5	34·4	30·3	26·3	22·5

Draw the graph. Have the intermediate points any meaning? Estimate the expectation of life of females aged 27, 38, 47 years.

5. Solve the equations (i) $\frac{x}{4}=\frac{x+1}{3}$, (ii) $\frac{1}{a}=-2$.

6. A man takes 10 minutes to travel a mile, partly by walking at 3 miles per hour, and partly by running at 8 miles per hour. How far does he run?

C

1. (i) Add together $5x^2-4xy+7y^2$, $3(2x^2+6xy-5y^2)$ and $2(-3x^2+6xy-y^2)$.

(ii) Multiply $4a^2b^3$ by $3ab^2$ and divide the result by $6a^3b$.

2. A car is travelling northwards at u miles per hour: x hours after passing a point P it is y miles north of P. Express y in terms of u and x. Find y, and interpret the results, if (i) $u=30$, $x=5$; (ii) $u=-30$, $x=3$; (iii) $u=50$, $x=-\frac{1}{2}$; (iv) $u=-40$, $x=-\frac{1}{4}$.

3. Solve the equations:

$$\text{(i) } 3-2(2-x)=3x, \quad \text{(ii) } \frac{2-3y}{2}-\frac{2+3y}{3}=-\frac{8+3y}{4}.$$

4. A man starts on a journey with a pounds, b half-crowns and c florins in his pocket. He buys $3a$ books at half a crown each and $10b$ shilling packets of cigarettes. What is the value in shillings of the money he has left?

5. Simplify (i) $\frac{3}{x^2}-\frac{2}{5xy}$, (ii) $\frac{x}{y}-\frac{x^2}{y^2}$.

6. If $x=3$, $y=-2$, $z=-1$, find the values of

(i) $3x^2y-2xy^2$, (ii) $3x^2y-(2xy)^2$, (iii) $3x^2yz-(2yz)^2$.

D

1. Simplify (i) $3a-2(2a-3b)+3(3a-2b)$,

(ii) $x\left(1-\frac{1}{x}\right)-y\left(1+\frac{1}{y}\right)$.

2. (i) What must be added to $2a-(3b+4c)$ to give $2a+(3b-4c)$?

(ii) Multiply $3x-2y+z$ by -3 and take the result from twice $2x+3y-4z$.

3. If $l=3$, $m=-2$, $n=0$, find the values of

(i) $l^2+m^2+n^2$, (ii) $3l^2-4m^2+2n^2$, (iii) $(3l-4m)^2$.

4. Solve the equations: (i) $\frac{6x+5}{3}+10x-3=\frac{21+x}{3}$,

(ii) $0\cdot7t+1\cdot1=0\cdot5t+0\cdot5$.

5. Out of a barrel three-quarters full 21 gallons are drawn, leaving the barrel two-fifths full. How many gallons does the barrel hold?

6. In the table below are given the greatest values of the load, W lb., which can be safely carried on a pine beam l ft. long, which is supported at its ends.

l -	8	12	16	20	24	28
W -	775	517	386	310	258	221

Draw a graph to show the relation between l and W. Find from your graph (a) whether a load of 360 lb. would be safe on a beam of length 18 ft., and (b) what is the greatest safe load for a beam of length 13 ft.

E

1. Simplify (i) $\dfrac{-12a^2}{-2a}$, (ii) $\dfrac{-12a^2}{(-2a)^2}$, (iii) $-12a^2-(-2a)^2$.

2. (i) Add together

$$4x^2-7x+11,\quad -2x+5-3x^2 \text{ and } -5-4x^2+3x$$

(ii) Subtract $-3xy-2x$ from $4y-2x+3xy$.

3. When $a=0$, $b=-1$, $c=2$, find the values of

(i) $(a-b)^2+(b-c)^2+(c-a)^2$, (ii) $3(a-b)^2-2(b-c)^2-4(c-a)^2$.

4. Solve the equations (i) $6(9-2c)=5(3-4c)$,

(ii) $8x-3(x-\frac{1}{3})+2(x+1)=0$.

5. Simplify, and arrange in descending powers of x:

(i) $6x-7x^2-2-7+3x-2x^2$, (ii) $x^4-5x+17-6x^4$.

6. A man takes $2\frac{3}{4}$ hours less to ride with the wind from A to B than from B to A against the wind. He rides at the rate of $12\frac{1}{2}$ miles per hour when going with the wind and at the rate of 7 miles per hour when going against it. Find the distance from A to B.

F

1. Simplify (i) $\dfrac{x}{2}-\dfrac{x}{3}+\dfrac{2x}{5}$, (ii) $x^2y^3\div\dfrac{x^2}{y^3}$, (iii) $3-\dfrac{x-y}{x}$.

2. If $p=4$, $q=-3$, find the values of

$$\text{(i) } \frac{p^2-q^2}{p-q},\quad \text{(ii) } \frac{p^2-2q^2}{p-2q},\quad \text{(iii) } \frac{3p^2-2q^2}{3p-2q}.$$

3. Simplify, and arrange in descending powers of a

$$ab^2-7a^2b+9a^3-b^3-7a^2b-5a^3-7ab^2+11b^3.$$

What is the coefficient of (i) a, (ii) ab^2? Find the value of the expression when $a=b=-1$.

4. Add together $-2a+3b-4c$ and $-4c-3b+2a$ and subtract $a-5b$ from the result.

5. Solve the equation $\dfrac{3s+5}{8} - 3 = \dfrac{2+s}{3} - 5s$.

6. A man bought 25 railway tickets for £7. There were some single tickets, each costing 5s., and the rest were return tickets, which cost 6s. 8d. each. How many single tickets were there?

G

1. Find the sum of $6x - (2x - y)$ and $y - (3x - 2y)$ and subtract $x - 2y$ from the result.

2. Find the H.C.F. and L.C.M. of $16a^2b$, $24a^3b^2c$ and $56a^2bc^2$.

3. Simplify (i) $(a - x) - (x - a)$, (ii) $\frac{1}{2}(6 + k) - \frac{1}{2}(2 - k)$.

4. Solve the equations (i) $x - 2 - \dfrac{x-3}{2} - \dfrac{x+2}{3} = 0$,

(ii) $\dfrac{x}{6} - \dfrac{1}{2} = \dfrac{x}{5}$.

5. Two motor-cars can run, one at 34 miles an hour, the other at 40 miles an hour. If the faster car sets out to catch the slower when the latter has 27 miles start, when will it catch it up?

6. At a given temperature, p lb. per sq. inch represents the pressure of a gas which occupies a volume of v cubic inches. The following table gives values of p for different values of v.

p -	40	33·3	28·6	25	22·2	20	18·2	16·7
v -	5	6	7	8	9	10	11	12

Draw the graph. Find (i) the volume when the pressure is 31 lb. per sq. in., (ii) the pressure when the volume is 9·2 cubic inches.

H

1. (i) What must be added to $(a + 2b - c)$ to make $(3a - 5b + c)$?
(ii) By what must $11l^2m^3n^5$ be multiplied to produce $77l^5m^4n^{12}$?

2. If $r = 3$, $s = \frac{1}{4}$, $t = \frac{1}{2}$, find the values of
(i) $r^2 + 4s^2 - 3t^2$, (ii) $r^2 + (4s)^2 - 3t^2$,
(iii) $(r + 4s)^2 - 3t^2$, (iv) $(r + 4s - 3t)^2$.

3. Find the H.C.F. and L.C.M. of $6a^2x^2y^2$, $9a^2xy^3$ and $12ax^3y^2$.

4. A man travels 152 miles in 5 hours. For part of the way he travels at 36 miles per hour; for the remainder of the distance he travels at 28 miles per hour. For how long did he ride at the former rate?

5. If $S=\frac{n}{2}(a+l)$, and $S=50$, $n=6$, $a=18$, find l.

6. What is the total bill for s lb. of sugar at r pence per lb., and $3r$ lb. of margarine at s shillings per lb.? How much change will there be out of a ten shilling note? Give the answers in pence.

I

1. A man receives during the day $8a$ shillings, $2b$ florins, and $6c$ half-crowns. He pays out at the end of the day $5b$ shillings and $2a$ half-crowns. How much money, in shillings, has he left?

2. Simplify:

(i) $4s^2 \div (4s)^2$, (ii) $4s^2 - (4s)^2$, (iii) $3c^5 \div (-3c)^5$, (iv) $3c^5 - (-3c)^5$.

3. (i) From the sum of $3x-4y+5z$ and $-2x-5z+4y$ take the sum of $6x-7y-3z$ and $4z-5y-2x$.

(ii) Simplify and arrange in ascending powers

$$8x^2-4x+9-3x-2-15x^2.$$

4. Solve the equations:

$$\text{(i) } k-5(k-1)=0, \quad \text{(ii) } \frac{2x-3}{3}-\frac{3x-2}{2}=0.$$

5. A person spent £7 7s. 0d. in buying geese and rabbits. If each goose cost 6s. and each rabbit 10d., and if the total number of geese and rabbits bought was 102, how many of each did he buy?

6. Draw a graph from the following table to show the relation between the number N of gallons of milk contained in a milk churn, which when full holds 31·1 gallons, and the vertical depth h inches of the milk in it.

h -	6	12	18	24	30	36
N -	8·9	16·0	21·6	25·9	29·0	31·1

From your graph find (i) the depth when the churn is filled to half its capacity, (ii) how much more it will hold when the depth is already 26·5″.

J

1. Simplify (i) $2(a-b)-3\{a+4[b-3(a-b)]+2a\}$,

$$\text{(ii) } \frac{1}{x}-\frac{x-1}{x^2}, \quad \text{(iii) } \frac{x-1}{x^2}\div\frac{1}{x}.$$

2. Multiply $6x-7y+3z$ by 4 and take it from 3 times

$$4z-5y-8x.$$

3. There are 30 bookcases in a library. One-third of the cases each contain $2x$ books, two-fifths of the cases each contain y books, and the remainder each contain $3z$ books. How many books are there in the library?

4. Solve the equations :

$$\text{(i) } 3(6-x)-\frac{1}{2}(4+3x)=7, \quad \text{(ii) } \frac{t}{7}-\frac{2}{3}=5.$$

5. If $a=\frac{1}{2}$, $b=-\frac{1}{3}$, find the values of (i) $\frac{1}{a}+\frac{1}{b}$, (ii) $\frac{2}{a}-\frac{3}{b}$.

6. A tramp set out to walk from P to Q, a distance of 103 miles. After he had walked a certain distance at the rate of 3 miles per hour he was given a lift by a motor lorry travelling at 27 miles per hour. As a result his total time for the whole journey was 5 hours. How far did he walk?

K

1. (i) Write down the square of $3a^2b$, and the cube root of $64x^6y^{12}$.

(ii) Simplify, and arrange in descending powers of x :

$$8-5x^3-9x-7x^2+20x.$$

2. Solve the equations (i) $\frac{x-5}{10}-\frac{x+5}{5}=2$,

(ii) $\frac{1}{2}z-3(5+\frac{1}{2}z)=3$.

3. Simplify (i) $24c^3 \div 33c^7$, (ii) $\frac{5r^2}{6s^3} \div \frac{20r^5}{9s}$, (iii) $\frac{5}{6a}-\frac{6b}{5a^2}$.

4. If $a=2$, $b=-3$, find the values of

(i) $(2a-3b)^2$, (ii) $2a^2-3b^2$, (iii) $(2a \div 3b)^2$,

(iv) $2a^2 \div 3b^2$, (v) $(2a)^2-(3b)^2$.

5. Two men, C and D, start on a holiday together, C with £38, and D with £26. During the holiday D spends £4 more than C, and at the end of the holiday C has five times as much as D. How much has each spent?

6. Draw a graph from the following table to show the relation between the speed v m.p.h. of a train, and P, the resistance which it experiences in lb. per ton weight of the train.

v -	10	20	30	40	50	60	70
P -	6·9	9·6	14·1	20·4	28·5	38·4	50·1

From your graph find (i) the total resistance in lb. experienced by a train of 360 tons when moving at 42 m.p.h., (ii) the speed at which the total resistance is 6·8 tons for this train.

L

1. Simplify :

(i) $\sqrt[3]{\dfrac{64l^6m^{15}}{27m^{18}}}$, (ii) $(-a^2b^3c)\div(-a^3c^3)$, (iii) $\left(-\dfrac{1}{3x}\right)^3(-3x)^2$.

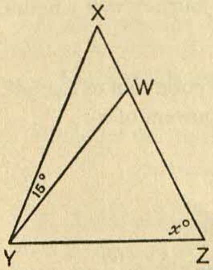
2. (i) Find the H.C.F. and L.C.M. of $7a^2bx$, $14a^2b^2x^2$, ab^2.

(ii) If $X=3a^2-4ab$ and $Y=4ab-3b^2$, express $4X-3Y$ in terms of a and b, and find the value of $4X-3Y$ when $a=2$, $b=-1$.

3. (i) Take $7a-3b$ from the sum of $4a+5b+6c$ and $6a-5b-4c$.

(ii) Simplify $(a^2-ab+b^2)-(a^2+ab+b^2)$.

4. A man rode a distance of 72 miles in $7\frac{1}{2}$ hours ; for part of the time he rode at 8 miles an hour, for the rest at 11 miles an hour. For how long did he ride at 8 miles an hour?



5. Solve the equations :

(i) $5(3x-1)-2(2x-5)=3x$,

(ii) $\dfrac{3t-1}{2}-\dfrac{4t-3}{3}=1-6t$.

6. XYZ is a triangle in which $XY=XZ$ and $YZ=YW$. If $\angle XYW=15°$, find x.

M

1. (i) Simplify $(4Q-2P+R)-(3Q+4P+5R)$.

(ii) Take $(2d+e)$ from $(4c-2d+5e)+(3d-2c-e)$.

2. Find the H.C.F. and L.C.M. of $36n^2$, $63m^3n^3$, $81n^4$.

3. Simplify (i) $(ab)\times(-bc)\times(ca)\div(-abc)$,

(ii) $3(l^2-m^2)-2[l^2-\{m^2+lm+m(m-\overline{l-m})\}]$.

4. A body is acted upon by a variable force. The following table shows the connection between the distance s feet travelled by the body from the starting point and the force P pounds weight acting upon it :

s -	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
P -	20	21	21	20	19	18·5	18	14	9	5	0

Draw the graph. How far has the body moved when P has the values (i) 15, (ii) 6?

5. Divide 16s. 7d. between X, Y, Z so that X may have twice as much as Z and 8 pence more than three times Y's share.

6. The sum S of a certain series is given by the formula

$$S=\tfrac{1}{2}n\{2a+(n-1)d\}.$$

Find a when $\quad S=-570, \quad d=-6, \quad n=15.$

N

1. (i) By selling a picture for $(3a-7b+5c)$ shillings, I gain $(b-2c)$ pence. What did I pay for it?

(ii) The perimeter of a rectangle is $(6s-3t)$ feet. One side is $(2s-t)$ feet. Find the other.

2. Simplify (i) $3s(s-4t)-2t(5s-t)$,

(ii) $2x+3y-(2x-3z)-\{3y+x-\overline{3z-x}\}$.

3. One-fourth of the subscribers to a certain fund each gave £1, one-sixth of the remainder each gave 10s., and the rest each gave 1s. If the three sets of subscribers raised their subscriptions to £1 1s., 10s. 6d., and 2s. respectively, the total increase in the subscriptions would be £5 12s. 6d. How many subscribers were there?

4. Solve (i) $0=5(2z-3)-3(7-4z)+8(z+12)$,

(ii) $0{\cdot}6(6x-0{\cdot}5)+1{\cdot}5x=0{\cdot}4+3x$.

5. (i) Find the H.C.F. of $14x^2y^3z^4$, $21xy^2z^3$, $35xy^3z^4$.

(ii) Find the L.C.M. of x^4, $4x^3y$, $6x^2y^2$, $4xy^3$, y^4.

6. (i) Simplify $\dfrac{2t-3}{8}-\dfrac{3t-2}{12}$.

(ii) Multiply $\dfrac{5-3n}{8}-\dfrac{2}{3}-\dfrac{4n-1}{6}$ by 24, and simplify the result.

F

1. Simplify (i) $(5p-3)-(3p+4q)-(5-6q)$,

(ii) $3t^2+5+t^4-t^2-3-3t^2+t^5+4-2t+3+t^3$

What is the coefficient of t^2?

2. Out of a collection of foreign stamps, $12xy+2y^2$ in all, $2xy-3y^2$ are found to be worthless forgeries. If the others are divided equally among $5y$ boys, how many stamps will each boy get?

3. Solve (i) $\dfrac{3a}{2}-\dfrac{4-a}{3}=2\dfrac{1}{3}-3(a-2)$,

(ii) $\dfrac{z}{6}-\dfrac{1}{3}\left(z-\dfrac{1}{2}\right)-\dfrac{1}{3}\left(\dfrac{2}{5}-\dfrac{z}{3}\right)=0$

4. A man buys two horses for £86. By selling one for three-quarters of its cost price, and the other for four-thirds of its cost price, he makes a profit of £3 on the whole transaction. Find the cost price of each horse.

5. Simplify (i) $\sqrt[7]{-x^{28}}$, (ii) $(x^2)^5-(2x^5)^2$, (iii) $\dfrac{(5c^5)\times(-8a^3c^3)}{20c^2}$.

6. The following table gives the areas of cross-sections of a body at right angles to its axis :

Area (A) in sq. in. - -	220	262	280	268	235	200	125
Distance (x in.) from one end	0	20	40	60	80	100	120

Draw the graph connecting A and x. What is the probable cross-section when $x=70$ in. ?

Q

1. (i) From $(6a-3b+3c)$ take $(4a-6b-2c)$.

(ii) Subtract $5a$ shillings from $3b$ florins. Give the answer in pounds.

2. Simplify (i) $\sqrt[3]{-27t^{15}}$, (ii) $-3(-x)^4$, (iii) $\dfrac{4x^2}{5y}\div\dfrac{y}{z}$.

3. (i) Find the H.C.F. and L.C.M. of $8a^2xy$, $6pxy^2$, $18p^2x^2y$, $9ax^2y^3$.

(ii) By what must $7a^2b^5$ be multiplied to produce $-28a^5b^9$?

(iii) To what power must $-a^{10}$ be raised to give a^{200}?

4. Solve (i) $8(4a-3)-3(7a-3)-12(2-3a)=60a$,

$$\text{(ii)}\ \frac{x}{1\cdot2}-\frac{x}{0\cdot8}+13=\frac{1}{9}\left(\frac{x}{0\cdot5}-4\right).$$

5. A man has two sons X and Y whose united ages just equal his own. In two years' time he will be twice as old as Y, and 16 years ago he was three times as old as X. Find the present ages of father and sons.

6. Simplify (i) $\dfrac{2(c-3)}{9}-\dfrac{7(c+5)}{6}+\dfrac{4(c-2)}{3}$,

$$\text{(ii)}\ 21\left[\frac{a-2b}{3}-\overline{a+3b}+\frac{5}{7}(a+8b)\right].$$

PART II

CHAPTER XIII

SIMPLE SIMULTANEOUS EQUATIONS. PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS

71. On a road running west from Cambridge **X** and **Y** are respectively 60 miles and 8 miles from Cambridge. A man **A**

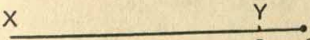


FIG. 8.

leaves **X** at 9 a.m. and travels east at 4 miles an hour. After t hours he will have travelled $4t$ miles, and he will then be $60-4t$ miles west of Cambridge. If we call this distance x miles, it is clear that

$$x=60-4t. \quad \text{(i)}$$

From consideration of the motion it is clear that this equation is satisfied by an unlimited number of pairs of values of x and t. For any value of t, there is a definite position reached by **A**, and a definite value of x which may be obtained from the equation (i). Thus, if $t=1$, $x=56$; if $t=2$, $x=52$; if $t=5\frac{1}{2}$, $x=38$; and so on.

In general, whenever we have a single equation in two unknowns, we can find innumerable pairs of values of the unknowns which satisfy it. Such equations are said to be **indeterminate.**

It often happens that two unknowns which are connected by an indeterminate equation also satisfy some other equation. If this is the case, it is usually possible to find a limited number of pairs of values of the unknowns which satisfy both equations.

In the example taken above, consider a man **B** who leaves **Y** at 11 a.m. and travels west at 28 miles an hour. After t hours (measured from 9 a.m.) he will have travelled $28(t-2)$ miles west, and he will then be $8+28(t-2)$ miles west of Cambridge. If we call this distance x miles, it is clear that

$$x=8+28(t-2)=28t-48. \quad \text{(ii)}$$

This equation is satisfied by an unlimited number of pairs of values

of x and t. It should be clear that for any value of t, the value of x for **A** will not in general be the same as the value of x for **B**. But at the moment when **A** meets **B**, for the particular value that t has when they meet (and for that value only), the values of x for A and **B** must be the same. We expect therefore to find one and only one pair of values of x and t which satisfy both the equations (i) and (ii)) **at the same time**, i.e. **simultaneously**.

To solve these equations by Algebra, we equate the two values of x. If the two men are at the same spot at the same moment, the values of x and t must, **at that moment**, be the same in (i) and (ii). We thus have

$$60 - 4t = 28t - 48,$$

$$\therefore \quad -4t - 28t = -48 - 60,$$

$$\therefore \quad -32t = -108,$$

$$\therefore \quad t = \frac{-108}{-32} = 3\frac{3}{8}.$$

But, from (i), $\quad x = 60 - 4t$;

$$\therefore \text{ when } t = 3\frac{3}{8}, \quad x = 60 - 13\frac{1}{2} = 46\frac{1}{2}.$$

It is easily verified that equation (ii) is satisfied by this pair of values. Thus the men meet $3\frac{3}{8}$ hours after 9 a.m., and they will then be $46\frac{1}{2}$ miles west of Cambridge.

We see that, although there are many pairs of values of x and t which satisfy (i), and many pairs which satisfy (ii), there is only one pair which occurs in both sets of values. This pair is $x = 46\frac{1}{2}$, $t = 3\frac{3}{8}$, and is called the solution of the **simultaneous equations**

$$x = 60 - 4t,$$

$$x = 28t - 48$$

GENERAL METHODS OF SOLVING SIMULTANEOUS EQUATIONS

72. Method 1. Elimination by equating equivalent values of one unknown.

Example 1. *Solve the simultaneous equations* :

$$5x - 3y = 28, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$10x + 25y = -6. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) $\quad 5x = 3y + 28, \quad \therefore x = \dfrac{3y + 28}{5}. \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$

From (ii) $10x = -25y - 6, \quad \therefore x = \frac{-25y-6}{10}$. ....(iv)

But the values of x obtained in (iii) and (iv) must be equal,

$$\therefore \frac{3y+28}{5} = \frac{-25y-6}{10}, \quad \text{.........................(v)}$$

$$\therefore 6y + 56 = -25y - 6 \text{ (multiplying each side by 10)},$$

$$\therefore 31y = -62, \quad \therefore y = -2.$$

Put $\quad y = -2$ in (iii), $\quad \therefore x = \frac{22}{5} = 4 \cdot 4$.

The solution is $\quad x = 4 \cdot 4, \quad y = -2$.

Check. When $\quad x = 4 \cdot 4, \quad y = -2$,

in (i), $\quad$ L.H.S. $= 22 + 6 = 28 =$ R.H.S.,

in (ii) $\quad$ L.H.S. $= 44 - 50 = -6 =$ R.H.S.

Note. In the above working it should be clearly understood that x and y stand for the values of x and y which satisfy **both** the equations (i) and (ii).

73. Method 2. The method of substitution.

Example 2. *Solve the simultaneous equations* :

$$6x - 5y = 7, \quad \text{..............................(i)}$$

$$9x + 4y = 22. \quad \text{.............................(ii)}$$

From (i) we can find an expression for x in terms of y; if we substitute this value of x in (ii) we shall have an equation in y only, which we can solve.

Thus, from (i), $\quad 6x = 7 + 5y, \quad \therefore x = \frac{7+5y}{6}$. ....(iii)

Substitute this value of x in (ii).

Then $\quad \frac{3\cancel{9}(7+5y)}{\cancel{6}\,2} + 4y = 22; \quad \therefore 21 + 15y + 8y = 44;$

$$\therefore 23y = 23; \quad \therefore y = 1.$$

Put $\quad y = 1$ in (iii), then $x = \frac{7+5}{6} = 2$.

The required pair of numbers is therefore $x = 2, y = 1$.

Check. If $x = 2, y = 1$, in (i) L.H.S. $= 12 - 5 = 7 =$ R.H.S.,
in (ii) L.H.S. $= 18 + 4 = 22 =$ R.H.S.

Note 1. The value of x was found by substituting $y = 1$ in (iii), i.e. in the equation previously used for substituting. The pupil should make a habit of doing this.

Note 2. It is necessary to verify that the pair of numbers satisfies **both** the **original** equations.

Note 3. The pupil should in each case consider whether it is easier to express x in terms of y, or y in terms of x. Thus, in solving the simultaneous equations $5x-3y=6$, $4x-y=9$, it is easier to express y in terms of x by means of the second equation. We thereby avoid the introduction of fractions. The process of getting rid of one of the unknowns is called **elimination**. In each of the above examples we eliminated x.

EXERCISE 31. a

Solve the following pairs of simultaneous equations and check your answers. In each case consider carefully which unknown can be more easily eliminated, or whether it makes no difference.

1. $x+y=14,$ $x-y=8.$

2. $x+y=24,$ $x-y=0.$

3. $x+2y=7,$ $2x+3y=12.$

4. $3x+2y=13,$ $7x-y=19.$

5. $2x+y=23,$ $3x-2y=3.$

6. $13+5y=19x,$ $2y=5x.$

7. $3a-2b=6,$ $6b-5a=30.$

8. $l+3m=-1,$ $3l+m=-11.$

9. $x=5y-7,$ $y=6x-16.$

10. $x-3y=0=20+y-2x.$

11. $5y-6x=0=8x-6y-1.$

12. $a-b=b-a+15=5b.$

EXERCISE 31. b

(*See instructions at the head of Exercise* 31 *a*)

1. $x-y=11,$ $x+y=17.$

2. $x+4y=10,$ $3x+5y=23.$

3. $s+t=14,$ $t-s=0.$

4. $4x+3y=20,$ $6x-y=8.$

5. $5y-7=2x,$ $3x=4y.$

6. $5x+y=26,$ $x-3y=2.$

7. $3x=10+y,$ $5x=7y+38.$

8. $y=3x-26,$ $x=5y+4.$

9. $2l+5m=-5,$ $4l+3m=-17.$

10. $x-4y=0=22+y-3x.$

11. $3y-7x=4x-2y+1=0.$

12. $2x-3y=4y-3x+39=16.$

74. The above methods are both useful. If both equations are simple equations the first method is better than the second, but the second—the method of substitution—is of fundamental importance, if one equation is of the first degree and the other equation is of higher degree. But when both equations are of the first degree, the third method, given below, is the best method. This method

has the advantage that fractions are not unnecessarily introduced, and the pupil should regard it as the standard method.

Method 3. Elimination by addition or subtraction after equalising coefficients.

If the given equations are such that the coefficients of one of the variables are numerically equal in both equations, it is possible to eliminate that variable by addition or subtraction. Thus, if the equations are

$$4x-3y=6, \quad 7x+3y=27,$$

we obtain at once, by addition, $11x=33$, and the remainder of the work is easy. Similarly, if the equations are

$$5x-3y=28, \quad 5x+8y=6,$$

we obtain at once, by subtraction, $-11y=22$, etc.

In general, the equations are not *given* in such a simple form. But it is always possible to get the given equations into the above form by multiplying each side of one or both equations by a suitable number.

Example 3. *Solve the simultaneous equations :*

$$6x-5y=7, \quad \text{.........................(i)}$$
$$9x+4y=22. \quad \text{.........................(ii)}$$

The L.C.M. of 6 and 9 is 18, $\therefore$ if we multiply each side of (i) by 3 and each side of (ii) by 2, we shall obtain equations in which the coefficients of x are numerically equal.

[Multiplication of (i) and (ii) by 9 and 6 respectively would, of course, lead to the same solution, but the working would be slightly heavier.]

Multiply each side of (i) by 3, $\therefore$ $18x-15y=21$. ....(iii)
Multiply each side of (ii) by 2, $\therefore$ $18x+8y=44$. ....(iv)
From (iii) and (iv) by subtracting, $-23y=-23$, $\therefore$ $y=1$.
Put $y=1$ in (i), $\therefore$ $6x-5=7$, $\therefore$ $6x=12$, $\therefore$ $x=2$.
The remainder of the work is as in Example 2.

75. Sometimes it will be necessary to simplify the equations before applying any one of the methods of solution.

If the equations contain fractions, it is usually best to get rid of the fractions as a first step, but this is not a universal rule and there is scope for considerable skill in avoiding unnecessary working. See Ex. 5, below.

Example 4. *Solve* $3x+\dfrac{y+5}{7}=\dfrac{4x-3}{2}$, ...(i)

$$\frac{4-3y}{5}-\frac{2x-5}{3}=-y. \quad \text{......(ii)}$$

To clear of fractions,

Multiply each side of (i) by 14,

$$\therefore\ 42x+2y+10=28x-21, \quad \therefore\ 14x+2y=-31. \quad \text{......(iii)}$$

Multiply each side of (ii) by 15,

$$\therefore\ 12-9y-10x+25=-15y, \quad \therefore\ -10x+6y=-37. \quad \text{(iv)}$$

Multiply each side of (iii) by 3,

$$\therefore\ 42x+6y=-93. \quad \text{......(v)}$$

From (iv) and (v) by subtraction,

$$52x=-56, \quad \therefore\ x=-1\tfrac{1}{13}.$$

We may obtain y, as hitherto, by substituting this value of x in (iv) or (v), but it may be considered preferable to obtain y directly, by eliminating x. Thus,

Multiply each side of (iii) by 5,

$$\therefore\ 70x+10y=-155. \quad \text{......(vi)}$$

Multiply each side of (iv) by 7,

$$\therefore\ -70x+42y=-259. \quad \text{......(vii)}$$

From (vi) and (vii) by addition,

$$52y=-414, \quad \therefore\ y=-7\tfrac{25}{26},$$

$$\therefore\ \text{the solution is} \quad x=-1\tfrac{1}{13}, \quad y=-7\tfrac{25}{26}.$$

In an example like this, when the answer contains awkward fractions, the work of checking would be just as likely to lead to mistakes as the work of solving. In such cases it is quicker to look over the working again to make sure that there are no errors.

Note. Whenever the values of the unknowns render substitution awkward, it is advisable to adopt the procedure of Ex. 4. It will be seen later that this is of particular importance in connection with literal equations.

Example 5. *Solve* $\dfrac{7x}{11}+\dfrac{3y}{8}=8$, ...(i)

$$\frac{5x}{11}-\frac{3y}{4}=22. \quad \text{......(ii)}$$

If we clear of fractions, we do an unnecessary amount of work. Instead we proceed thus:

Multiply each side of (i) by 2,

$$\therefore \frac{14x}{11}+\frac{3y}{4}=16. \quad\text{......................(iii)}$$

From (ii) and (iii) by addition,

$$\frac{19x}{11}=38, \quad \therefore x=22.$$

The completion of the work is left to the pupil.

76. The work of this chapter is summed up in the following instructions :

1. First decide which unknown it is easier to eliminate.

2. When one unknown has been found, it is usually best to obtain the other by substituting in one of the equations containing both unknowns. If you use the first or second methods substitute in the equation you obtained for the purpose of substitution. It occasionally happens that the value of the first unknown is an awkward fraction ; it may then be easier to obtain the second unknown by repeating the process of elimination.

3. When checking, check in BOTH equations in their ORIGINAL form.

4. If the answers involve awkward fractions or decimals, it is better to look over your working again, instead of checking by a substitution which may lead to errors owing to the complexity of the working.

5. Number your equations, so that the explanation of your work may be clear.

EXERCISE 32. a

(*See instructions at the head of Exercise* 31 *a*)

1. $5x+6y=28,$ $4x+11y=41.$

2. $3a+7b=26,$ $5b+4a=13.$

3. $5l-9m=17,$ $8m-3l=-5.$

4. $9P-11Q=15,$ $13Q+25=7P.$

5. $12s-t=-9,$ $36t-10s=113.$

6. $4X+3Y=0,$ $11X-5Y=53.$

7. $15x-4y=6,$ $9x-2y=5.$

8. $14x+2y=1,$ $38x+5y=7.$

9. $44x+3y=62,$ $20x-9y=4.$

10. $20l-15m=33,$ $15l-6m=16.$

11. $30x+14y=27,$ $75x+6y=53.$

12. $100x+65y=1,$ $175x+55y=37$

13. $7c-11d=32,$ $2c=7+d.$

14. $3x-5y=30,$ $11x-27y=32.$

15. $22x+39y=24,$ $24-14x=15y.$

EXERCISE 32. b

(*See instructions at the head of Exercise* 31 *a*)

1. $4a+5b=41,\quad 3a-2b=2.$

2. $5l-3m=1,\quad 3l=m+5.$

3. $3x+7y=22,\quad 7x+9y=44.$

4. $2P-9Q=0,\quad 18Q-7P=27.$

5. $3A+2B=6,\quad 5A+6B+30=0.$

6. $5a+7b=4,\quad 13a+21b=2.$

7. $22x+27y=3,\quad 44x+51y=2.$

8. $12x-10y=7,\quad 36x-18y=3.$

9. $21x+22y=41,\quad 24x+66y=149.$

10. $40x+9y=14,\quad 72x+51y=2.$

11. $25x+63y=1,\quad 15x+84y=16.$

12. $30x+8y=9,\quad 140x-36y=97.$

13. $3a+2b=2,\quad 30a=6b+59.$

14. $6x+20y=7,\quad 8y=3x-8.$

15. $11l-7m=32,\quad 19l=43+8m.$

EXERCISE 32. c

Solve Nos. 1–16 :

1. $2(x-3)-(y-5)=(x-4)-(y-6)=10.$

2. $2-2(3x-y)=10(4-y)-5x=4(y-x).$

3. $4(a-2)-5(1-b)=0=29a-3(a-b)+4.$

4. $2y-3x=12(y-3x)=39-6y.$

5. $5(x-3y)-(3x-14y)=14,\quad 10x-6y-3(2x-3y)=38.$

6. $7x+2-6(y-1)=24,\quad 7y+17-6(x+2)+18=0.$

7. $\frac{x}{6}-\frac{y}{3}=4,\ \frac{x}{12}-\frac{2y}{3}=4.$

8. $\frac{x}{3}-\frac{y}{2}=1,\ \frac{x}{4}-\frac{y}{5}=\frac{3}{4}.$

9. $\frac{a}{5}+\frac{b}{3}=0,\ \frac{a}{4}-\frac{b}{3}=\frac{3}{20}.$

10. $\frac{2x-3y}{21}=\frac{x-11}{8}=\frac{y+3\frac{1}{2}}{20}.$

11. $\frac{3x+2y}{4}+\frac{2(y-x)}{11}=0,\quad 17x+20y+1=0.$

12. $2x+15y=\frac{2}{3}(x-3y),\quad 7x-27y+77\frac{1}{2}=0.$

13. $1\cdot5x+2\cdot4y=1\cdot4,\quad 1\cdot8x-14y=5\cdot9.$

14. $2x+19y=21,\quad 38(3x-y)=1079.$

15. $0\cdot2x+2\cdot5y=3\cdot28,\quad 10x+1\cdot5y=15\cdot8.$

16. $3\cdot75x-1\cdot5y=0\cdot75,\quad 1\cdot4x+0\cdot7y=9\cdot1.$

Solve Nos. 17–22, correct to two decimal places, :

17. $0\cdot4x+0\cdot7y=1\cdot3,\quad 0\cdot9x-0\cdot2y=1\cdot8.$

18. $0\cdot3x+0\cdot8y=1\cdot7,\quad 0\cdot8x-0\cdot3y=1\cdot1.$

19. $0\cdot2x+1\cdot1y=-2\cdot3,\quad 0\cdot5x-0\cdot7y=-1\cdot2.$

20. $0\cdot5x+0\cdot6y=2\cdot8,\quad 0\cdot6x-0\cdot5y=1\cdot6.$

21. $1\cdot3x+0\cdot7y=0\cdot2,\quad 0\cdot3x+0\cdot6y=0\cdot4.$

22. $1\cdot2x+1\cdot1y=-3\cdot1,\quad 0\cdot5x+0\cdot4y=-2\cdot7.$

23. If $a-2b=3$ and $3a+b=19{\cdot}5$, find the value of $3a-8b$.

24. If $4l-5m=5{\cdot}2$ and $7l-8m=6{\cdot}5$, prove that $7m=8l$.

25. If $3l+m=1$ and $5l-2m=20$, find the value of $5l+2m$.

26. The equation $ax^2+bx+4=0$ is satisfied by $x=0{\cdot}5$ and $x=4$. Find the values of a and b.

27. If $2s+4t=3{\cdot}3$ and $5s-7t=5{\cdot}5$, prove that $5s=41t$.

28. The equation $ax^2+bx-10=0$ is satisfied by $x=\frac{2}{3}$ and $x=-5$. Find the values of a and b.

29. Show that it is impossible to find a pair of numbers x, y to satisfy the three equations $7y-3x=2$, $5x=8y+4$, $3x+2y=17$.

30. The values of x and y are connected by the equation $x=ly+m$. When $x=3$, $y=4$, and when $x=-7$, $y=-1$. Find l and m. Find x when $y=-3$, and y when $x=6$.

31. Show that it is impossible to find a pair of numbers a, b to satisfy the three equations $3a+b=1$, $5a+2b=4$, $2a+3b=10$.

32. The values of x and y are connected by the equation $y=ax+b$. When $x=4$, $y=5$, and when $x=7$, $y=14$. Find a and b. Find x when $y=2$, and y when $x=-3$.

Further and harder examples are given in Ch. XXI, Exs. 63. a and b.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS

77. Example 6. *If 2 is subtracted from the numerator of a fraction and 3 is added to the denominator, it reduces to $\frac{1}{4}$; if 6 is added to the numerator and the denominator is multiplied by 3, it reduces to $\frac{2}{3}$. Find the fraction.*

Let x be the numerator of the fraction and y the denominator; then the fraction is $\dfrac{x}{y}$.

From the first condition $\dfrac{x-2}{y+3}=\dfrac{1}{4}$, ...(i)

from the second $\dfrac{x+6}{3y}=\dfrac{2}{3}$. ....(ii)

To clear of fractions multiply each side of (i) by $4(y+3)$, and each side of (ii) by $3y$.

Thus, from (i) $4(x-2)=y+3$, or $4x-y=11$, ...(iii)

from (ii) $x+6=2y$, or $x-2y=-6$. ....(iv)

Solving (iii) and (iv) in the usual way, we get $x=4$, $y=5$; and the fraction is $\frac{4}{5}$.

Check (by using the data of the problem):

$$\text{(i)}\ \frac{4-2}{5+3}=\frac{2}{8}=\frac{1}{4};\quad \text{(ii)}\ \frac{4+6}{5\times 3}=\frac{10}{15}=\frac{2}{3}.$$

Example 7. *Two men, $8\frac{1}{4}$ miles apart, set out at the same time and are together in 11 hours, if they walk in the same direction, but in 1 hour, if they walk in opposite directions; find their speeds, supposed uniform.*

Let the faster walker go at the rate of x miles per hour,
and let the slower " " " " y " "

When they walk in the same direction, the faster walker gains on the other $(x-y)$ miles per hour; $\therefore$ in 11 hours he gains $11(x-y)$ miles,

$$\therefore\ 11(x-y)=8\tfrac{1}{4}. \quad\text{.........................(i)}$$

When they walk in opposite directions they approach one another at the rate of $(x+y)$ miles per hour;

$$\therefore\ x+y=8\tfrac{1}{4}. \quad\text{.........................(ii)}$$

Solving (i) and (ii) in the usual way, we get $x=4\frac{1}{2}$, $y=3\frac{3}{4}$.

Thus, the rates of walking are $4\frac{1}{2}$ and $3\frac{3}{4}$ miles per hour respectively.

The pupil should check the result by using the data of the problem.

EXERCISE 33. a

1. Find two numbers whose sum is 92 and whose difference is 34.

2. Find two numbers which are such that three times the less exceeds twice the greater by 18; and such that one-third of the less plus one-fifth of the greater equals 21.

3. 6 cows and 8 sheep cost £148; 11 cows and 7 sheep cost £233; find the cost of a cow and of a sheep.

4. I buy 3 tables and 5 chairs for £30, and 4 tables and 8 chairs for £42. Find the price of a table and of a chair.

5. 8 lb. of apples and 5 lb. of pears cost 4s. 9d.; 6 lb. of apples and 3 lb. of pears cost 3s. 3d. Find the cost of 1 lb. of pears.

6. 15 lb. of tea and 17 lb. of coffee cost £3 11s. 6d.; 25 lb. of tea and 13 lb. of coffee cost £4 8s. 6d. Find the price of each per lb.

7. A number is formed of two digits whose sum is 9. If the digits are reversed the number is increased by 27. Find the number. [Note. If the tens digit is x and the units digit is y, the number is $10x+y$.]

8. A number of two digits is equal to 3 times the sum of its digits. The number formed by reversing its digits plus 9 equals 3 times the original number. Find it.

9. A number of two digits is five-sixths of what it would be if its digits were reversed. If the number is increased by 3 times the sum of its digits, the result is 72. Find the number.

10. A is 7 years older than B. 15 years ago B's age was three-quarters of A's age. Find their ages now.

11. Three years ago a man was eight times as old as his son, and in two years time he will be four and a half times as old as the son. What are their present ages?

12. The sum of the ages of a father and his two sons is 57 years, and one son is 2 years older than the other. If the elder son lives to be as old as his father is now, and if all three are then alive, the sum of their ages will be 162. Find their present ages.

13. Find a fraction which reduces to $\frac{1}{2}$, if the numerator and denominator are each diminished by 1, and reduces to $\frac{2}{3}$, if the numerator and denominator are each increased by 2.

14. Find a fraction such that, if its numerator is diminished by 1, it reduces to $\frac{1}{2}$, and becomes equal to 3 when the numerator is increased by 3 and the denominator diminished by 12.

15. My quarterly electricity bill is made up of a fixed charge of a/- plus a charge of b pence per unit used. In two successive quarters I use 124 and 76 units and pay £2 8s. and £1 10s. Find a and b.

16. A cricketer has made $(y+5)$ runs for $(x+3)$ times out and his average is exactly 13. If he makes 45 in his next innings and gets out, his average will then be exactly 15. Find x and y.

EXERCISE 33. b

1. Find two numbers whose sum is 95 and whose difference is 51.

2. Find two numbers which are such that one-fifth of the greater exceeds one-sixth of the less by 4; and such that one-half of the greater plus one-quarter of the less equals 38.

3. 5 horses and 7 cows cost £223; 7 horses and 9 cows cost £301. Find the price of a horse and of a cow.

4. 4 lb. of tea and 6 lb. of coffee cost £1; 1 lb. of tea and 8 lb. of coffee cost £1 0s. 2d. Find the price of each per lb.

5. 12 pencils and 3 fountain-pens cost 15s. 9d.; 36 pencils and 5 fountain-pens cost £1 7s. 3d. Find the cost of a pencil.

6. 25 lb. of apples and 33 lb. of plums cost 16s. 7d.; 35 lb. of apples and 48 lb. of plums cost £1 3s. 8d. Find the price of each per lb.

7. A number of two digits whose sum is 14 exceeds by 36 the number formed by reversing its digits. Find the number.

8. A number of two digits exceeds 4 times the sum of its digits by 3; if the number is increased by 18, the result equals the number formed by reversing its digits. Find the number.

9. A number of two digits is three-eighths of the number formed by reversing its digits. If the number is increased by three times the sum of its digits, the result is 54. Find it.

10. A's age exceeds twice B's age by 6 years. In 4 years' time one-third of B's age will exceed one-tenth of A's age by one year. What are their present ages?

11. Last year a father was 7 times as old as his son; in 2 years' time he will be 5 times as old as his son. Find their present ages.

12. A is 5 times as old as her son B, while in 3 years' time she will be 4 times as old as B. What is her present age?

13. Find a fraction which reduces to $\frac{2}{3}$ when 1 is subtracted from both numerator and denominator, and which is equal to 1, if the numerator is increased by 2 and the denominator diminished by 2.

14. Subtract 3 from both numerator and denominator of a fraction and it reduces to $\frac{4}{5}$; subtract 9 from each and it reduces to $\frac{1}{2}$. Find the fraction.

15. In the $\triangle XYZ$, $\angle Y = 4\angle Z$ and $3\angle X - 5\angle Y = 15°$. Find the $\angle$s of the $\triangle$.

16. If A gives B £3, B will have 3 times as much as A. If B gives A £4, A will have £20 less than B. How much has each?

EXERCISE 33. c

1. If A gives B 4/-, B will have twice as much as A; if B gives A 15/-, A will have 10 times as much as B. How much has each?

2. In a $\triangle PQR$, $\angle P = 3\angle Q$ and $2\angle Q - \angle R = 30°$. Find the $\angle$s of the $\triangle$.

3. A heap of shillings and half-crowns is worth £1 2s. If there were half the number of shillings and twice the number of half-crowns, it would be worth 4s. more. How many coins of each kind are there?

4. When petrol costs 2s. 4d. a gallon and oil 3s. a quart, a motorist finds that the cost of petrol and oil on a given journey is £3 12s. 3d. When the prices are each reduced by 4d., the cost for the same journey is £3 2s. How much petrol does he use on the journey?

5. P, Q, and R travel from the same place in the same direction at the rates of 25, 30 and 40 miles per hour respectively. If Q

starts 2 hours after P, how long after Q must R start in order that they may overtake P at the same moment?

6. A cricketer has made $(y-7)$ runs for $(x-5)$ times out, and his average is exactly 15. If he makes 34 in his next innings and gets out, his average will then be exactly 16. Find x and y.

7. My quarterly telephone bill is made up of a fixed charge of l/- plus a charge of m pence per call. In two successive quarters I make 144 and 96 calls and pay £2 10s. and £2 5s. Find l and m.

8. I have 8s. to spend on football matches. If I pay tram fares and entrance money each time, I can go 8 times; if I walk once out of every 3 times, I can go 9 times. What is the entrance money for one match?

9. A dealer sells 7 horses and buys 9 cows, thus increasing his cash by £88. He then at the same prices buys 9 horses and sells 13 cows, thus decreasing his cash by £91. Find the price of each cow.

10. Two men, 11 miles apart, set out at the same time and are together in 1 hr. 20 min., if they walk in opposite directions, but in 14 hr. 40 min., if they walk in the same direction. Find their speeds, supposed uniform.

11. A man sold apples at 5 for 2d. and pears at 16 for 1/-. He would have received 1d. less, had he sold both at $\frac{1}{2}$d. each, and 9d. more, had he sold both at 3 for 2d. How many of each sort did he sell?

12. I spend 7s. in buying pears at 7 for 6d., and bananas at 9 for 6d. I sell two-thirds of my pears and three-eighths of my bananas for 4s., making a profit of 6d. on them. How many of each do I buy?

13. A, B, C travel from the same place in the same direction at the rates of 20, 25 and 40 miles per hour respectively. If B starts half an hour after A, how long after A must C start in order that B and C may overtake A at the same moment?

14. I spend 5s. in buying eggs at 3 for 2d. and oranges at 4 for 3d. If I were to sell them all at the rate of 14 for 1/-, I should gain 1/-. How many of each do I buy?

15. A boy has 4/- to spend on two kinds of note-book. If he buys 5 of one size and 7 of a larger size, he will require 2d. more; if he buys 7 of the smaller size and 5 of the larger, he will have 2d. too much. Find the price of each kind.

16. I spend 5s. 8d. in buying apples at 5 for 3d. and oranges at 5 for 2d. I sell three-fourths of my apples and four-fifths of my oranges for 4s. 9d., making a profit of 5d. on them. How many apples do I buy?

CHAPTER XIV

GRAPHS OF FUNCTIONS. GRAPHICAL SOLUTION OF EQUATIONS. LINEAR GRAPHS. GRADIENT. UNIFORM SPEED GRAPHS

78. Functions. If one variable changes when another variable is changed, we say that the first (or dependent variable) is a function of the second (or independent variable).

Thus, a boy's weight is a **function** of his age; the income-tax paid by a man is a **function** of his income; the time of swing of a pendulum is a **function** of its length. In using the word "function" we do not imply the existence of an algebraic expression from which values of the function can be calculated; thus, although a boy's weight is a function of his age, there is no algebraic expression from which we can calculate his weight when we know his age. But when there is such a function we call it an algebraic function of the independent variable, e.g. $4x^3-3x$, $\frac{x-5}{2x+3}$ are each algebraic functions of x. The graph showing the connection between the variables is called the graph of the function. In this chapter we extend the work of Ch. VIII, and consider graphs of algebraic functions. We then proceed to consider the graphical solution of equations, linear graphs, the gradient of a straight line and uniform speed graphs.

79. Example 1. *It is known that a stone projected upwards from a point on the ground with velocity* 48 *ft. per sec. is s ft. above the ground after t sec., where* $s=16t(3-t)$. *Draw the graph of s for values of t from* 0 *to* 3.

Start by calculating the values of s when $t=0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$. Arrange the work as shown below:

t - -	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
$16t$ - -	0	8	16	24	32	40	48
$3-t$ - -	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	0
$16t(3-t)=s$	0	20	32	36	32	20	0

After this table has been made, choose a convenient scale, so that the graph will fill as much of your paper as possible.

On a standard sheet of graph paper, we may take values of t along the short side and 1″ to represent 1 unit. Along the other side we take values of s and 0·1″ to represent 1 unit.

The points may now be plotted and a curve drawn through them. It is permissible to do this, since the intermediate points have a meaning. But before drawing the curve it is as well to consider whether we have sufficient points to enable us to do this accurately. In this case greater accuracy is desirable and can be obtained by calculating values for $t=\frac{1}{4}, 1\frac{1}{4}, 1\frac{3}{4}, 2\frac{3}{4}$. Accordingly we add these values to our table and plot the new points.

t - -	$\frac{1}{4}$	$1\frac{1}{4}$	$1\frac{3}{4}$	$2\frac{3}{4}$
$16t$ - -	4	20	28	44
$3-t$ - -	$2\frac{3}{4}$	$1\frac{3}{4}$	$1\frac{1}{4}$	$\frac{1}{4}$
$16t(3-t)=s$	11	35	35	11

The curve may now be drawn with a fair degree of accuracy. (See p. 150.) As previously, care should be taken to label the axes and to mark the scales along them. The graph should be given a title.

GRAPH OF $16t(3-t)$ FOR VALUES OF t BETWEEN 0 AND 3

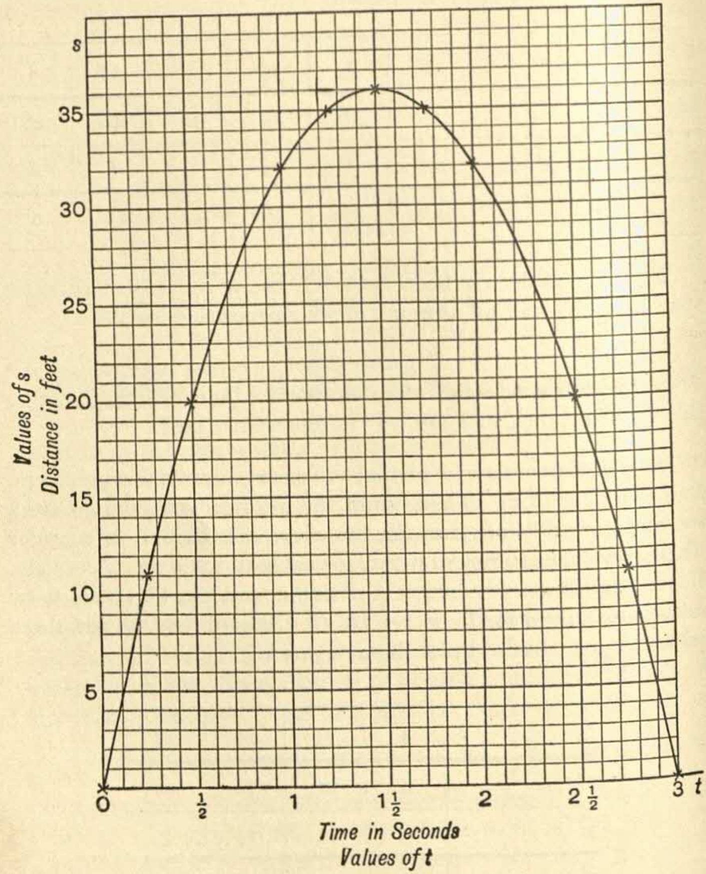


FIG. 9.

If the graph has been accurately drawn the value of s may be read off for any value of t. Thus, to obtain the value of s when $t=\frac{3}{4}$, take, on the axis across the page, the point corresponding to $t=\frac{3}{4}$, and note the reading for the upright from this point to the curve, i.e. 27. The pupil will find it instructive to draw his own

curve and take several readings, e.g. for $t = 0{\cdot}2$, $1{\cdot}8$, $2{\cdot}3$ etc. He may then estimate the accuracy of his drawing by comparing the values of s obtained from the graph with those obtained by substituting the values of t in the expression $16t(3-t)$. It should be clear that the curve gives approximately the value of s corresponding to any value of t between 0 and 3. The degree of accuracy of the result depends upon the accuracy of the drawing; this can be increased by increasing the number of points plotted. For most purposes it will be sufficient to start by plotting from 6 to 8 points; three or four more points may then be added where they appear to be most useful. If a sufficient degree of accuracy is not then attained, other points should be added. The following questions may now be discussed and the answers obtained from the graph:

(i) What is s when $t = 0{\cdot}2$, $1{\cdot}8$, $2{\cdot}3$?

(ii) For what values of t is $s = 15$, 27, 34?

(iii) How long is the stone in the air?

(iv) The stone going upwards passes at a certain instant a point 10 ft. from the ground. What time elapses before the stone again passes the same point?

(v) What is the greatest height the stone reaches?

(vi) How long does the stone take to reach its highest point?

(vii) For how long is the stone more than 32 ft. above the ground?

(viii) Is it possible to find values for t for which $16t(3-t) = 27$, 40, 22, -4?

80. In the above question the value of the independent variable t is restricted by the nature of the problem; it cannot be greater than 3 or less than 0. In other questions the function can be calculated for all values of the independent variable. We proceed to consider the graphs of such functions.

We now require a pair of axes on which positive and negative values of the independent variable and of the function can be represented. We take our axes as before; their point of intersection is called the **origin**. The axes are labelled to the right and upwards as before, but also to the left and downwards to represent negative numbers, using the same scale as for the positive numbers.

Thus, in Fig. 10, P is the point where $x=3$, $y=10$; this is usually written "the point (3, 10)"; Q is the point where $x=2$,

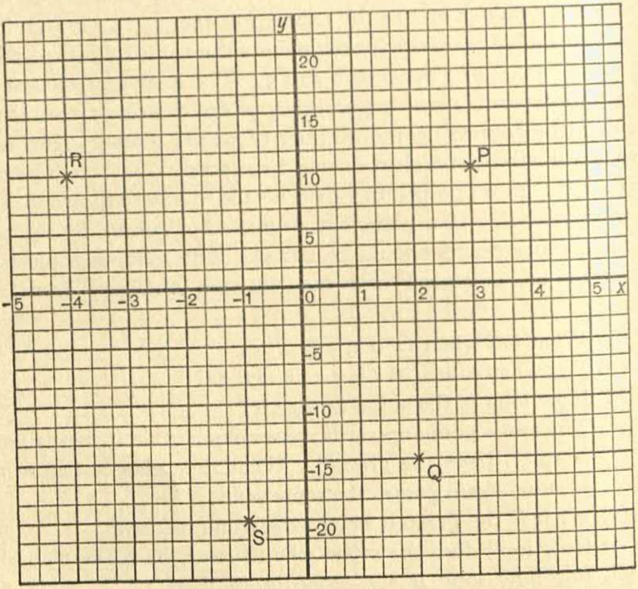


FIG. 10.

$y=-15$, i.e. the point $(2, -15)$; R is the point where $x=-4$, $y=10$, i.e. the point $(-4, 10)$; S is the point where $x=-1$, $y=-20$, i.e. the point $(-1, -20)$.

The values of x and y are called **coordinates**. The value of x is always written first; it is called the **x-coordinate or abscissa**. The value of y is called the **y-coordinate** or **ordinate**. It should be noted that the point of intersection of the axes, or origin, is the point (0, 0). Coordinates may clearly be used to fix the position of a point in a plane with reference to any chosen axes.

[Exs. 34 *a* and *b*, Nos. 1–4, may now be worked.]

81. Example 2. *Draw the graph of* x^2-5x-6 *from* $x=-2$ *to* $x=8$.

Make a table of values for whole numbers from $x=-2$ to 8.

x	-2	-1	0	1	2	3	4	5	6	7	8	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$
x^2	4	1	0	1	4	9	16	25	36	49	64	$5\frac{1}{16}$	$6\frac{1}{4}$	$7\frac{9}{16}$
$-5x$	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	$-11\frac{1}{4}$	$-12\frac{1}{2}$	$-13\frac{3}{4}$
-6	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6
x^2-5x-6 $(=y)$	8	0	-6	-10	-12	-12	-10	-6	0	8	18	$-12\frac{3}{16}$	$-12\frac{1}{4}$	$-12\frac{3}{16}$

The pupil should now plot the graph, taking $\frac{1}{2}''$ as the unit for x, $\frac{1}{4}''$ as the unit for y. After plotting these points it seems desirable to plot a few extra points between $x=2$ and $x=3$. These have been inserted in the above table on the right of the thick line and the pupil should plot them on his graph. The curve may then be drawn fairly accurately.

The following questions may now be discussed and the answers obtained from the pupils' graphs:

(i) What is (*a*) the least value of y, (*b*) the greatest value of y within the limits taken, i.e. from $x=-2$ to $x=8$?

(ii) Is it possible to find values of x for which $x^2-5x-6=0$, 14, -8, -18? How many values are there in each case? Could more values be obtained by continuing the curve either way?

(iii) Does x^2-5x-6 increase or decrease as x increases (*a*) from -2 to $2\frac{1}{2}$, (*b*) from $2\frac{1}{2}$ to 8? If the curve were continued to the right, would x^2-5x-6 continue to increase?

(iv) What can we say about the value of x^2-5x-6 when x is (*a*) a very large positive number, (*b*) a very large negative number?

(v) How can the graph be used to find solutions of the equations $x^2-5x-6=0, 4, -8, -16$?

(vi) How can the graph be used to find solutions of the equations $x^2-5x=6, 10, -2, -10, 1$?

It should be noticed that from the graph we can solve equations of the type $x^2-5x=$ any number. Similarly, if we wish to solve an equation such as $6x^2-7x-11=0$, we can draw the graph of $6x^2-7x$ and read the values for which this function equals 11.

This is a general method for finding approximately the roots of

any equation of the form $ax^2+bx+c=0$, where a, b, c are given numbers.

EXERCISE 34. a

(*Approximate answers should be given correct to one decimal place*)

In Nos. 1–4, plot the points and draw a smooth curve through them. In each case choose a suitable scale.

1. $(-6, 18)$, $(-5, 14)$, $(-4, 6)$, $(-3, 0)$, $(-2, -4)$, $(-1, -6)$.

2. $(1, 0)$, $(2, 1)$, $(3, 4)$, $(4, 10)$, $(5, 20)$.

3. $(-4, -6)$, $(-3, -2\frac{1}{2})$, $(-1, -\frac{3}{4})$, $(0, -\frac{2}{5})$, $(2, 0)$, $(5, \frac{3}{10})$.

4. $(-3, -5)$, $(-2, -8)$, $(-1\frac{1}{2}, -8\frac{3}{4})$, $(-1, -9)$, $(-\frac{1}{2}, -8\frac{3}{4})$, $(0, -8)$, $(1, -5)$.

5. It is 200 miles from King's Cross to Sutton-on-Hull.

(i) What will be the average speeds (v) of aeroplanes covering the distance in $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, t hours?

(ii) Draw the graph of $v=\frac{200}{t}$, for values of t between $\frac{1}{2}$ and 3.

(iii) Read off the value of v for which $t=2{\cdot}6$.

(iv) What meaning do you attach to this value of v?

(v) Read off the value of t for which $v=110$.

(vi) What meaning do you attach to this value of t?

(vii) Compare your answers to (iii) and (v) with those obtained by calculation.

(viii) What would be the value of $\frac{200}{t}$, (*a*) if t were very large, (*b*) if t were very small? Interpret the meaning of this for the above journey.

6. From a sheet of tin, 24 in. by 18 in., equal squares, with sides x in., are cut away at each corner; the sides are then turned up to make a rectangular box. Prove that the volume of the box is $4x(12-x)(9-x)$ cu. in.

Draw the graph of this function for values of x from 0 to 9. From the graph find (i) the volume of the box (correct to the nearest integer) when $x=3{\cdot}2$, $8{\cdot}1$; (ii) the values of x for which the volume is 600 cu. in.; (iii) the greatest volume of the box and the corresponding value of x; (iv) the values of x which satisfy the equation $4x(12-x)(9-x)=525$.

7. Draw the graph of $y=2(x-1)(2-x)$ from $x=-1$ to $x=5$. From the graph find (i) the values of $2(x-1)(2-x)$ when $x=-0{\cdot}2$, $1{\cdot}7$, $4{\cdot}3$; (ii) the values of x for which $y=-8$; (iii) the maximum value of y; (iv) the solutions of the equations

(*a*) $2(x-1)(2-x)=-3$, (*b*) $2(x-1)(2-x)=-17$,

(*c*) $2(x-1)(2-x)=5$.

Discuss the number of solutions in each case. Could more solutions be obtained by extending the curve?

8. Draw the graph of $y=2+3x-x^2$ from $x=-1$ to $x=5$. Use it to solve the equations (i) $2+3x-x^2=3$, (ii) $2+3x-x^2=-1$, (iii) $2+3x-x^2=4{\cdot}25$, (iv) $3x-x^2=-8$.

9. Draw the graph of $y=4x^2-6x+3$ from $x=-3$ to $x=3$. Use it to solve the equations

(i) $4x^2-6x+3=10$, (ii) $4x^2-6x+3=0{\cdot}75$,
(iii) $4x^2-6x+3=17$, (iv) $4x^2-6x=3$.

Solve graphically the equations in Nos. 10–15.

10. $(2x+5)(x-3)=3$. **11.** $(2x+3)(x+1)=-4$.

12. $(3x-5)(x+1)=3$. **13.** $5x^2+3x=1$.

14. $2x^2-x=5$. **15.** $x^2+4x=-2$.

16. Draw the graph of $y=\dfrac{3x}{2+x}$ from $x=1{\cdot}5$ to $x=4$.

EXERCISE 34. b

In Nos. 1–4, plot the points and draw a smooth curve through them. In each case choose a suitable scale.

1. $(-6, 21)$, $(-5, 12)$, $(-4, 5)$, $(-3, 0)$, $(-2, -3)$.

2. $(-2, -7)$, $(-1, -2\frac{1}{2})$, $(0, -1)$, $(1, -\frac{1}{4})$, $(2, \frac{1}{8})$.

3. $(-1, 0)$, $(-2, -1)$, $(-3, -4)$, $(-4, -10)$, $(-5, -20)$.

4. $(-3, 0)$, $(-2, -3)$, $(-1\frac{1}{2}, -3\frac{3}{4})$, $(-1, -4)$, $(-\frac{1}{2}, -3\frac{3}{4})$, $(0, -3)$, $(1, 0)$.

5. A ball projected from a point 50 ft. above the ground is s ft. above its starting point after t sec., where $s=16t(5-t)$. Draw the graph of s for values of t from 0 to 6. Use the graph to find :

(i) the values of s when $t=0{\cdot}4$, $2{\cdot}8$, $5{\cdot}2$;
(ii) the values of t when $s=10$, 90, -12 ;
(iii) the greatest height the ball reaches and the time taken to reach this height ;
(iv) how long the ball is in the air ;
(v) the values of t which make $16t(5-t)=-4$;
(vi) how long the ball is more than 80 ft. from the ground.

6. Draw the graph of $y=5(x-1)(x-2)$ from $x=-1$ to $x=6$. From the graph find (i) the values of $5(x-1)(x-2)$ when $x=-0{\cdot}7$, $1{\cdot}9$, $3{\cdot}3$; (ii) the values of x for which $y=10$; (iii) the minimum value of y ; (iv) the solutions of the equations (*a*) $5(x-1)(x-2)=8$, (*b*) $5(x-1)(x-2)=-3$, (*c*) $5(x-1)(x-2)=84$. Discuss the number of solutions in each case. Could more solutions be obtained by extending the curve?

7 From a sphere of radius 10 cm. a segment of height h cm. is cut off; the volume of this segment equals $\pi h^2\left(10-\frac{h}{3}\right)$ c.c. Draw the graph of this function for values of h from 0 to 20. [Along the V-axis take 1″ to represent 100π c.c.; along the h-axis take 1″ to represent 4 cm.] From the graph find: (i) the volume of the segment when $h=12{\cdot}8$; (ii) the values of h for which the volume is 200π c.c.; (iii) the greatest volume of the segment and the corresponding value of h; (iv) the values of h which satisfy the equation $h^2\left(10-\frac{h}{3}\right)=400$.

If the curve is drawn for values of h greater than 20, has it any meaning with reference to the above question?

8. Draw the graph of $y=25-2x^2$ from $x=-4$ to $x=4$. Use it to solve the equations:

(i) $25-2x^2=10$ (ii) $25-2x^2=-5$,

(iii) $25-2x^2=18$, (iv) $15=2x^2$.

9. Draw the graph of $y=5x^2+2x-3$ from $x=-2$ to $x=3$. Use it to solve the equations:

(i) $5x^2+2x-3=11$, (ii) $5x^2+2x-3=-1$,

(iii) $5x^2+2x-3=-3{\cdot}2$, (iv) $5x^2+2x=1$.

Solve graphically the equations in Nos. 10–15:

10. $(7+x)(1-x)+6=0$. **11.** $(5x-8)(x+1)=18$.

12. $(2x+1)^2=2$. **13.** $2x^2-x=11$.

14. $2x^2+5x+0{\cdot}5=0$. **15.** $4x^2-3x=3$.

16. Draw the graph of $y=\dfrac{5x}{3x-7}$ from $x=-5$ to $x=2$.

EXERCISE 34. c

1. Draw the graph of $y=\frac{1}{2}(x-3)(x+1)$ for values of x from -3 to 5. From your graph read off the two values of x that make the expression equal to 1.

2. Draw the graph of $y=\frac{1}{5}(x^2-2x+2)$ for values of x from -3 to 3. On the same diagram and with the same scales draw the graph of $4x+15y=14$, and state the coordinates of the points common to the two graphs.

3. Draw the graph of $6-x-2x^2$ for values of x from -3 to $2\frac{1}{2}$. Use your graph to determine (*a*) the values of x for which the expression becomes equal to -1; (*b*) the range of values of x for which the expression assumes positive values.

4. Draw the graph of $y=(x+2)(2-3x)$ for values of x from -4 to 2. Use the graph to find (*a*) the range of values of x for which

y exceeds unity; (*b*) the range of negative values of x for which y is positive; (*c*) the roots of $3x^2+4x=9$.

5. Draw the graph of $y=\frac{1-x}{2+x}$ from $x=-5$ to $-2{\cdot}5$ and from $x=-1{\cdot}5$ to 3. Find from your graph the values of x for which $y=2x$.

6. Plot in one figure and with the same scales the graphs of $8y=4x^2-4x-3$ and $3y=x+4$ for values of x from -3 to 3. Find, approximately, between what values of x the expression $\frac{1}{8}(4x^2-4x-3)-\frac{1}{3}(x+4)$ is negative.

7. Draw the graph of $y=5+2x-x^2$ for values of x from -2 to 4. Using your graph find: (*a*) the value of y when $x=1{\cdot}8$, (*b*) the greatest value of y, (*c*) the values of x between which the expression $5+2x-x^2$ is always positive.

8. Draw the graphs of $y=x^2$ and $x=2(y-1)$ on the same diagram and with the same scales. Use your graph to find (i) $\sqrt{3{\cdot}6}$, (ii) the range of values of x between -2 and 2 for which x^2 is greater than $\frac{x}{2}+1$.

9. Draw the graph of $y=x-2+\frac{7}{x+5}$ for values of x from -4 to 3. For what values of x is y zero?

10. Draw the graph of $y=(x-1)(4-x)$ for values of x from 0 to 5; and determine graphically where it intersects $5y=4x-6$.

11. Find graphically the maximum value of $3-5x-x^2$, and the values of x between which the expression is positive.

12. Draw the graph of $x^2+0{\cdot}8x-5$ for such values of x as make the expression negative, and find the least value which the expression can have.

13. Draw the graph of $y=3+6x-x^2$. For what values of x is y zero? What is the maximum value of y?

14. A piece of wire 24″ long is cut into two pieces $4x''$ and $(24-4x)''$ long respectively; each piece is then bent into the form of a square. Find an expression for the sum of the areas enclosed by these squares. Calling this sum A sq. in., plot A against x for values of x from 0 to 6 inclusive. Find from your graph the values of x for which the area enclosed by the two squares is 27 sq. in.

15. Draw the graphs of $y=1+2x-x^2$, and $y=x^2-2x-5$ between the points where they intersect, using the same axes and the same scales.

16. The base of a tank is a square of side x ft., and the total area of the bottom and sides (excluding the lid) is 36 sq. ft. Prove that

its volume is $\frac{1}{4}x(36-x^2)$ cu. ft. By drawing a graph of this expression for values of x between 0 and 6, find the side of the base of the largest tank with a square base which has this superficial area.

STRAIGHT LINE GRAPHS

82. (a) **Lines through the origin.** Consider the graph of $y=5x$. Plotting points as usual, we have

$$x=0,\ 1,\ 2,\ 3,\ -1,\ -2;$$
$$y=0,\ 5,\ 10,\ 15,\ -5,\ -10 \text{ etc.}$$

Let the points be $O, P_1, P_2, P_3, P_4, P_5$ etc. (see Fig. 11) and draw the perpendiculars $P_1N_1,\ P_2N_2,\ P_3N_3,\ P_4N_4,\ P_5N_5$ etc., to the axis Ox.

Then, in the $\triangle$s $P_1ON_1,\ P_2ON_2,\ P_3ON_3,\ P_4ON_4,\ P_5ON_5$ etc., we have

(i) $\dfrac{P_1N_1}{ON_1}=\dfrac{P_2N_2}{ON_2}=\dfrac{P_3N_3}{ON_3}=\dfrac{P_4N_4}{ON_4}=\dfrac{P_5N_5}{ON_5}=\ldots=5,$

(ii) $\angle P_1N_1O=\angle P_2N_2O=\angle P_3N_3O$
$$=\angle P_4N_4O=\angle P_5N_5O=\ldots=90^\circ;$$

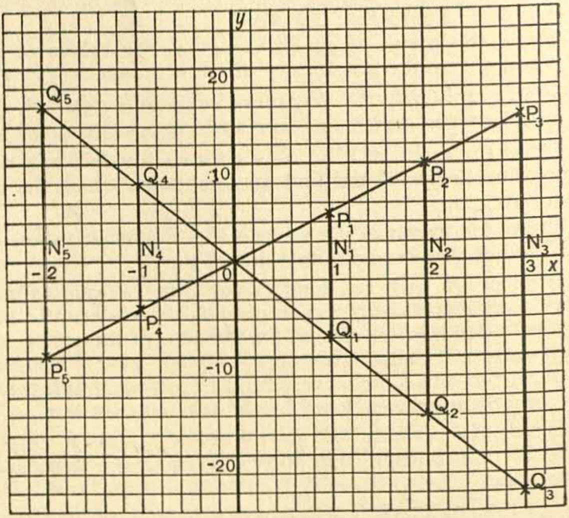


FIG. 11.

$\therefore$ these $\triangle$s are similar and, in particular,

$$\angle P_1ON_1 = \angle P_2ON_2 = \angle P_3ON_3 = \angle P_4ON_4 = \angle P_5ON_5 \text{ etc.}$$

It follows that P_1, P_2, P_3, P_4, P_5 etc. lie on a straight line which passes through O.

Similarly the graph of $y = -8x$ may be shown to be a straight line passing through O, Q_1, Q_2, Q_3, Q_4, Q_5 etc.

In general, the graph of $y = kx$, where k is any constant, positive or negative, represents a straight line passing through the origin.

Conversely, if P_1, P_2, P_3, P_4, P_5 ... represent points on any straight line through the origin, it is easily shown that

$$\frac{P_1N_1}{ON_1} = \frac{P_2N_2}{ON_2} = \frac{P_3N_3}{ON_3} = \frac{P_4N_4}{ON_4} = \frac{P_5N_5}{ON_5} = \dots \text{ etc.}$$

If, then, P_1 is a *fixed* point (x_1, y_1) on the line, and P is a variable point (x, y) on the line, we have $\frac{x}{y} = \frac{x_1}{y_1} =$ a constant, say k, i.e. the equation of the line is $\frac{x}{y} = k$, or $x = ky$.

Note 1. If k is positive, the line slopes upwards to the right; if k is negative, the line slopes upwards to the left.

Note 2. If a particular straight line is drawn, k may be found by taking the coordinates of any one point on the line, and finding the ratio $x : y$ at that point.

83. (b) Lines not through the origin. Consider the graphs of $y = 3x - 4$, $y = 3x$, $y = 3x + 2$. In Fig. 12 these are drawn on the same axes and with the same scales.

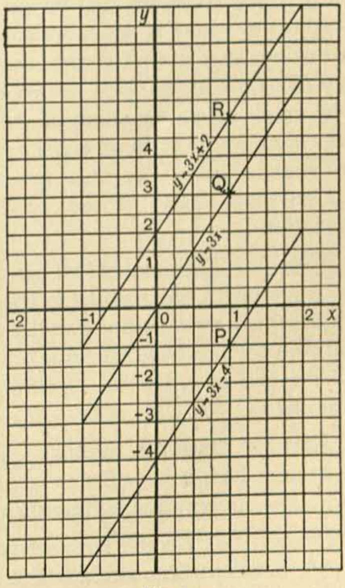


FIG. 12.

If P, Q, R are points on these graphs corresponding to the same value of x, it is clear that $PQ = 4$ units, $QR = 2$ units, and that this

is true for *any* positions of P, Q, R. It is clear that the graph of $y=3x-4$ may be obtained from the graph of $y=3x$ by diminishing each ordinate by 4 units. The line so obtained is parallel to the line $y=3x$, and is this line displaced a distance 4 units downwards. Similarly the graph of $y=3x+2$ may be obtained from the graph of $y=3x$ by increasing each ordinate by 2 units. The line so obtained is parallel to the line $y=3x$, and is this line displaced a distance 2 units upwards.

In general, the graph of $y=kx+l$ is the graph of $y=kx$ displaced upwards through a distance l units (or downwards through a distance $-l$ units). If k is kept constant, and different values of l are taken, we obtain a set of lines parallel to $y=kx$.

It is clear from the above that the graph of any expression of the form $y=kx+l$, where k and l are constants, is a straight line. Similarly it may be shown that the graph of any expression of the form $x=my+n$, where m and n are constants, is a straight line. But all equations of the first degree may be written either in the form $y=kx+l$, or in the form $x=my+n$. We therefore conclude that the graph of any equation of the first degree in x, y, i.e. of any equation of the form $ax+by+c=0$, where a, b, c are constants, is a straight line. For this reason an expression (or equation) of the first degree is sometimes called **a linear expression (or equation).**

Note. 1. The graph of all points which have the same abscissa, h, is a straight line parallel to the axis of y, i.e. the equation $x=h$ represents a straight line parallel to $x=0$. Similarly, the graph of all points which have the same ordinate, k, is a straight line parallel to the axis of x, i.e. the equation $y=k$ represents a straight line parallel to $y=0$.

Note 2. Since the graph of a linear expression (or equation) is a straight line, it is sufficient to plot two points on the graph and draw the straight line joining them; but it is wiser to plot a third point to serve as a check.

84. Gradient of a straight line. Take a fixed straight line and two points P, Q on it. Consider the value of the expression:

$$\frac{\text{The value of } y \text{ at the point } Q - \text{the value of } y \text{ at the point } P}{\text{The value of } x \text{ at the point } Q - \text{the value of } x \text{ at the point } P}.$$

This may conveniently be written $\dfrac{y_Q - y_P}{x_Q - x_P}$, and we shall denote it by G. In Fig. 13 it is equal to

$$\frac{QM - PN}{OM - ON} = \frac{6-3}{4-1} = \frac{3}{3} = 1.$$

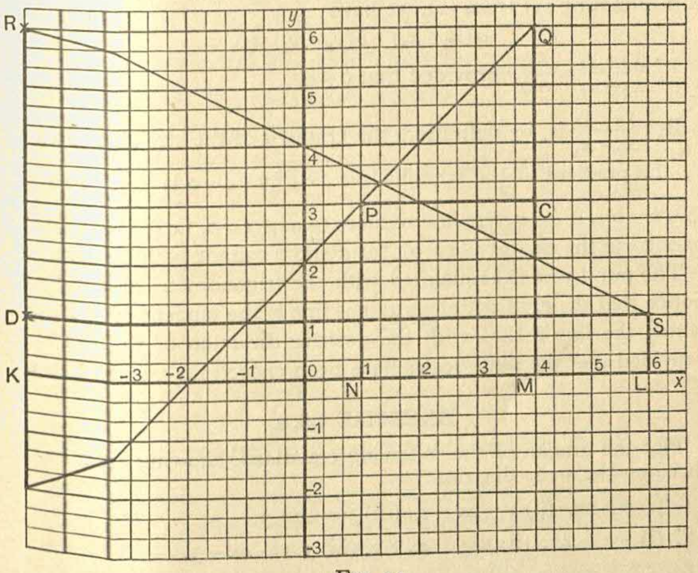


FIG. 13.

The pupil who is familiar with similar triangles will have no difficulty in proving that the expression G is the same for all positions of P and Q on the given line. For, $\dfrac{QM - PN}{OM - ON} = \dfrac{CQ}{PC}$, and all such triangles as PCQ are similar. The expression G is called **the gradient of the straight line PQ**. Since G is the same for all positions of P and Q, it follows that the gradient of a straight line is constant; it may be calculated by choosing any two convenient points on it and applying the definition.

Thus the gradient of the straight line SR

$$= \frac{y_R - y_S}{x_R - x_S} = \frac{RK - SL}{OK - OL} = \frac{6 - 1}{-4 - 6} = \frac{5}{-10} = -\frac{1}{2}.$$

Let us now consider the gradient of $ax+by+c=0$, where a, b, c are constants.

Let P, Q be two points (x_1, y_1), (x_2, y_2) respectively, on the line.

Since P lies on the graph, $ax_1+by_1+c=0$. ....(1)

Since Q lies on the graph, $ax_2+by_2+c=0$. ....(2)

Subtracting (1) from (2), we have

$$a(x_2-x_1)+b(y_2-y_1)=0,$$

$$\therefore \frac{y_2-y_1}{x_2-x_1}=-\frac{a}{b}, \text{ provided that } b\neq 0.$$

But $\dfrac{y_2-y_1}{x_2-x_1}$ is, by definition, the gradient of the straight line PQ,

$\therefore$ the gradient of the straight line $ax+by+c=0$ is $-\dfrac{a}{b}$, provided that $b\neq 0$. If $b=0$, the line reduces to $ax+c=0$, i.e. a straight line parallel to the axis of $y\,(x=0)$, for a and b cannot both be zero.

The pupil who is acquainted with the trigonometrical ratios will notice that the gradient of a straight line is the tangent of the angle which the line makes with the positive direction of the axis Ox, provided that the same scales have been chosen for x and for y.

EXERCISE 35. a

Plot the graphs of the following equations, showing each set of three on the same diagram.

1. (i) $y=2x$, (ii) $y=2x-1$, (iii) $y=2x+3$.

2. (i) $y=-4x$, (ii) $y=-4x+2$, (iii) $y=-4x-3$.

3. (i) $y+3x=0$, (ii) $y+3x=5$, (iii) $3y=x+2$.

4. (i) $3y-4x=0$, (ii) $3y-4x=7$, (iii) $4y+3x=5$.

5. (i) $y=8$, (ii) $y=-7$, (iii) $y=5{\cdot}5$.

6. (i) $x=-3$, (ii) $x=5$, (iii) $x+4{\cdot}5=0$.

Find the gradients of the following straight lines :

7. $3x-4y=11$. **8.** $2x+5y=8$. **9.** $y+4=0$.

10. $3x=5y$. **11.** $7y=a$. **12.** $mx+6y=n$.

Find the gradients of the straight lines joining the following pairs of points :

13. $(1, 2)$; $(7, 10)$. **14.** $(6, 3)$; $(4, 7)$.

15. $(-1, -3)$; $(4, 0)$. **16.** $(4, 6)$; $(-3, 6)$.

17. $(-3, -7)$; $(-5, -1)$. **18.** $(0, 3)$; $(5, 0)$.

EXERCISE 35. b

Plot the graphs of the following equations, showing each set of three on the same diagram.

1. (i) $y=-3x$, (ii) $y=-3x-4$, (iii) $y=-3x-5$.

2. (i) $y=5x$, (ii) $y=5x+3$, (iii) $y=5x-7$.

3. (i) $y+2x=0$, (ii) $y+2x+5=0$, (iii) $2y-x=3$.

4. (i) $2y-5x=0$, (ii) $5y+2x=8$, (iii) $2y+4=5x$.

5. (i) $x=2$, (ii) $x=-6$, (iii) $x+3{\cdot}3=0$.

6. (i) $y=-2$, (ii) $y=3{\cdot}8$, (iii) $y+4{\cdot}2=0$.

Find the gradients of the following straight lines :

7. $3x+4y=7$. **8.** $5x-6y=10$. **9.** $4x+5y=0$.

10. $3y-7=0$. **11.** $5y+b=0$. **12.** $ax-2y=c$.

Find the gradients of the straight lines joining the following pairs of points :

13. (4, 3) ; (8, 6). **14.** (10, 2) ; (5, 4).

15. (2, −3) ; (0, −3). **16.** (−5, 0) ; (6, −3).

17. (−2, −5) ; (−8, −1). **18.** (−7, −8) ; (−1, −11).

UNIFORM SPEED GRAPHS

85. If a train is travelling at a uniform speed u m.p.h., the distance, s miles, travelled in t hours is given by $s=ut$, i.e. $s=t\times$ some constant. It follows that the graph of $s=ut$ is a straight line whose gradient is u. In general, if $y=kx+l$ (k and l being constants), the gradient k represents the rate of increase of y compared with the rate of increase of x. Conversely, if the rate of increase of y compared with the rate of increase of x is constant, x and y are connected by an equation of the form $y=kx+l$ and the graph representing this equation is a straight line,

Again, we have previously seen that if (x_1, y_1), (x_2, y_2) are any two points on $ax+by+c=0$, then $a(x_2-x_1)+b(y_2-y_1)=0$. It follows that everywhere on the graph equal increases, A, in x cause equal increases, $-\dfrac{Aa}{b}$, in y. Also equal increases, B, in y cause equal increases, $-\dfrac{Bb}{a}$ in x. Conversely, it is easily seen that if at

all points of a graph equal increases in x cause equal increases in y, and vice-versa, then the graph is a straight line, and x and y are connected by a linear equation. This fact is of great importance; thus, in the following instances, equal increments in one letter cause equal increments in the other. The graph connecting the letters is therefore in each case a straight line.

(1) The extension (e in.) of a spring balance and the load (L lb.)

(2) A temperature $F°$ on the Fahrenheit scale expressed as $C°$ on the Centigrade scale.

(3) The marks obtained in an examination (x) and the same marks expressed on a different scale (y).

(4) A price of x shillings per lb. expressed as y francs per kilogram.

86. We now consider some problems which may be solved graphically by using the principles of this chapter.

Example 3. *A man starts to go from a place X to Y, 6 miles away, at 12 m.p.h.; after travelling 2 miles he stops for 5 minutes and then proceeds at a steady pace of 16 m.p.h. Another man starts from Y at the same time as the first leaves X, and travels to X at a uniform rate of 30 m.p.h. Where and when do they meet, and at what time will they be 1 mile apart?*

Take 0·5″ along XY to represent 1 mile, and 1″ along XZ to represent 10 minutes (see Fig. 14 on p. 165).

Each portion of the first man's journey is travelled at uniform speed and is therefore represented by a straight line. Similarly the second man's journey is represented by a straight line. The first man starts at X and goes 2 miles in 10 minutes, then reaching A; the graph of the first part of his journey is therefore XA. He remains for 5 minutes at the same spot, 2 miles from X. This wait is represented by AB. He then completes the journey in 15 minutes; the graph of this part of the journey is BC.

The second man completes the journey in 12 minutes; the graph of this journey is YD. The two graphs meet at K, which is

1·7 miles (approx.) from X, and they meet $8\frac{1}{2}$ minutes (approx.) after the start.

To find when they are 1 mile apart, draw through M, L (points 1 mile from Y on XY) lines parallel to the path of the second man, meeting the path of the first man at P and A. At these points the

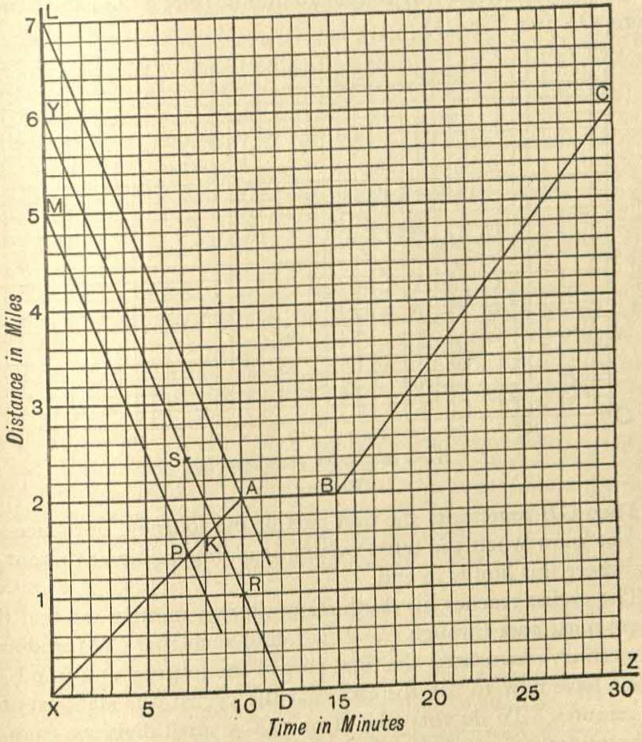


FIG. 14.

distance apart (PS, AR) is one mile, i.e. the men are 1 mile apart 7 minutes (approx.) and 10 minutes after the start.

Example 4. *A cyclist starts at* 10 *a.m. to ride to a place* 8 *miles away, riding at* 12 *m.p.h. till one of his tyres bursts. After spending* 10 *minutes in seeing whether he can repair it, he decides to walk the rest of the way, and walking at* 4 *m.p.h., he reaches his destination at* 11.15 *a.m. Determine by means of a graph how far he had ridden when the burst occurred.*

Let 5 small divisions represent 10 minutes along OX, and 2 miles along OY (see Fig. 15).

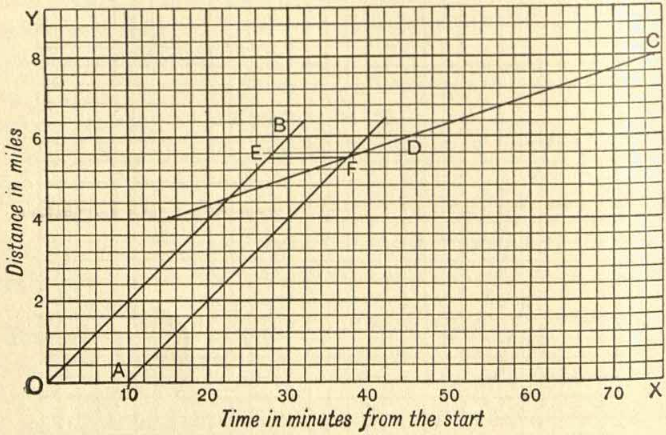


FIG. 15.

Then OB represents the first part of the journey, but since we do not know when the burst occurs, it is impossible at present to say where this line is to end.

The cyclist reaches his destination after 75 minutes, so that the graph must pass through C. D is found such that CD (produced if necessary) represents the final part of the journey at 4 m.p.h.

We have now to use the fact that the cyclist was stationary for 10 minutes. To do this take A on OX 5 small divisions from O (representing 10 minutes), and draw AF parallel to OB, meeting CD produced at F. Then FE drawn parallel to AO is equal to AO, i.e. it represents 10 minutes ; $\therefore$ the complete graph of the journey is $OEFC$.

It is easily seen from the graph that E, where the burst occurred, is $5\frac{1}{2}$ miles from the starting point.

EXERCISE 36. a

(*In this exercise, unless otherwise stated, all speeds are uniform*)

1. A goods train starts from A at 11.30 a.m., travels at 20 m.p.h., stops for 15 min. at B, 10 miles from A, and then proceeds at the former speed. A passenger train on a separate track leaves A at 12.10 p.m. and travels at 40 m..p.h., running through B without a stop. Where and when does the latter train pass the former?

2. A starts cycling at 10 a.m. at the rate of 12 m.p.h. and rests for 10 min. after each hour's cycling. B, starting at 11 a.m., cycles steadily at 15 m.p.h. When and where does B overtake A?

3. A man travels from X to Y at 24 m.p.h. amd returns at 18 m.p.h. If he takes 5 hr. 15 min. in all, find graphically the distance from X to Y.

4. P and Q walk towards one another from two places 18 miles apart; P walks at the rate of $4\frac{1}{4}$ m.p.h. and Q at $3\frac{3}{4}$ m.p.h. Find graphically when they will meet and when they will be 1 mile apart.

5. In a motor-car race, car B gives car A 10 min. start, travels at 90 m.p.h. and overtakes A 70 miles from the start. If A's time for the whole course is 81 min., find graphically (*a*) the length of the course, (*b*) the number of minutes by which B won, (*c*) the distance of A from the finishing post at the instant B finished, (*d*) the speed of A.

6. Water freezes at 0° Centigrade and 32° Fahrenheit; it boils at 100° Centigrade and 212° Fahrenheit. If F° Fahrenheit is the same temperature as C° Centigrade, draw a graph to show the connection between F and C for all values of C from 0 to 100. From the graph read off (i) the reading on the centigrade scale corresponding to 59° F., (ii) the reading on the Fahrenheit scale corresponding to 45° C.

7. X and Y are two towns 45 miles apart; two cyclists start at 9 a.m., one from each town, and ride towards one another; they pass at a point 25 miles from X, and the first cyclist reaches Y 1 hr. $7\frac{1}{2}$ min. before the second reaches X; find the rate at which each rides and the time at which they pass one another.

8. Alternative rates of payment for electricity are allowed to a man as follows: (*a*) an initial charge of £4 and a further charge of $\frac{1}{2}$d. for each unit consumed, (*b*) a charge of 6d. each for the first 60 units, 3d. each for the next 200 units, and $1\frac{1}{2}$d. for each subsequent unit consumed. In one diagram and with the same scales draw graphs showing the two rates and deduce (to the nearest 5 units) the largest number of units the man can pay for more economically at rate (*b*) than at rate (*a*).

9. A man sets out to walk to a certain town and takes 5 hours to walk there and back at a uniform speed. A second man sets out 1 hr. 50 min. later, cycles to the town at three times the pace, stays there for 10 min. and cycles back at his former speed, passing the first man 2 miles from the town. How far away is the town?

Draw *rough graphs* to illustrate the following, Nos. 10–12:

10. The fare for a journey by taxi-cab: charge, up to 1 mile, 1s. 6d.; for each additional $\frac{1}{4}$ mile (or part of it), 3d. more.

11. The inland postage for parcels of various weights: charge, up to 3 lb., 6d.; 3-4 lb., 7d.; 4-5 lb., 8d.; 5-6 lb., 9d.; 6-7 lb., 10d.; 7-8 lb., 11d.; 8-15 lb., 1s.

12. A man's daily earnings: rate of payment, at the rate of 2s. an hour up to 7 hours; at the rate of 3s. an hour for any time worked in excess of 7 hours, such overtime not to exceed 3 hours in any one day.

13. The marks of a Form range from 27 to 67. They are scaled so as to range from 0 to 100. Find graphically the new mark corresponding to 43. If the new mark is 85, what was the old mark?

14. A man starts from A at 10 a.m. to walk to B (12 miles off) at 4 m.p.h. After 45 min. he meets a motor-bus coming from B to A. When the 'bus reaches A, it waits 10 min. and then returns to B. The 'bus travels at 10 m.p.h. Draw graphs showing the positions of the man and the 'bus at different times. Find from your graph when and where the man is caught up by the returning 'bus.

EXERCISE 36. b

(*In this exercise, unless otherwise stated, all speeds are uniform*)

1. A cyclist, who rides at a steady 8 m.p.h., leaves his house at 9 a.m. A man in a car, which does 24 m.p.h., leaves the house by the same road at 9.30 a.m.; he goes to a town 30 miles off, stops there 10 min. and starts back. At what time and how far from the town will he pass the cyclist the second time?

2. Two trains start from X and go to Y. The first starts at 8 a.m. and travels at 40 m.p.h., while the second starts at 8.15 a.m. and travels at 50 m.p.h. If they arrive at Y simultaneously, find graphically the distance XY.

3. P starts from a certain place and walks to another place 14 miles away in $3\frac{1}{2}$ hours, while Q, starting 30 minutes later, arrives 12 minutes sooner. Find graphically (i) where Q passes P, (ii) what Q's rate is, (iii) when they are $\frac{1}{2}$ mile apart.

4. A man travels from A to B at the rate of 30 m.p.h., and 20 min, after his start a second man starts from A to B at 36 m.p.h. and reaches his destination 8 minutes after the first man. Find graphically the distance from A to B.

5. The salary of a clerk is increased each year by a fixed sum until a maximum salary of £350 per annum is reached. After 8 years' service his salary is raised to £204, and after 12 years' service to £254. Draw a graph from which his salary may be read off for any year, and determine from it (i) his initial salary, (ii) the salary he should receive for his 19th year of service.

6. Draw a graph to convert "miles per hour" into "feet per second" for speeds up to 45 m.p.h. Read off speeds of 5 and 40 m.p.h. as ft. per sec. (to the nearest integer); also speeds of 20 and 49 ft. per sec. as m.p.h. (to the nearest integer).

7. At noon A starts to cycle at 18 km. per hour along a certain road. At the same time B starts to walk along it from the same point and in the same direction as A at 7 km. per hour. After A has ridden for 45 min. he waits for 15 min. and then leaves his cycle and walks on at 5 km. per hour. When B arrives at the place where A left his cycle he mounts it and rides after A, overtaking him at 2.15 p.m. Find graphically (*a*) the interval between the time at which A leaves the cycle and that at which B finds it; (*b*) how far A is ahead at the latter instant; (*c*) how fast B rides.

8. The marks of a Form range from 7 to 93. They are scaled so as to range from 0 to 200. Find graphically the new mark corresponding to 50. If the new mark is 151, what was the old mark?

9. A person cycles at the rate of 12 m.p.h., taking a rest of 15 minutes at the end of each hour. A second man starting from the same place $1\frac{1}{2}$ hours later in a motor-car goes by the same road as the first and catches him up after 30 miles. At what rate does the motor-car travel?

Draw *rough graphs* to illustrate the following, Nos. 10–12:

10. The inland postage for letters of various weights: charge, up to 2 oz., $1\frac{1}{2}$d.; for each additional 2 oz. (or part of it), $\frac{1}{2}$d. more.

11. A travel-graph; a boy walks for 15 minutes, runs for 3 minutes, rests for 2 minutes, and then rides by motor-car for 5 minutes.

12. The cost of sending a telegram: charge, up to 9 words 6d.; for each additional word, 1d. more.

13. A boy sets out from Exeter to walk towards Launceston, from which place a car is being sent to meet him. The car starts 40 min. after he leaves Exeter, and travels at 24 m.p.h. He walks at 3·5 m.p.h. and rests for 5 min. after each hour's walking. The car meets him when he is just about to rest for the second time. Show graphically the position of the boy and the car at any time and deduce the distance from Exeter to Launceston.

9. A man sets out to walk to a certain town and takes 5 hours to walk there and back at a uniform speed. A second man sets out 1 hr. 50 min. later, cycles to the town at three times the pace, stays there for 10 min. and cycles back at his former speed, passing the first man 2 miles from the town. How far away is the town?

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EXERCISE 36. b

(*In this exercise, unless otherwise stated, all speeds are uniform*)

1. A cyclist, who rides at a steady 8 m.p.h., leaves his house at 9 a.m. A man in a car, which does 24 m.p.h., leaves the house by the same road at 9.30 a.m. ; he goes to a town 30 miles off, stops there 10 min. and starts back. At what time and how far from the town will he pass the cyclist the second time?

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3. P starts from a certain place and walks to another place 14 miles away in $3\frac{1}{2}$ hours, while Q, starting 30 minutes later, arrives 12 minutes sooner. Find graphically (i) where Q passes P, (ii) what Q's rate is, (iii) when they are $\frac{1}{2}$ mile apart.

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6. Draw a graph to convert "miles per hour" into "feet per second" for speeds up to 45 m.p.h. Read off speeds of 5 and 40 m.p.h. as ft. per sec. (to the nearest integer); also speeds of 20 and 49 ft. per sec. as m.p.h. (to the nearest integer).

7. At noon A starts to cycle at 18 km. per hour along a certain road. At the same time B starts to walk along it from the same point and in the same direction as A at 7 km. per hour. After A has ridden for 45 min. he waits for 15 min. and then leaves his cycle and walks on at 5 km. per hour. When B arrives at the place where A left his cycle he mounts it and rides after A, overtaking him at 2.15 p.m. Find graphically (*a*) the interval between the time at which A leaves the cycle and that at which B finds it; (*b*) how far A is ahead at the latter instant; (*c*) how fast B rides.

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CHAPTER XV

PRODUCTS AND QUOTIENTS

87. In Chapter XI it was shown that

$$x(y+z)\equiv xy+xz, \quad a(p-q+r)\equiv ap-aq+ar;$$

i.e. to multiply a polynomial by a monomial, each term in the polynomial must be multiplied by the monomial. Thus,

$$3x(4x^2-2x+3)=(4x^2\times 3x)-(2x\times 3x)+(3\times 3x)$$
$$=12x^3-6x^2+9x.$$

88. Multiplication by a binomial.

Example 1. *Multiply* $3x-2$ *by* $5x-4$.

We have to multiply $3x-2$ by $5x$ and by -4 and then add together the results. Thus,

$$(3x-2)(5x-4)=(3x-2)(5x)+(3x-2)(-4).$$

[At this stage, the brackets containing the monomial may be omitted as soon as the pupil has mastered the work. Thus it is more usual to write

$$(3x-2)(5x-4)=5x(3x-2)-4(3x-2)]$$
$$=15x^2-10x-12x+8=15x^2-22x+8.$$

Example 2. *Expand the expression* $(x-3)(3x^2-x-2)$, *i.e. find the product of the expressions contained in the brackets.*

We may regard the expression either as $x-3$ multiplied by $3x^2-x-2$, or as $3x^2-x-2$ multiplied by $x-3$; it is more usual to multiply by the expression containing fewer terms. Thus,

$$(3x^2-x-2)(x-3)=x(3x^2-x-2)-3(3x^2-x-2)$$
$$=3x^3-x^2-2x-9x^2+3x+6=3x^3-10x^2+x+6.$$

Note. Any two polynomials may be multiplied together in this way.

EXERCISE 37. a

Multiply:

1. $4x^3-3x^2-7x+1$ by $-2x$.

2. $2t^2-5t-3$ by -2.

3. $3a^2-5ab-2b^2$ by ab.

4. $2x^3-3x^2y-5xy^2+3y^3$ by $-3xy$.

5. $x-8$ by $5x+4$.

6. $5t+2$ by $5t-2$.

7. $2x-6$ by $5x-9$.

8. $x+y$ by $3x-y$.

9. $a-3b$ by $3a+2b$.

10. $2-3u$ by $3-u$.

11. $a+bc$ by $a-bc$.
12. a^2+2b^2 by $2a^2-5b^2$.
13. $2x^2-3y^2$ by $3x^2+4y^2$.

EXERCISE 37. b

Multiply :

1. $4x^2-3x-5$ by $2x$.
2. $3l^2-7l+1$ by -5.
3. $2c^2-cd+3d^2$ by $-cd$.
4. $3c^3-2c^2d+4cd^2-d^3$ by $-2cd$.
5. $5x-3$ by $3x+2$.
6. $x-7$ by $3x-4$.
7. $2t-7$ by $2t+7$.
8. $a+2b$ by $5a-3b$.
9. $2x-3y$ by $x-y$.
10. $5-u$ by $3-2u$.
11. $x-2yz$ by $x+2yz$.
12. $2x^2-y^2$ by $5x^2-3y^2$.
13. a^2-3bc by $3a^2+2bc$.

EXERCISE 37. c

Expand and simplify :

1. $(x^2-3x+1)(x-4)$.
2. $(c^2-3c+9)(c+3)$.
3. $(x^2-2xy+y^2)(x-y)$.
4. $(6c^2-5cd+2d^2)(3c-2d)$.
5. $(3s^2-7st+4t^2)(3s-5t)$.
6. $(1-3x)(4+7x-2x^2)$.
7. $(a^2-3a+5)(2a^2-a+2)$.
8. $(5x^2-6x+3)(3x^2-5)$.
9. $(1-8t^2)(2t^3-3t^2+4t-3)$.
10. $(7x^2-3x+2)(3x^2-2x-5)$.
11. $(2x-3)(x+2)+5(x^2-2)(x+3)$.
12. $(2a-b)(2a+3b)-(a+b)(a-4b)$.
13. $(2x+3)(x-1)+(x-3)(3x-4)-2(x-3)(x+1)$.
14. $(x^2-x+1)(x+1)-(1+y+y^2)(1-y)$.

As soon as the pupil has had sufficient practice to master the general principle, he should learn to do the work mentally.

Example 3. *Expand* $(4x-3)(2x+5)$.

It is clear that the highest term, i.e. the x^2 term, is obtained by multiplying $4x$ by $2x$ and is $8x^2$, the x term by multiplying -3 by $2x$ and $4x$ by 5 and then adding the results ; it is $-6x+20x$, i.e. $14x$.

The constant term (i.e. the term which does not contain x) is obtained by multiplying -3 by $+5$ and is -15 ;

$$\therefore\ (4x-3)(2x+5)=8x^2+14x-15.$$

It is essential that the pupil should have plenty of oral drill in the multiplication of binomials. The working of the above example may be read out as follows :

$(4x-3)(2x+5)$ equals $8x^2$, $-6x+20x$, that is $+14x$, -15.

Example 4. *Expand* $(2+3x+4x^2)(2-x-3x^2)$.

The constant term is 2×2, i.e. 4.

The x term is $3x \times 2 + 2 \times (-x)$, i.e. $6x - 2x = 4x$.

The x^2 term is

$$4x^2 \times 2 + 3x \times (-x) + 2 \times (-3x^2), \text{ i.e. } 8x^2 - 3x^2 - 6x^2 = -x^2.$$

The x^3 term is

$$4x^2 \times (-x) + 3x \times (-3x^2), \text{ i.e. } -4x^3 - 9x^3 = -13x^3.$$

The x^4 term is $4x^2 \times (-3x^2) = -12x^4$.

Thus the expansion is $4 + 4x - x^2 - 13x^3 - 12x^4$.

It is important for the pupil to learn to pick out a particular term, e.g. the x^3 term, without working out the whole expansion.

When long expressions have to be multiplied together the working may be arranged as in Arithmetic.

Example 5. *Multiply* $5x^3 - 4 + 3x^2$ *by* $3x^2 - 2x + 5$.

$$\begin{array}{rrrrrr}
5x^3 + & 3x^2 + & 0x & - 4 & & \\
3x^2 - & 2x + & 5 & & & \\
\hline
15x^5 + & 9x^4 + & 0x^3 & - 12x^2 & & \\
& -10x^4 - & 6x^3 + & 0x^2 + & 8x & \\
& & 25x^3 + & 15x^2 + & 0x & - 20 \\
\hline
15x^5 - & x^4 + & 19x^3 + & 3x^2 + & 8x & - 20
\end{array}$$

The steps in the working are as follows:

(1) Arrange each of the expressions to be multiplied together so that the powers are in ascending (or descending) order. If any intermediate power does not occur, insert a term with coefficient zero, or leave a gap.

(2) Form the partial products by multiplying the upper expression by the different terms of the lower. Write each of these partial products on a separate line, and place like terms in a vertical column. Then add the partial products.

89. Multiplication is sometimes necessary in order to reduce an equation to its simplest form.

Example 6. *Solve*

$$3(x+1)(x+3) - 2(x+1)(x-1) = (x-1)^2 + 3(5x+1).$$

Expanding each product, we have

$$3(x^2+4x+3) - 2(x^2-1) = (x^2-2x+1) + 3(5x+1),$$

$$\therefore\ 3x^2 + 12x + 9 - 2x^2 + 2 = x^2 - 2x + 1 + 15x + 3,$$

$\therefore\ 3x^2 - 2x^2 - x^2 + 12x + 2x - 15x = -9 - 2 + 1 + 3,$

$$\therefore\ -x = -7, \quad \therefore\ x = 7.$$

The check is left to the pupil.

EXERCISE 38. a (*Oral*)

Expand :

1. $(x+2)(x+3)$. **2.** $(t-3)(t+7)$. **3.** $(t+5)(t-2)$.
4. $(c-7)(c+3)$. **5.** $(l+2)(l-5)$. **6.** $(d-3)(d-7)$.
7. $(b-11)(b+5)$. **8.** $(k-8)(m+9)$. **9.** $(t-5)(w+5)$.
10. $(n-5)(n-5)$. **11.** $(3x+5)(2y-3)$. **12.** $(3x-5)(2x-3)$.
13. $(3x-5)(2x+3)$. **14.** $(4c+5d)(3c-2d)$.
15. $(7s-5t)(5s+2t)$. **16.** $(2l-9m)(3l-2m)$.
17. $(7s-5t)(5s-2t)$. **18.** $(4c-5d)(3c-2d)$.
19. $(7s+5t)(5s-2t)$. **20.** $(2l+9m)(3l-2m)$.
21. $(2l-9m)(3l+2m)$. **22.** $(7s+5t)(5s+2t)$.
23. $(4c-5d)(3c+2d)$. **24.** $(3a^2-2b^2)(2a^2+3b^2)$.
25. $(2a^2-5bc)(a^2-bc)$. **26.** $(3a^2+2b^2)(2a^2-3b^2)$.
27. $(2a^2+5bc)(2a^2-3bc)$. **28.** $(2a^2-5bc)(3a^2+2bc)$.
29. $(2a^2+5bc)(-a^2+5bc)$. **30.** $(3a^2-2b^2)(2a^2-3b^2)$.

EXERCISE 38. b (*Oral*)

Expand :

1. $(x+3)(x+7)$. **2.** $(x-2)(x+5)$. **3.** $(a+6)(a-3)$.
4. $(y+2)(y-5)$. **5.** $(t-6)(t+3)$. **6.** $(x-2)(x-3)$.
7. $(u+4)(u-4)$. **8.** $(a+10)(b-3)$. **9.** $(m-3)(m-3)$.
10. $(y+8)(z-9)$. **11.** $(2a+5)(3a-1)$. **12.** $(2c-5)(3d-1)$.
13. $(2a-5)(3a+1)$. **14.** $(9a+2b)(a-4b)$. **15.** $(7x-6y)(x+y)$.
16. $(6x-5y)(2x+3y)$. **17.** $(9a-2b)(a+4b)$.
18. $(6x+5y)(2x-3y)$. **19.** $(9a+2b)(a+4b)$.
20. $(7x+6y)(x-3y)$. **21.** $(9a-2b)(a-4b)$.
22. $(6x-5y)(2x-3y)$. **23.** $(7x-6y)(3x-2y)$.
24. $(4x^2-5y^2)(2x^2-3y^2)$. **25.** $(3x-8yz)(-2x+3yz)$.
26. $(3x+8yz)(3x-yz)$. **27.** $(4x^2+5y^2)(2x^2-3y^2)$.
28. $(3x-8yz)(-2x-3yz)$. **29.** $(4x^2-5y^2)(2x^2+3y^2)$.
30. $(3x+8yz)(5x+2yz)$.

EXERCISE 38. c

(*Some of these may be taken orally*)

Find the coefficient of :

1. x^2 in $(4-x-3x^2)(5+3x-2x^2)$.

2. x^3 in $(5-3x-4x^2)(2+5x+3x^2)$.

3. x in $(3+2x-x^2-3x^3)(1+2x)$.

4. x^2 in $(x^2+2x+1)(x^2+2x+1)$.

5. x^3 in $(1-3x+3x^2-x^3)(1-2x+x^2)$.

6. a^3b in $(2a^2-3ab+4b^2)(-5a^2+3ab+4b^2)$.

7. t^3 in $(1-9t^2+20t^3)(1-3t)$.

8. n in $(l^2+3-n)(l^2-3+n)$.

9. ab^3 in $(a^2-2ab+b^2)(a^2+2ab+b^2)$.

10. x^2 in $(2x^2-3ax+a^2)(3ax-2a^2-x^2)$.

Expand :

11. $(3c^4-4c^2-1)(3c^4+4c^2+1)$.

12. $(1-3k+3k^2-k^3)(1-2k+k^2)$.

13. $(y-5+3y^2)(7-y^2-3y)$.

14. $(x^4+3x^3+7x^2-9x-5)(3x^3-x-4)$.

15. $(2-3x-2x^3)^2$.

16. $(x^2+2x-7)(3x^2-x-5)(2x-3)$.

Solve the equations :

17. $(2x+1)(2x+2)=2x(2x+7)-6$.

18. $15+x(8+x)=(x+5)^2$.

19. $2(x-3)(x+3)=2x(x-9)+144$.

20. $(2x-3)(2x-4)-4x(2x-3)=2x(11-2x)$.

21. $2(3x-8)(2x+1)-(7x+4)(x-2)=(5x+6)(x+4)-4$.

22. $(15x+1)(3x-2)-3(4x-1)(9x-1)=10-(21x+2)(3x+1)$.

23. $(3x+1)^2+(3x-2)^2=18x^2-5$.

24. $(x+5)^2-4(3+x)=(2-x)^2-8x$.

TWO IMPORTANT EXPANSIONS

90. By multiplication in the usual way we obtain :

$$(\mathbf{a}+\mathbf{b})^2=(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b})=\mathbf{a}^2+\mathbf{ab}+\mathbf{ab}+\mathbf{b}^2=\mathbf{a}^2+\mathbf{2ab}+\mathbf{b}^2,$$
$$(\mathbf{a}-\mathbf{b})^2=(\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{ab}-\mathbf{ab}+\mathbf{b}^2=\mathbf{a}^2-\mathbf{2ab}+\mathbf{b}^2.$$

It is so frequently necessary to square a binomial expansion that the above results should be committed to memory. They may be expressed in words as follows :

The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

The square of the difference of two quantities is equal to the sum of their squares minus twice their product.

Example 7. *Write down the squares of* $(5x+3y)$ *and* $(7a-2b)$.

$$(5x+3y)^2 = (5x)^2 + (3y)^2 + 2(5x)(3y)$$
$$= 25x^2 + 9y^2 + 30xy = 25x^2 + 30xy + 9y^2.$$
$$(7a-2b)^2 = (7a)^2 + (2b)^2 - 2(7a)(2b)$$
$$= 49a^2 + 4b^2 - 28ab = 49a^2 - 28ab + 4b^2.$$

After a little practice the result may be written down in one step. These results also enable us to find mentally the squares of numbers which occur in Arithmetic.

Example 8. *Evaluate* 102^2.

$$(102)^2 = (100+2)^2 = 100^2 + 2^2 + 2 \,.\, 100 \,.\, 2 = 10000 + 4 + 400 = 10404.$$

It should be particularly noted that $(a+b)^2$ is not equal to a^2+b^2, that $(a-b)^2$ is not equal to a^2+b^2, that $\sqrt{a^2+b^2}$ is not equal to $a+b$ or $a-b$, that $\sqrt{a^2-b^2}$ is not equal to $a+b$ or $a-b$.

EXERCISE 39. a (*Mainly oral*)

Write down the squares of :

1. $a+x$.	**2.** $2+x$.	**3.** $c+3$.	**4.** $2x+3$.
5. $x-3$.	**6.** $b-c$.	**7.** $6-x$.	**8.** $3l-1$.
9. $4x-3$.	**10.** $2c-3d$.	**11.** $3a+5b$.	**12.** $3x-7y$.
13. $9l+4m$.	**14.** $3x-10y$.	**15.** $2x^2+5y^2$.	**16.** $5a^2-2a$.
17. $7-3c^2$.	**18.** $3c^2+d$.	**19.** a^2-2b^2.	**20.** $-3x^2+2y^2$.
21. $2a-5bc$.	**22.** $2a^3+3b^3$.	**23.** $4x^3-5$.	**24.** $6-5x^4$.
25. $3a^2b-7c^3$.	**26.** $-9a^2-4a^3$.	**27.** $-5c-6c^4$.	**28.** $5c-12d^2$.

Without doing the actual multiplication, evaluate :

29. 202^2.	**30.** 410^2.	**31.** 199^2.	**32.** $100{\cdot}5^2$.
33. $99{\cdot}5^2$.	**34.** 1005^2.	**35.** $50{\cdot}2^2$.	**36.** $49{\cdot}9^2$.

EXERCISE 39. b (*Mainly oral*)

Write down the squares of :

1. $b+h$.	**2.** $5+x$.	**3.** $t+7$.	**4.** $3t+1$.
5. $4-a$.	**6.** $8-t$.	**7.** $s-r$.	**8.** $2a-3$.
9. $5x-2$.	**10.** $x-4y$.	**11.** $3a-5b$.	**12.** $2x+7y$.
13. $8l-5m$.	**14.** $4x+3y$.	**15.** $2x^2-y^2$.	**16.** $3x^2-x$.
17. $2-9x^2$.	**18.** $5c-d^2$.	**19.** r^2-s^2.	**20.** $5a+2bc$.
21. c^2-3cd.	**22.** $5n^3-7$.	**23.** x^3+2z^3.	**24.** $2x^3-5y^3$.
25. $2rs^2-9t^3$.	**26.** $3-7a^4$.	**27.** $5l-7m^2$.	**28.** $-2x^2-5x^3$.

Without doing the actual multiplication, evaluate :

29. 305^2.	**30.** 203^2.	**31.** 99^2.	**32.** $20{\cdot}5^2$.
33. $19{\cdot}5^2$.	**34.** 1001^2.	**35.** $40{\cdot}6^2$.	**36.** $39{\cdot}8^2$.

91. Division. We have already seen in Ch. XI that to divide a polynomial by a monomial, each term of the polynomial must be divided by the monomial. Thus, $(12t^3 - 8t^2 - 4t) \div 2t$

$$= (12t^3 \div 2t) - (8t^2 \div 2t) - (4t \div 2t) = 6t^2 - 4t - 2.$$

[The remainder of this chapter may be postponed, if desired. Knowledge of it is not presumed in Chs. XVI to XXI.]

92. If we wish to divide by a polynomial, the work is arranged as in Arithmetic. The method is shown in Example 9.

Example 9. *Divide* $20x^4 + 16 - 43x^2 - 7x^3$ *by* $5x^2 - 3 + 2x$.

$$\begin{array}{lrrrrrl}
 & 5x^2 + 2x - 3\ ) & 20x^4 - & 7x^3 - 43x^2 & + 0x & + 16 & (\ 4x^2 - 3x - 5 \\
\dagger & & \underline{20x^4 +} & \underline{8x^3 - 12x^2} & & & \\
 & & & -15x^3 - 31x^2 & + 0x & & \\
\dagger & & & \underline{-15x^3 - 6x^2} & \underline{+ 9x} & & \\
 & & & -25x^2 & - 9x & + 16 & \\
\dagger & & & \underline{-25x^2} & \underline{- 10x} & \underline{+ 15} & \\
\dagger & & & & x & + 1 &
\end{array}$$

Thus the quotient is $4x^2 - 3x - 5$ and the remainder is $x + 1$. We do not continue the division, for the result of dividing x by $5x^2$ is a fraction.

The steps in the working are as follows :

(1) Arrange each of the expressions in descending (or ascending) order ; if any intermediate power does not occur, insert a term with coefficient zero, or leave a gap.

(2) Divide the first term of the dividend by the first term in the divisor, i.e. $20x^4$ by $5x^2$; write the result, $4x^2$, in the quotient. Multiply the divisor by this result and subtract the product from the dividend. As in Arithmetic, it is unnecessary to bring down the whole dividend ; only the term (or terms) immediately required need be brought down.

(3) Repeat the process until it is no longer possible to divide without bringing in fractions.

It may easily be verified that the sum of the expressions marked † is the dividend. But the first expression is $4x^2$ times the divisor, the second expression is $-3x$ times the divisor, the third expression is -5 times the divisor.

Therefore, when we divide the sum of the four expressions by the divisor, the first three divisions are exact and the quotient is the

sum of the partial quotients, i.e. $4x^2 - 3x - 5$. The fourth expression is the remainder; if the remainder is zero, the division is exact.

EXERCISE 40. a

Divide:

1. $7a^2 - 9ab$ by $-a$.
2. $18c^3d - 27c^2d^2$ by $-9cd$.
3. $14x^4 - 4x^2$ by $-2x$.
4. $9l^4 - 6l^2m^2$ by $3l^2$.
5. $t^2 + 13t + 42$ by $t + 6$.
6. $a^2 - 15a + 36$ by $a - 3$.
7. $x^2 + 8x - 65$ by $x - 5$.
8. $2k^2 + 9k + 4$ by $2k + 1$.
9. $3a^2 + 8a + 6$ by $3a + 2$.
10. $4x^2 - 4x - 5$ by $2x - 3$.
11. $6t^2 - 7t + 5$ by $3t - 2$.
12. $5x^2 - 17x + 6$ by $5x - 2$.
13. $-21c^2 + c + 20$ by $3c + 2$.
14. $25a^2 - 4b^2$ by $5a + 2b$.
15. $3t^2 - 13t + t^3 - 15$ by $t - 3$.
16. $x^3 - x^2 - 41x + 100$ by $x - 5$.

EXERCISE 40. b

Divide:

1. $7a - 21b$ by -7.
2. $8c^2d - 6cd^2$ by $-2cd$.
3. $py + qy - ry$ by $-y$.
4. $-15x^4 + 9x^3$ by $3x^2$.
5. $d^2 + 4d + 3$ by $d + 1$.
6. $c^2 - 13c + 36$ by $c - 9$.
7. $t^2 - 10t - 39$ by $t + 3$.
8. $3x^2 - x - 2$ by $x - 1$.
9. $5x^2 + 16x - 3$ by $x + 3$.
10. $12a^2 - 17a + 6$ by $4a - 3$.
11. $15t^2 + 14t - 16$ by $3t - 2$.
12. $6x^2 + 13x - 6$ by $3x + 2$.
13. $-21c^2 + 58c - 21$ by $-7c + 3$.
14. $-4a^2 + 49$ by $2a - 7$.
15. $20 - x^3 - 26x + 9x^2$ by $4 - x$.
16. $9c^2 + 27c^3 - 3c + 10$ by $3c - 2$.

EXERCISE 40. c

Divide:

1. $2x^3 - 7x^2 - x + 2$ by $x^2 - 3x - 2$.
2. $1 - 6x + 11x^2 - 6x^3$ by $1 - 3x + 2x^2$.
3. $21a^3 + 26a^2 - 27a - 20$ by $7a^2 - 3a - 4$.
4. $x^4 - 4x^3 - 18x^2 - 11x + 2$ by $x^2 - 7x + 1$.
5. $7x + 12x^4 - 8x^2 - 2 + x^3$ by $3x^2 + 1 - 2x$.
6. $3z^5 - 3z^4 + 2z^3 - 1$ by $3z^3 - z - 1$.
7. $t^5 + 2t^4 - 4t^3 - 19t^2$ by $t^3 - 7t - 5$.
8. $30a - 71a^3 + 1 - 35a^2 + 28a^4$ by $4a^2 + 6 - 13a$.
9. $6x^4 - 11x^2 + 13x + x^3 - 2$ by $2x^2 - 3x + 2$.
10. $5c^4 - 28c^2 - 20c + 25 + 2c^3$ by $c^2 - 5$.
11. $6x^4 - 21x^2 + 9x + 7x^3 - 3$ by $2x^2 - 1 + 5x$.
12. $16x^4 - 2x^3 - 39x^2 - 15x - 8$ by $2x^2 - x - 5$.
13. $12a^4 - 27a^2b^2 - 16a^3b - 15b^4 + 46ab^3$ by $3b^2 - 5ab + 2a^2$.
14. $4x^4 - 2x^3 + 2x^2 - x + 5$ by $2x^2 - 3x + 4$.
15. $27a^4 - 39a^2b^2 + 21a^3b - 25ab^3 + 3b^4$ by $9a^2 + 7ab - b^2$.

CHAPTER XVI

EASY FACTORS

93. In the previous chapter we have shown how to find the product of two given algebraical expressions. We shall now consider the inverse operation: given an integral algebraical expression, to find two or more integral expressions which are such that their product is the given expression. Such integral expressions are called **factors** of the given expression. When the factors have been found the given expression is said to be **resolved** into factors, and the process of finding factors is called **resolution into factors.**

Resolution into factors is an inverse operation, and it differs from the direct operation of multiplication in two ways:

(1) Any two expressions can be multiplied together and their product found, but, in general, an algebraical expression written down at random has no simple factors, e.g. $3x+4y$ cannot be expressed as the product of two simple expressions.

(2) There is no general method of factorisation. In multiplication we have a definite process which always gives us the product, but in order to factorise a given expression we have to learn a number of special devices. **It should be noted that when one factor of an expression or number is known, the other factor can be obtained by division.**

There are many important types of expression which can be factorised, and we now proceed to the consideration of these.

94. Type I. Expressions in which each term has a common monomial factor.

Example 1. *Resolve* $14x^3-21x^2y+49xy^2$ *into factors.*

The H.C.F. of the terms $14x^3$, $-21x^2y$ and $49xy^2$ is $7x$.

$7x$ is therefore one factor, and the other is obtained by dividing $7x$ into the whole expression. We thus obtain the result

$$14x^3-21x^2y+49xy^2=7x(2x^2-3xy+7y^2).$$

The principle of factors may be used to simplify arithmetical calculations.

Example 2. *Evaluate* $575 \times 24 - 75 \times 24$.

It is clear that 24 is a factor of each term of the given expression, which equals

$$24(575 - 75) = 24 \times 500 = 12{,}000.$$

EXERCISE 41. a

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $9a + 3b$. **2.** $10c^2 - 12c$. **3.** $z^2 - z$.
4. $22 - 63t$. **5.** $3c^2 + 3cd$. **6.** $x^2 + xy$.
7. $x^2 - 2x^3$. **8.** $6x^2 + 18xy$. **9.** $x^4 + 2x^3$.
10. $5c^2 - 20c^2d$. **11.** $12a - 36a^2b$. **12.** $57c^2d^2 - 17$.
13. $3a - 10b - 20c$. **14.** $2x - 4y + 2z$. **15.** $6a^3 - 12a^2 + 24a$.

Evaluate, by using factors :

16. $89 \times 117 - 86 \times 117$. **17.** $35 \times 1013 - 35 \times 713$.
18. $42 \times 718 + 42 \times 282$. **19.** $289 \times 33 + 211 \times 33$.
20. $\frac{3}{5}$ of $839 - \frac{3}{5}$ of 339. **21.** 3% of £616 + 3% of £584.

EXERCISE 41. b

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $18c - 15d$. **2.** $14t^2 - 20t$. **3.** $3l^2 - l$.
4. $33 - 56a$. **5.** $r^2 + 5rs$. **6.** $x^2 - 2xy$.
7. $9t^2 - 3k$. **8.** $5l^2 - 10lm$. **9.** $2s^5 - s^4$.
10. $24a^2b^2 - 35b^4$. **11.** $7x^2 + 42x^2y$. **12.** $17t - 68t^2w$.
13. $7k - 21l - 28m$. **14.** $5a + 15b + 9c$.
15. $a^4 - 3a^3b - 6a^2b^2 - b^4$.

Evaluate, by using factors :

16. $439 \times 39 + 61 \times 39$. **17.** $234 \times 97 - 234 \times 37$.
18. $1197 \times 88 - 1072 \times 88$. **19.** $638 \times 113 + 638 \times 87$.
20. $\frac{2}{9}$ of $1634 + \frac{2}{9}$ of 166. **21.** 4% of £1392 − 4% of £942.

EXERCISE 41. c

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $7x^4 - 14x^3 + 21x$. **2.** $a^3 - a^2b - ab^2$.
3. $15bc^2 - 9b^2c - 3c^3$. **4.** $3x^2y^2 - 9x^2y - 3x^2y^3$.
5. $2x^5 - 6x^4y - 2x^3y^2$. **6.** $3x^4 - 9x^2y^2 + 4y^4$.

$\ldots xy - acxy.$

$4a^2 - 7a^3b + 28a^3b^4.$ 　 **12.** $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3.$

12. $8x^2y - 5x^3y + 3xy^3.$

14. $14x^3 - x^2y + 42y^3.$

95. Type II. Expressions in which the terms can be arranged in groups which have a common factor.

In the preceding exercises the common factor has been a monomial. We now proceed to consider cases in which the common factor is a binomial or a polynomial. In some cases this common factor is obvious; in others it is first necessary to rearrange the terms in groups having common factors. The procedure will be clear from the following examples.

Example 3. *Factorise* $(a+2b)x+(a+2b)y.$

Here it is obvious that $(a+2b)$ is a common factor.

We obtain the other factor by division.

The expression equals $(a+2b)(x+y).$

—**The common factor should always be written down first.**

...nners often find difficulty with this type. The process may ... clearer to them by using a single letter to represent the ... factor. Thus in the above example we may write P ...).

...ression is then $Px+Py$, a type which has already been ... It is equal to $P(x+y)$, i.e. $(a+2b)(x+y)$, as before. ... should dispense with the use of the subsidiary letter P ...ssible.

... *Factorise* $a(b-3)+4(3-b).$

... it appears that there is no common factor. But ... and the expression may be written

$\ldots(b-3)-4(b-3)=(b-3)(a-4).$

...**ation between (a − b) and (b − a).**

$$(a-b)=-(b-a)$$

$$(b-a)=-(a-b)$$

...ry rules for removing brackets.

---

s...

star...

rule...

powe...

Exa...

Takin...

EITHER

OR

OR

In each case ... further, **for the**... **between (a − d) a**... **pupil.**

There are no fac... expression itself.

Example 7. *Facto*...

The expression equ...

EITHER $ax-bx-c\ldots$

$=x(a-b-c)\ldots$

OR $ax-ay-bx\ldots$

$=a(x-y)-b(x\ldots$

It is most important that the pupil should learn to recognise this relationship; it is frequently needed in factorisation.

Example 5. *Factorise* $ab-bd-ac+cd$.

Here there is no obvious common factor, but it is possible to rearrange the terms in groups having a common factor.

Select one letter, say a, and bring the terms containing it together:

$$ab-ac-bd+cd=a(b-c)-d(b-c).$$

[*N.B.*—If this line is written $a(b-c)+d(c-b)$, it will be neces-
ary to get it into the form given above by replacing $(c-b)$ by
$(b-c)$.]

$$=(b-c)(a-d).$$

The pupil may verify that the same result is obtained, if at the
t we select one of the other letters in place of a. As a general
it is best to select, if possible, a letter which occurs to the first
only.

mple 6. *Factorise* $ab+ac-bd+cd$.

g the terms in pairs, we have:

$$ab+ac-bd+cd$$
$$=a(b+c)-d(b-c).$$
$$ab-bd+ac+cd$$
$$=b(a-d)+c(a+d).$$
$$ab+cd+ac-bd.$$

we reach a stage where it is impossible to proceed
e is no relationship between $(b+c)$ and $(b-c)$, or
nd $(a+d)$. This should be carefully noted by the

ctors of the expression other than unity and the

rise $ax-bx+by+cy-cx-ay$.

als

$x-ay+by+cy$

$-y(a-b-c)=(a-b-c)(x-y)$.

$+by-cx+cy$

$-y)-c(x-y)=(x-y)(a$

EXERCISE 42. a

Factorise the following expressions if possible. If it is not possible, say so.

1. $(c+d)s+(c+d)t$.
2. $(s+t)a-(s+t)b$.
3. $3(a-b)-(a-b)c$.
4. $2a^2(c-3d)+7(c-3d)$.
5. $2x(a^2+b^2)-y(a^2-b^2)$.
6. $xy(s+t)-z(s+t)$.
7. $a(u+v)+b(v+u)$.
8. $c(x-y)+d(y-x)$.
9. $3(x^2+y^2)+6a(x^2+y^2)$.
10. $4(a-b)-3(a+b)c$.
11. $k(a-b)-l(b-a)$.
12. $c(d-4)+7(4-d)$.
13. $x^2+xy+xz+yz$.
14. $5l^2+3l+5lm+3m$.
15. $4s-t^2+4t-st$.
16. $abc-b-1+ac$.
17. $4c^3-3c^2-3d^2+4cd^2$.
18. $2x^2-2xy+xz-yz$.
19. $3x^3-x^2y-2y^3+6xy^2$.
20. $6x^4-9x+15-10x^3$.
21. $2x^2-2xy-3xz-3yz$.
22. a^4-a^3+2a-2.
23. $24-xy-6y+4x$.
24. $10+3xy-15x-2y$.

EXERCISE 42. b

Factorise the following expressions if possible, If it is not possible, say so.

1. $(l-m)x+(l-m)y$.
2. $a(c-d)-b(c-d)$.
3. $3k^2(u+4v)-2(u+4v)$.
4. $5(x+y)-(x+y)t$.
5. $2ab(d+5)-9(d+5)$.
6. $2s(c^2-d^2)+3t(c^2+d^2)$.
7. $x(u-v)-y(v-u)$.
8. $c(p+q)+l(q+p)$.
9. $m(t-2)+5(2-t)$.
10. $5(l^2+m^2)+4x(m^2+l^2)$.
11. $5(c+d)-9(d-c)k$.
12. $m(y-z)-n(z-y)$.
13. $b^2-bd+bc-cd$.
14. $k^2m^2+kmn+klm+ln$.
15. $2d-cd+2c-c^2$.
16. $2b^2+2bd+bc+cd$.
17. $4x^2-5x+5+4x^3$.
18. $x+1-txy-ty$.
19. $2a^3-b^3+2ab^2-a^2b$.
20. $12l^3-8lm^2-10m^3+15l^2m$
21. $2t^4-2t^3+3t-3$.
22. $3b^2+3bd-2bc+2cd$.
23. $15+xy-3x-5y$.
24. $14-xy-7y+2x$.

EXERCISE 42. c

Factorise the following expressions if possible. If it is not possible, say so.

1. $3x^2(2x^2-x+5)+7(2x^2-x+5)$.
2. $3y(3x^2-2x+4)-5z(3x^2-2x+4)$.
3. $3a(x^2-2x-7)-2b(x^2-2x-7)+4c(x^2-2x-7)$.
4. $l(4x-3y+7z)-3m(4x-3y+7z)-2n(4x-3y+7z)$.

5. $3x^3+21x^2-5x-35$.

6. $7x^3-5x^2-14x+10$.

7. $cd(l^2+1)-l(c^2+d^2)$.

8. $4ab(x^2+1)-x(16a^2+b^2)$.

9. $l^2c-(m^2c+l^2d^2)-m^2d^2$.

10. $x^2a^2-(y^2b+x^2b)+y^2a^2$.

11. $a^2l+abl+ac+abm+b^2m+bc$.

12. $2c^2-2cd+2cx+c+x-d$.

13. $a^2l+ac-abl-abm-bc+b^2m$.

14. $2c^2+2cd-d-c+2cx-x$.

15. $3c^2-2c-9cd+6d+2x+3cx$.

16. $2x^2+6x-15z-2xy+5yz-5xz$.

96. Type III. ax^2+bx+c, where a, b, c stand for any integers, positive or negative. The integral expression ax^2+bx+c is called a **quadratic expression** in x.

There has been much controversy about the best method of factorising expressions of the type ax^2+bx+c, and it has been traditional to start by considering expressions of the type x^2+bx+c, i.e. the special case in which $a=1$, and afterwards to use Method 2 or Method 3 for the type ax^2+bx+c. Those who wish to retain the traditional method may do so by taking first Exs. 45 *a* and *b*. They may then use either Method 2 or Method 3 for Exs. 44 *a*, *b*, *c*.

It is, however, very strongly recommended that Method 1 be used as the principal method. When this has been mastered the pupil may be encouraged to use Method 2 in all **simple** cases. Those who adopt this recommendation may omit the greater part of Exs. 45 *a* and *b*.

97. Method 1. If l, m, n, p stand for any integers, positive or negative, we have

$$\begin{aligned}(lx+m)(nx+p)&=(lx+m)nx+(lx+m)p \quad\text{......(i)}\\&=lnx^2+mnx+lpx+mp \quad\text{......(ii)}\\&=lnx^2+(mn+lp)x+mp \quad\text{......(iii)}\\&=ax^2+bx+c, \quad\text{......(iv)}\end{aligned}$$

where $a=ln$, $b=(mn+lp)$, $c=mp$.

To factorise the expression ax^2+bx+c, it is necessary to reverse the above process of multiplication. If we try to do this the difficulty lies in proceeding from (iv) to (iii). It is necessary to replace b by two equivalent numbers,* mn and lp, whose product is $mn \times lp$, i.e. $lmnp$, which is the same as $ln \times mp$ or ac.

* For brevity in this chapter 'number' is used to denote a positive or negative integer.

We have therefore the following rule for factorising ax^2+bx+c, where a, b, c stand for any integers, positive or negative :

Replace b by two equivalent numbers whose product is ac. The given expression can then be factorised by grouping terms.

Note. The method is also applicable to expressions of the type $ax^2+bxy+cy^2$, $ax^2y^2+bxyz+cz^2$, etc.

The important step is the replacement of b by two equivalent numbers whose product is ac. It is recommended that the pupil be given considerable practice in this before the complete factorisation is attempted. **It is essential that this stage of the work should be done systematically and completely.** In most examples the following procedure is both short and certain :

Express the product, ac, of the end coefficients in prime factors. Make a table showing all the possible pairs of numbers whose product is ac, placing in one column unity and multiples of the **largest prime factor** and in another column the other factor of the complete product. Select the pair whose sum is equal to the required sum b.

Example 8. *Find two numbers whose sum is* -31 *and whose product is* 240.

Express 240 in prime factors, i.e. $2^4 . 3 . 5$. The largest prime factor is 5. We therefore make a table showing all the possible pairs of numbers whose product is 240, placing in one column unity and multiples of 5 (except those which do not divide exactly into 240). We also notice that since the given product is positive, the numbers must be both positive or both negative. Since the sum is negative, they are both negative The table is :

-1	-240
-5	-48
-10	-24
-15	-16, which are the required numbers.

Note. The pupil should test the sum of each pair as soon as it is written down, so that he may stop as soon as he has found the required numbers.

Example 9. *Find two numbers whose sum is* -11 *and whose product is* -60.

Express -60 in prime factors, i.e. $-2^2 \cdot 3 \cdot 5$. The largest prime factor is 5. We therefore make a table showing all the possible pairs of numbers whose product is -60, placing in one column unity and multiples of 5 (except those which do not divide exactly into -60). We also notice that, since the product is negative, one number must be positive and the other negative. Also, since the sum is negative, the larger number must be negative. In the table we therefore place the sign $-$ before the larger number and the sign $+$ before the smaller. The table is:

$+1$	-60
$+5$	-12
-10	$+6$
-15	$+4$, which are the required numbers.

Example 10. *Find two numbers whose sum is* -25 *and whose* product is 60.

Proceeding as before, the table is:

-1	-60
-5	-12
-10	-6
-15	-4
-20	-3
-30	-2
-60	-1

(The numbers -25, -35, -40, -45, -50, -55 are omitted from the first column because they do not divide exactly into 60.)

The table has now been completed and there is no pair of numbers with the sum -25. We conclude that there are no numbers satisfying the given conditions. In such a case the pupil must check the working carefully to make sure that he has not omitted a possible pair of numbers.

EXERCISE 43. a

Find, if possible, two numbers satisfying the following conditions. If it is not possible, say so. (P = product, S = sum.)

1. $P=15$, $S=8$.

2. $P=24$, $S=11$.

3. $P=22$, $S=13$.

4. $P=24$, $S=-11$.

5. $P=-15$, $S=-8$.

6. $P=22$, $S=-13$.

7. $P = -63$, $S = 2$.
8. $P = -42$, $S = -1$.
9. $P = -45$, $S = -4$.
10. $P = -42$, $S = 1$.
11. $P = -63$, $S = -2$.
12. $P = -45$, $S = 4$
13. $P = 9$, $S = 9$.
14. $P = 48$, $S = 19$.
15. $P = -40$, $S = 3$.
16. $P = -96$, $S = 10$.
17. $P = 45$, $S = 18$.
18. $P = -9$, $S = 0$.
19. $P = -56$, $S = -10$.
20. $P = 88$, $S = 19$.
21. $P = 88$, $S = -26$.
22. $P = -66$, $S = 5$.
23. $P = -18$, $S = -3$.
24. $P = 18$, $S = -9$.
25. $P = -48$, $S = -13$.
26. $P = 96$, $S = 22$.
27. $P = -18$, $S = 4$
28. $P = 56$, $S = 18$.

EXERCISE 43. b

Find, if possible, two numbers satisfying the following conditions. If it is not possible, say so. (P = product, S = sum.)

1. $P = 63$, $S = 16$.
2. $P = 42$, $S = 13$.
3. $P = 45$, $S = 14$.
4. $P = 42$, $S = -13$.
5. $P = 45$, $S = -14$.
6. $P = 63$, $S = -16$.
7. $P = -15$, $S = -2$
8. $P = -24$, $S = 5$.
9. $P = -22$, $S = 9$.
10. $P = -24$, $S = -5$.
11. $P = -22$, $S = -9$.
12. $P = -15$, $S = 2$.
13. $P = 18$, $S = 9$.
14. $P = 40$, $S = 13$.
15. $P = -96$, $S = -10$
16. $P = -40$, $S = -3$.
17. $P = 9$, $S = 10$.
18. $P = -45$, $S = -12$.
19. $P = -9$, $S = -6$.
20. $P = 48$, $S = -19$.
21. $P = 18$, $S = 12$
22. $P = 96$, $S = -22$.
23. $P = -9$, $S = -8$.
24. $P = -18$, $S = 3$.
25. $P = -48$, $S = 13$.
26. $P = 45$, $S = -18$.
27. $P = -88$, $S = 3$.
28. $P = -56$, $S = 10$.

Example 11. *Factorise* $21x^2 - 5xy - 6y^2$.

We have to find two numbers whose product is -126, or $-2 \cdot 3^2 \cdot 7$, and whose sum is -5. Proceeding as usual, we find that the numbers are -14 and $+9$. We therefore write the expression in the form

$$\begin{array}{rr} +1 & -126 \\ +7 & -18 \\ -14 & +9 \end{array}$$

$$21x^2 - 14xy + 9xy - 6y^2$$
$$= 7x(3x - 2y) + 3y(3x - 2y) = (3x - 2y)(7x + 3y).$$

Example 12. *Factorise* $-4x^2-15+16x$.

The expression must be rearranged in descending (or ascending) powers of x. It equals $-4x^2+16x-15$. We must therefore find two numbers whose product is $+60$ and whose sum is $+16$. They are $+10$ and $+6$,

$+1$	$+60$
$+5$	$+12$
$+10$	$+6$

$$\therefore\ -4x^2+16x-15=-4x^2+10x+6x-15$$
$$=-2x(2x-5)+3(2x-5)=(2x-5)(-2x+3).$$

Note. The signs in the expression must not be changed. Beginners sometimes change the sign throughout and write $4x^2-16x+15$ in place of the given expression. This is wrong.

The given procedure applies in all cases.

Example 13. *Factorise* $63x^2-15xy-18y^2$.

We notice that 3 is a factor of each term,

$\therefore$ the expression $=3(21x^2-5xy-6y^2)$
$=3(3x-2y)(7x+3y)$, as above, Ex. 11.

Note. If each term has a common factor, this should always be taken out first.

98. Method II. Factors by inspection. Easy quadratic expressions can often be factorised by inspection, without using the method of grouping terms.

Whichever method is used the answer should always be checked mentally by multiplication.

Let us consider the following results, which are obtained by ordinary multiplication :

$$(2x+3)(3x+4)=6x^2+17x+12, \quad \text{.................(i)}$$
$$(2x-3)(3x-4)=6x^2-17x+12, \quad \text{...............(ii)}$$
$$(2x+3)(3x-4)=6x^2+x-12, \quad \text{.................(iii)}$$
$$(2x-3)(3x+4)=6x^2-x-12. \quad \text{.................(iv)}$$

We notice that if, as in (i) and (ii), the third term of the quadratic expression is positive, then the second terms of its factors have the same sign, and this sign is the same as that of the middle term of the expression. If, as in (iii) and (iv), the third term of the expression is negative, then the second terms of its factors have opposite signs.

Again, consider the result (i). The first term $6x^2$ is the product of $2x$ and $3x$; the third term $+12$ is the product of $+3$ and $+4$. The middle term $+17x$ is the result of adding together the two

products $+3 \times 3x$ and $+4 \times 2x$. Similarly, in the result (iv), the first term $6x^2$ is the product of $2x$ and $3x$; the third term -12 is the product of -3 and $+4$. The middle term $-x$ is the result of adding together the two products $-3 \times 3x$ and $+4 \times 2x$.

If we try to reverse the process and find the factors of one of these expressions, say $6x^2 - x - 12$, it is clear that the factors must either start with $6x$ and x or with $3x$ and $2x$, and end with 12 and 1 or 6 and 2 or 4 and 3. Also the second terms have opposite signs. The possible pairs of factors are therefore :

$(6x+1)(x-12)$,	$(6x-1)(x+12)$,	$(6x+12)(x-1)$,
$(6x-12)(x+1)$,	$(6x-6)(x+2)$,	$(6x+6)(x-2)$,
$(6x+2)(x-6)$,	$(6x-2)(x+6)$,	$(6x+4)(x-3)$,
$(6x-4)(x+3)$,	$(6x+3)(x-4)$,	$(6x-3)(x+4)$,
$(3x+1)(2x-12)$,	$(3x-1)(2x+12)$,	$(3x+12)(2x-1)$,
$(3x-12)(2x+1)$,	$(3x+6)(2x-2)$,	$(3x-6)(2x+2)$,
$(3x+2)(2x-6)$,	$(3x-2)(2x+6)$,	$(3x+4)(2x-3)$,
$(3x-4)(2x+3)$,	$(3x+3)(2x-4)$,	$(3x-3)(2x+4)$.

We then test mentally by multiplication until we find the right pair. This process seems long, but a little consideration will show that a number of the possible factors may at once be rejected. Thus $(6x+12)(x-1)$ is impossible, for 6 is a factor of $6x+12$ and therefore of the whole product, but it is not a factor of $6x^2 - x - 12$. In the same way all the other pairs may be rejected except

$(6x+1)(x-12)$, $(6x-1)(x+12)$, $(3x+4)(2x-3)$, $(3x-4)(2x+3)$, which must be tested by multiplication.

After a little practice, the pupil will be able to reject mentally the impossible pairs of factors, and the process is quicker than the above working would suggest. With simple numbers it is slightly quicker than the method of grouping terms, **but the grouping method is the method to rely upon. It should always be used whenever the pupil is not able to obtain the factors quickly by inspection.**

99. Method III. The method will be clear from the following example.

Example 14. *Factorise* $6x^2 - x - 12$.

We write the expression in the form $\frac{1}{6}[36x^2 - 6x - 72]$

$$= \frac{1}{6}[(6x)^2 - (6x) - 72].$$

We now write $Y=6x$, and the expression becomes

$$\frac{1}{6}[Y^2 - Y - 72].$$

We thus reduce the problem to the easier problem of finding the factors of $Y^2 - Y - 72$.

These are found by inspection to be $(Y-9)(Y+8)$,

$$\text{i.e. } (6x-9)(6x+8) = 3(2x-3).\,2(3x+4)$$
$$= 6(2x-3)(3x+4).$$

The expression is thus equal to $\frac{1}{6} \times 6(2x-3)(3x+4)$,

$$\text{i.e. } (2x-3)(3x+4).$$

EXERCISE 44. a

Factorise, if possible, the following expressions. If it is not possible, say so. Check your answers mentally.

1. $2x^2+3x+1$.	**2.** $2d^2+7d+3$.	**3.** $3t^2+5t+2$.
4. $2a^2-5a+3$.	**5.** $2x^2-7xy+6y^2$.	**6.** $3a^2-11a+6$.
7. $2c^2+9c-5$.	**8.** $2x^2-3xy-5y^2$.	**9.** $3c^2+7c-6$.
10. $3z^2-2z-8$.	**11.** $15p^2-7pq-2q^2$.	**12.** $3a^2+5a-8$.
13. $2-c-6c^2$.	**14.** $3-10z+3z^2$.	**15.** $3-8x-3x^2$.
16. $3x^2-7x-5$.	**17.** $a^2-3a-10$.	**18.** $a^2+7a+10$.
19. $2x^2-32xy+126y^2$.	**20.** $5a^2-4a-3$.	**21.** $2x^4-3x^2-2$.
22. $2a^2-abc-b^2c^2$.	**23.** $3k^2-3k-168$.	**24.** $3a^2+6a-189$.
25. $x^2+3xy-40y^2$.	**26.** $9t^2+36t-108$.	**27.** $4-3st-10s^2t^2$.
28. $5a^2-5ab-280b^2$.	**29.** $6y^2+4y+3$.	**30.** $4-13xy+10x^2y^2$.
31. $4c^2+9c-8$.	**32.** $8l^2-lm^2-7m^4$.	**33.** $2x^2-xyz-15y^2z^2$.

EXERCISE 44. b

Factorise, if possible, the following expressions. If it is not possible, say so. Check your answers mentally.

1. $3x^2+4x+1$.	**2.** $3t^2+8t+4$.	**3.** $2c^2+9c+4$.
4. $2x^2-11xy+5y^2$.	**5.** $2c^2-5c+2$.	**6.** $2a^2-9a+10$
7. $3t^2+5t-2$.	**8.** $5t^2+29t-6$.	**9.** $3a^2+ab-2b^2$.
10. $2d^2-d-3$.	**11.** $15s^2+13s-2$.	**12.** $15a^2+7ab-2b^2$.
13. $2-7x+6x^2$.	**14.** $2+7t+5t^2$.	**15.** $2-3k-5k^2$.
16. x^2+8x+7.	**17.** $3x^2+7x+5$.	**18.** $-x^2-x+20$.
19. $5a^2-4a+3$.	**20.** $2a^2b^2+3abc-2c^2$.	**21.** $2x^4+x^2-1$.

22. $2x^4 - 11x^2y^2 + 15y^4$. **23.** $7x^2 - 28x - 84$. **24.** $-8d^2 - 9d + 2$.
25. $4a^2 + 60ab + 224b^2$. **26.** $2c^2 - cd - 28d^2$.
27. $4a^2 + ab - 14b^2$. **28.** $40 - 13p + p^2$.
29. $5p^2 - 75pq + 280q^2$. **30.** $4a^2 + 15abc + 14b^2c^2$.
31. $8 - 15ab + 7a^2b^2$. **32.** $4 + 3z^2 - 10z^4$. **33.** $3c^2 - 3c - 189$.

EXERCISE 44. c

Factorise, if possible, the following expressions. If it is not possible, say so. Check your answers mentally.

1. $2a^2 - 17a + 8$. **2.** $3c^2 + 13c - 30$. **3.** $8 - 17x + 7x^2$.
4. $18x^2 + 129x + 21$. **5.** $15a^2 + 99a - 42$. **6.** $21 - 19t - 2t^2$
7. $3t^2 + 23t + 30$. **8.** $21l^2 - 17lm + 2m^2$. **9.** $12c^2 - 23cd + 10d^2$.
10. $30k^2 + 87k + 63$. **11.** $2a^2 - 15a - 8$. **12.** $28x^2 + 31xy - 5y^2$.
13. $15 + 16xy - 15x^2y^2$. **14.** $7a^2 - 9a - 8$. **15.** $21p^2 - 11pq - 2q^2$.
16. $12x^2 - 7x - 10$. **17.** $24t^2 + 22t - 21$. **18.** $15 - 34c + 15c^2$.
19. $28a^2 + 39ab + 5b^2$. **20.** $14z^4 + 48z^2 - 7$. **21.** $5n^2 - 104n - 21$
22. $42x^2 - 77xy - 49y^2$. **23.** $40a^2 - 108abc + 72b^2c^2$.
24. $24x^2 - 50x + 21$. **25.** $20 - 9z - 20z^2$. **26.** $21a^2 - 23ab + 2b^2$.
27. $105 + 5p - 50p^2$. **28.** $15l^2 - 35l + 16$. **29.** $12x^2 - 2x - 70$.
30. $12d^2 - 82d - 14$. **31.** $14a^4 + 96a^2 - 14$. **32.** $x^2 - x - 90$.
33. $20x^2 - 32x - 84$. **34.** $40t^4 + 82t^2 + 40$. **35.** $7x^2 + 60x + 24$.
36. $5x^2 + 37xy + 14y^2$. **37.** $30m^2 - 9m - 54$. **38.** $60a^2 - 65a - 70$.
39. $42k^2 - 81k - 6$. **40.** $4x^4 + 37x^2 - 30$.
41. $30x^2 - 85xy + 35y^2$. **42.** $16 + 11c - 15c^2$.
43. $7c^2 + 50cd + 7d^2$. **44.** $30m^2 - 57mn + 21n^2$.
45. $40c^2 - 4c - 84$. **46.** $33a^2 - 26ab + 4b^2$.
47. $10x^2 - 212xy + 42y^2$. **48.** $33z^2 - 14z - 4$

Note. Harder examples of this type will be found in Chapter XXIV.

Note. Those who have adopted Method 1 may omit Exs. 45 *a* and *b*, or they may do only a small selection from them.

EXERCISE 45. a

Factorise, it possible, the following expressions If it is not possible, say so. Check your answers mentally.

1. $a^2 + 3a + 2$. **2.** $a^2 + 7a + 12$. **3.** $c^2 + 9cd + 18d^2$.

4. $s^2 - 16st + 15t^2$ 5. $d^2 - 5d + 4$. 6. $x^2 - 9x + 20$.
7. $a^2 - a - 2$. 8. $h^2 + 3h - 10$. 9. $t^2 + 2t - 18$.
10. $n^2 - n - 20$ 11. $m^2 + m - 42$. 12. $x^2 - x - 12$.
13. $a^2 + 8a + 16$. 14. $14b^2 + 9bc + c^2$. 15. $z^2 - 4z - 21$.
16. $x^2 - 11xy + 28y^2$ 17. $1 - 10xy - 24x^2y^2$. 18. $x^2 + 13x + 33$.
19. $a^2 - 5ab - 24b^2$ 20. $1 - 16k + 63k^2$ 21. $y^2 - 12y + 40$.
22. $11 + 10t - t^2$ 23. $x^2 - 25x + 136$. 24. $c^2 + 4c - 35$.
25. $y^2 - 23yz + 102z^2$. 26. $a^2 - 4a - 24$. 27. $x^2 - 7x - 48$.
28. $a^2 + 4ac - 45c^2$. 29. $1 + 12xyz + 35x^2y^2z^2$. 30. $t^2 - 7t - 78$.

EXERCISE 45. b

Factorise, if possible, the following expressions. If it is not possible, say so. Check your answers mentally.

1. $b^2 + 4b + 3$. 2. $n^2 + 6n + 8$. 3. $d^2 + 13dk + 42k^2$.
4. $y^2 - 5y + 6$. 5. $x^2 - 7x + 10$. 6. $m^2 - 8mn + 15n^2$.
7. $z^2 - 3z - 4$. 8. $y^2 - 2y - 15$. 9. $x^2 + 2x - 3$.
10. $t^2 + t - 6$. 11. $x^2 - 3xy - 18y^2$ 12. $b^2 - 14b - 15$.
13. $x^2 + 10x + 21$ 14. $d^2 - 10d + 25$. 15. $z^2 - 2z - 24$.
16. $x^2 + 11xy + 24y^2$. 17. $b^2 - 10b + 18$. 18. $x^2 + 3xy - 28y^2$.
19. $14 - 5xy - x^2y^2$ 20. $c^2d^2 + 15cd + 44$ 21. $c^2 - 14cd + 24d^2$.
22. $a^2 + 20ab - 69b^2$. 23. $t^2 + 11t + 42$. 24. $1 - 9k - 136k^2$.
25. $1 - 8x^2 - 65x^4$. 26. $c^2 + 12c - 72$. 27. $c^2 + 24c - 81$.
28. $x^2 + 6xyz - 91y^2z^2$. 29. $h^2 + 3h - 88$. 30. $t^6 - 2t^3 - 35$.

100. Type IV. The difference of two squares. The factors of $x^2 - y^2$ may be found as a special case of Type III in which the middle term is zero. Thus

$$x^2 - y^2 = x^2 + xy - xy - y^2$$
$$= x(x+y) - y(x+y) = (x+y)(x-y).$$

This result is so important that it should be committed to memory in words as follows :

(1) **The difference of the squares of any two quantities is equal to the product of the sum and the difference of the two quantities.**

(2) **The product of the sum and the difference of any two quantities is equal to the difference of their squares.**

By means of the first rule any expression which is the difference of two squares may at once be resolved into factors.

Example 15. *Factorise* $9a^2 - 16b^2$.

$9a^2$ is the square of $3a$, $16b^2$ is the square of $4b$.

Therefore the first factor is $3a$ plus $4b$, and the second factor is $3a$ minus $4b$; $\therefore$ $(9a^2 - 16b^2) = (3a + 4b)(3a - 4b)$.

The intermediate steps may usually be omitted, but it is a good plan to write in the margin the numbers which are squared. The work may be arranged as in the following example.

Example 16. *Factorise* $4x^2 - 25(y-x)^2$. $2x$

The expression equals $[2x + 5(y-x)][2x - 5(y-x)]$ $5(y-x)$

$$= [2x + 5y - 5x][2x - 5y + 5x] = [5y - 3x][7x - 5y].$$

It should be carefully noted that $25(y-x)^2$ is the square of $5(y-x)$.

It is sometimes necessary to take out a common factor before applying the rule.

Example 17. *Factorise* $3A^2 - 48B^2$.

Here there is a common factor 3, so that the expression equals $3(A^2 - 16B^2)$. We can now apply the rule to get the complete result: $3(A + 4B)(A - 4B)$. A $4B$

The first rule may also be used to shorten arithmetical calculations.

Example 18. *Evaluate* $\sqrt{(625)^2 - (375)^2}$.

$$\sqrt{(625)^2 - (375)^2} = \sqrt{(625 + 375)(625 - 375)} = \sqrt{1000 \times 250}$$
$$= \sqrt{250000} = 500.$$

The second rule may be used to do certain multiplications mentally.

Example 19. *Multiply* $(a + b - c)$ *by* $(a - b + c)$.

The first bracket is a plus $(b - c)$.

The second bracket is a minus $(b - c)$.

Therefore the product is $a^2 - (b-c)^2 = a^2 - (b^2 - 2bc + c^2)$
$$= a^2 - b^2 + 2bc - c^2.$$

EXERCISE 46. a

(*Some of these may be done orally*)

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $a^2 - 25$. **2.** $c^2d^2 - 16n^2$. **3.** $9 - 49b^2$.

4. $49 - 25t^2$. **5.** $x^2 - 1$. **6.** $4d^2 - 49$.

7. $25x^2 - 100$. **8.** $121a^2 - 64b^6$. **9.** $144 - 25t^4$.

10. $4x^4y^4 - 9z^6$. **11.** $81l^2 - 100p^2q^2$ **12.** $25a^4 - 9x^2$.

13. $5x^2 - 7y^2$. **14.** $3k^2 - 75$. **15.** $169a^2 - 121b^2$.

16. $8 - 2t^2$. **17.** $9a^2 + 16b^2$. **18.** $5x^2 - 80y^4$.

Find, by factors, the value of :

19. $(157)^2 - (156)^2$. **20.** $(483)^2 - (283)^2$. **21.** $(264)^2 - (36)^2$.

22. $(81)^2 - (19)^2$. **23.** $(96)^2 - 16$. **24.** $(873)^2 - (823)^2$

25. $(42{\cdot}8)^2 - (42{\cdot}3)^2$. **26.** $(999)^2 - 1$.

27. $(6{\cdot}7)^2 - (3{\cdot}3)^2$. **28.** $\sqrt{(1{\cdot}945)^2 - (0{\cdot}945)^2}$.

29. $(475)^2 - (225)^2$. **30.** $\sqrt{(125)^2 - (120)^2}$.

Find the value of :

31. $(a - 2b + c)(a + 2b - c)$. **32.** $(3a + b - 2c)(3a + b + 2c)$.

33. $(4a + b + 3c)(4a - b - 3c)$. **34.** $(5x^2 + x + 2)(5x^2 - x - 2)$.

35. $(c^2 - 3c + 4)(c^2 + 3c - 4)$. **36.** $(2x^2 + 3x - 4)(2x^2 - 3x - 4)$.

EXERCISE 46. b

(Some of these may be done orally)

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $c^2 - 9$. **2.** $1 - 49n^2$. **3.** $x^2 - 36$.

4. $100 - x^2y^2z^2$. **5.** $9k^2 - 64$. **6.** $36a^2 - 25b^2c^2$.

7. $4a^2b^4 - 9b^2c^4$. **8.** $144a^2 - 36b^2$. **9.** $49x^4 - 9y^6$.

10. $36m^2n^2 - 49x^4$. **11.** $81 - 25p^6$. **12.** $121a^4 - 1$.

13. $18x^2 - 50y^2$. **14.** $63a^2 - 7b^4$. **15.** $a^2 + 4b^2$.

16. $45 - 20c^2$. **17.** $2x^2 - 3y^2$. **18.** $27b^2 - 3c^2$

Find, by factors, the value of :

19. $(101)^2 - 1$. **20.** $(97)^2 - (87)^2$. **21.** $(97)^2 - 9$.

22. $(107)^2 - (93)^2$. **23.** $(785)^2 - (215)^2$. **24.** $(252)^2 - (52)^2$.

25. $(12{\cdot}5)^2 - (7{\cdot}5)^2$. **26.** $\sqrt{(6{\cdot}5)^2 - (6{\cdot}3)^2}$. **27.** $\sqrt{(101)^2 - (99)^2}$.

28. $(1{\cdot}437)^2 - (0{\cdot}437)^2$. **29.** $\sqrt{(6{\cdot}8)^2 - 36}$. **30.** $(3{\cdot}14)^2 - (3{\cdot}05)^2$.

Find the value of :

31. $(2a - b + c)(2a + b - c)$. **32.** $(2a + 3b + c)(2a - 3b - c)$.

33. $(5a + 2b - 3c)(5a + 2b + 3c)$. **34.** $(3x^2 + x + 3)(3x^2 - x + 3)$.

35. $(2x^2 + x + 5)(2x^2 - x - 5)$. **36.** $(t^2 - 5t + 2)(t^2 + 5t + 2)$.

*EXERCISE 46. c

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $(a+2b)^2-(c+d)^2$.
2. $(3m+2n)^2-(2x-y)^2$.
3. $(a-b)^2-(c-d)^2$.
4. $(4b-3c)^2-(2m+n)^2$.
5. $(5b-c)^2-(m-2n)^2$.
6. $(m+4n)^2-(3x-y)^2$.
7. $(a-b)^2-(c+d)^2$.
8. $(a+b)^2-(c-d)^2$.
9. $(a+b)^2-n^2$
10. $2a^2-(s-t)^2$.
11. $c^2-(y+z)^2$.
12. $(a-b)^2-x^2$.
13. $(4c-3d)^2-4l^2$.
14. $9a^2-(b-c)^2$.
15. $(3c+2d)^2-16l^2$.
16. $(6x-7y)^2-(3x+3y)^2$
17. $25x^2-(y-z)^2$.
18. $(3b-4c)^2-(b+3c)^2$.
19. $25(3l-m)^2-16(r+2s)^2$.
20. $25c^2-4(2x+y)^2$.
21. $9x^2-16(y+2z)^2$.
22. $4k^2-49(m-2n)^2$.
23. $9(4l-3m)^2-4(3r-s)^2$
24. $64t^2-25(2x+3z)^2$.
25. $9(x-2y)^2-(4x-7y)^2$.
26. $4(2l-3m)^2-(3l-7m)^2$.
27. $25x^2+(3x+7y)^2$.
28. $25c^2-(5c-7d)^2$.
29. $16x^2-(3x-5y)^2$.
30. $16c^2-(4c-3d)^2$.
31. $(a-2b)^2-9(a-3b)^2$.
32. $16x^2-(y-x)^2$.
33. $9(l+m)^2-4(l-m)^2$.
34. $(x-4y)^2-16(x-y)^2$.
35. $64t^2-(t-4)^2$.
36. $4(c+2d)^2-25(3c-d)^2$.

101. Type V. $\mathbf{x^2+2xy+y^2}$ and $\mathbf{x^2-2xy+y^2}$.

It was shown in the previous chapter that

(1) **The square of the sum of two quantities is equal to the sum of their squares plus twice their product.**

(2) **The square of the difference of two quantities is equal to the sum of their squares minus twice their product.**

In other words
$$\mathbf{x^2+2xy+y^2=(x+y)^2,}$$
$$\mathbf{x^2-2xy+y^2=(x-y)^2.}$$

It should also be recognised that $\mathbf{x^2+y^2}$ has no factors.

These results may also be obtained as special cases of Type III.

It is important that the pupil should learn to recognise a perfect square at sight.

Example 20. *Which of the following expressions are perfect squares* (1) $9x^2-30xy+25y^2$, (2) $9x^2+26xy+16y^2$?

(1) $9x^2$ is the square of $3x$; $25y^2$ is the square of $5y$;

$30xy$ is twice the product of $3x$ and $5y$;

$\therefore\ 9x^2 - 30xy + 25y^2 = (3x - 5y)^2$ or $(5y - 3x)^2$.

(2) $9x^2$ is the square of $3x$;

$16y^2$ is the square of $4y$;

$26xy$ is NOT twice the product of $3x$ and $4y$;

$\therefore\ 9x^2 + 26xy + 16y^2$ is NOT a perfect square.

Note. Any positive number has two square roots; thus

$$\sqrt{49} = +7 \text{ or } -7, \quad \sqrt{x^2 + 2xy + y^2} = x + y \text{ or } -x - y,$$
$$\sqrt{x^2 - 2xy + y^2} = x - y \text{ or } y - x.$$

EXERCISE 47. a (*Mainly oral*)

In the following examples, state whether the expression is a perfect square; if it is, give one of the square roots.

1. $a^2 + 4a + 4$. **2.** $a^2 - 10a + 25$. **3.** $4a^2 + 4a + 1$.
4. $4a^2 - 28a + 49$. **5.** $9a^2 - 6a + 1$. **6.** $9d^2 + 30d + 25$.
7. $9x^2 - 24xy + 16y^2$. **8.** $16x^2 - 8x + 1$.
9. $a^2 - 6ac + 16c^2$. **10.** $25l^2 + 20lm + 4m^2$.
11. $9a^2 + 24a + 4$. **12.** $16x^2 + 56xy + 49y^2$.
13. $25m^2 + 30mn + 9n^2$. **14.** $4a^2 + 18a + 81$.
15. $25t^4 - 60t^2 + 36$. **16.** $4a^4 - 44a^2 + 121$.
17. $9a^2 + 6abc + 4b^2c^2$. **18.** $25c^2 + 110cd + 121d^2$.

EXERCISE 47. b (*Mainly oral*)

In the following examples, state whether the expression is a perfect square; if it is, give one of the square roots.

1. $a^2 + 8a + 16$. **2.** $a^2 - 6a + 9$.
3. $4a^2 - 12a + 9$. **4.** $4a^2 + 20a + 25$.
5. $9x^2 - 12x + 4$. **6.** $9c^2 - 42c + 49$.
7. $9a^2 + 48ab + 64b^2$. **8.** $9x^2 - 25xy + 25y^2$.
9. $16x^2 + 24x + 9$. **10.** $16x^2 - 40xy + 25y^2$.
11. $25c^2 - 10cd + d^2$. **12.** $4a^2 + 32ab + 49b^2$.
13. $25m^2 - 80mn + 64n^2$. **14.** $a^2 - 8a + 64$.
15. $25x^4 - 40x^2 + 16$. **16.** $9a^4 + 60a^2 + 100$.
17. $16x^2 - 72xy + 81y^2$. **18.** $4x^2 + 22xy + 121y^2$.

102.* Type VI. Harder cases of the difference of two squares. By suitably grouping together the terms, certain expressions may be expressed as the difference of two squares, and so be factorised.

Example 21. *Factorise* $4x^2+9y^2-81a^2-12xy$.

$4x^2$, $9y^2$ and $81a^2$ are perfect squares; this suggests that the expression may be resolved into the difference between two squares. If so, it is clear that the term $-12xy$ must be grouped with $4x^2$ and $+9y^2$.

Thus the expression equals

$$4x^2-12xy+9y^2-81a^2=(2x-3y)^2-81a^2 \qquad 2x-3y$$
$$=(2x-3y+9a)(2x-3y-9a). \qquad 9a$$

Note. The pupil should notice which terms are *not* perfect squares, and then decide which terms must be grouped with them.

*EXERCISE 48. a

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $x^2-2xy+y^2-z^2$.
2. $x^2-y^2-2yz-z^2$.
3. $x^2-2xy+y^2-25$.
4. $4-y^2+2yz-z^2$.
5. $1-k^2+6kl-9l^2$.
6. $4a^2+4ab+b^2-64$.
7. $4l^2+12lm+9m^2-81$.
8. a^2-b^2-4b-4.
9. $36-9y^2+6yz-z^2$.
10. $c^2-8c+16-9d^2$.
11. $36-y^2+7yz-49z^2$.
12. $x^2-8x+16-x^4$.
13. $x^2+2xy+y^2-l^2-2lm-m^2$.
14. $1+4c+2xy+4c^2-x^2-y^2$.
15. $a^2-6ab+9b^2-p^2+4pq-4q^2$.
16. $c^2+3cd+9d^2-25l^2-10l-1$.
17. $a^2-14a+49-l^2+6lm-9m^2$.
18. $4c^2-12c+9-x^2-6xy-9y^2$
19. $-x^2+4l^2+9m^2-y^2-2xy-12lm$.
20. $4l^2+24yz-9y^2+m^2+4lm-16z^2$.
21. $7a^2-14ab+7b^2-63c^2$.
22. $9a^2-16x^2-16xy-4y^2$.
23. $4c^4+9d^4-a^4-25b^4-12c^2d^2+10a^2b^2$.
24. $4m^2-25t^2-28mn+49n^2$.

*EXERCISE 48. b

Factorise, if possible, the following expressions. If it is not possible, say so.

1. $x^2+2xy+y^2-z^2$.
2. $x^2-y^2+2yz-z^2$.
3. $x^2+2xy+y^2-16$.
4. $49-y^2-2yz-z^2$.
5. $x^2-y^2-10y-25$.
6. $25-k^2-6kl-9l^2$.
7. $25-y^2-14yz-49z^2$.
8. $4a^2-4ab+b^2-1$.
9. $c^2+6c+9-4d^2$.
10. $4y^2-12yz+9z^2-36$.
11. x^4-x^2+6x-9.
12. $c^2+4c+16-25d^2$.

13. $c^2-2cd+d^2-x^2+2xy-y^2$. **14.** $49a^2-14a+1-4b^2+4b-1$.
15. $x^2-10x+25-4c^2-12cd-9d^2$.
16. $4l^2+4lm+m^2-p^2+10pq-25q^2$.
17. $2ab-2pq+p^2-a^2-b^2+q^2$. **18.** $x^2-y^2+5y-25+4z^2-4xz$.
19. $9x^4-y^2-9z^2+6yz$. **20.** $x^4+4yz-z^2-4y^2$.
21. $4x^2+49-36y^2+28x$. **22.** $5x^2-10xy+5y^2-45$.
23. $9c^2-4l^2+16lm-16m^2$. **24.** $x^4+y^4-l^4-m^4-2x^2y^2-2l^2m^2$.

103.* We conclude this chapter with a few miscellaneous examples illustrating all the types dealt with in this chapter. It should be particularly noticed that, after factorising the given expression by one of the rules, it is frequently possible to factorise the factors so obtained. **Care must be taken to find as many factors as possible of the given expression.**

Example 22. *Factorise* x^5-81x.

$$x^5-81x=x(x^4-81)\text{ (Type I)}=x(x^2+9)(x^2-9)\text{ (Type IV)}$$
$$=x(x^2+9)(x+3)(x-3)\text{ (Type IV)}.$$

Example 23. *Factorise* $(x^2+x)^2-18(x^2+x)+72$.

The expression equals

$$(x^2+x)^2-12(x^2+x)-6(x^2+x)+72 \qquad -1-72$$
$$=(x^2+x)[(x^2+x)-12]-6[(x^2+x)-12] \qquad -3-24$$
$$=[(x^2+x)-12][(x^2+x)-6]. \qquad -6-12$$

Similarly, we obtain

$$x^2+x-12=(x+4)(x-3) \quad\text{and}\quad x^2+x-6=(x+3)(x-2).$$

Hence the complete factors are $(x+4)(x-3)(x+3)(x-2)$.

Example 24. *Factorise* $(2x^2-3x-5)^2-(x^2-3x-4)^2$.

The expression equals

$$[2x^2-3x-5+x^2-3x-4][2x^2-3x-5-x^2+3x+4]\text{ (Type IV)}$$
$$=(3x^2-6x-9)(x^2-1). \qquad 2x^2-3x-5$$

But $3x^2-6x-9=3(x^2-2x-3)\text{ (Type I)}$ $\qquad x^2-3x-4$

$$=3(x-3)(x+1)\text{ (Type III)}.$$
$$x^2-1=(x+1)(x-1)\text{ (Type IV)}.$$

Hence the complete factors are $3(x-3)(x+1)^2(x-1)$.

Example 25. *Factorise* $18(x-y)^3-2x+2y$. $\qquad 3(x-y)$

The expression equals $18(x-y)^3-2(x-y)$ $\qquad 1$

$$=2(x-y)[9(x-y)^2-1]\text{ [Type II]}$$
$$=2(x-y)(3x-3y+1)(3x-3y-1)\text{ [Type IV]}.$$

EXERCISE 49

Miscellaneous Factors

(*Nos.* 1–50 *are easy. Nos.* 51–100 *are harder*)

Express as the product of as many factors as possible :

1. $18x^3 - 9xy^2$. **2.** $3x^2 + 7x + 2$. **3.** $16a^2 - 49x^2$.
4. $y^2 - cy + 5y - 5c$. **5.** $75x^2 - 48$. **6.** $t^2 + 7t - 78$.
7. $3a^2 - 19a + 6$. **8.** $4t^3 - t^2 + t$. **9.** $(2c + d)^2 - (l - m)^2$.
10. $6ln - 2ns - 3l + s$. **11.** $2a^3 - 50a$. **12.** $2y^2 + 11y - 6$.
13. $9l^2 - (3l - 4m)^2$. **14.** $7 - 28c^2$. **15.** $110 - a - a^2$.
16. $3a^2 - ab - 4b^2$. **17.** $2xc^2 + 3xcd - 2ycd - 3yd^2$.
18. $(3x - y)^2 - (x + 2y)^2$. **19.** $2y^5 - 6y^4z + 2y^3z^2$.
20. $3 - 46l + 15l^2$. **21.** $20xy^2 + 15x^2y + 5x^3$. **22.** $3x^2 - x - 10$.
23. $3a^4 - 300$. **24.** $9a^2 - 30ab + 25b^2$. **25.** $13 - 52z^2$.
26. $5x^2 + 17x + 6$. **27.** $x^5 - 25x$. **28.** $9a^3 - 9a^2b + 27ab^5$.
29. $1 + 8x^2 - 65x^4$. **30.** $l^2 + 15lm + 26m^2$.
31. $l^2r^2 + l^2s^2 - cs^2 - cr^2$. **32.** $1 - 14a + 49a^2$.
33. $a(bc)^2 - (ab)^2c$. **34.** $500x^2y - 20y^3$. **35.** $s^2 - 4st - 45t^2$.
36. $5x^2 + x - 6$. **37.** $4c^2 + c - 1 - 4c^3$. **38.** $(3d)^2 - 3d$.
39. $7a^3 - 3a^2 - 21a + 9$. **40.** $4x^2 - x - 5$.
41. $81a^2 - (a - 2b)^2$. **42.** $4d^4 + 28d^2 + 49$. **43.** $c^2(c + 1) + c + 1$.
44. $y^2 + 9y - 36$. **45.** $10x^2 + 23x - 21$. **46.** $15x^3 + 38x^2 + 7x$.
47. $144t^2 + 120t + 25$. **48.** $x(y + 1) - y - 1$.
49. $2z^3 + 14z^2 - 3z - 21$. **50.** $t(l - m) - l + m$.
51. $3a^2b^2 - 3a^2 - 3b^2 + 3$. **52.** $a^2 - 2ab + b^2 - 9$.
53. $18r^4 - 8s^4$. **54.** $6a^3 - 5a^4 + a^5$.
55. $(2x - 7)^2 - 6x + 21$. **56.** $4l^2 - 4lm + m^2 - 49$.
57. $3(t + 1)^2 - 7(t + 1) + 2$. **58.** $5(2c - 3)^2 - (2c - 3) - 6$.
59. $(4x - 5)^2 - 12x + 15$. **60.** $16 - a^2 - 6ab - 9b^2$.
61. $x^2 - y^2 + 2ax + 2ay$. **62.** $(7x + 2)^3 - 7x - 2$.
63. $(a + b)^2 + 19c(a + b) + 78c^2$. **64.** $x^2 - y^2 + 8y - 16$.
65. $6x^4 - 17x^2(y + z) - 10(y + z)^2$. **66.** $5(x + 7)^2 + 4(x + 7) - 12$.
67. $(3x - 1)^3 - 27x + 9$. **68.** $9c^9 - 3c^3$.
69. $5(a - c)^2 + 8t(a - c) + 3t^2$. **70.** $a^2 - 4a(x - y) - 5(x - y)^2$.
71. $(3 + d^2)^2 - 16d^2$. **72.** $(x + 2)(x + 3)^2 - 5x - 10$.
73. $x^2 - 7x(l - m) + 10(l - m)^2$. **74.** $ax + by - cx - cy + bx + ay$.
75. $3(7x - 3)^2 + 4(7x - 3) - 15$. **76.** $al - bl + bm + cm - cl - am$.
77. $(l - m)^2 - l + m$. **78.** $x^4 + 12x - 4x^2 - 9$.

79. $4y^2z^2-(y^2+z^2-x^2)^2$.

80. $(x^2-xy)^2-(xy-y^2)^2$.

81. $a^2+4b^2-4ab-l^2-9n^2-6ln$.

82. $(l^2-l)^2-8(l^2-l)+12$.

83. a^4-a^2+2a-1.

84. $(6x^2-9x-3)^2-(2x^2-9x-2)^2$.

85. $(x+3)(x+4)(x+2)^2-(x-1)(x+2)(x+4)^2$.

86. $36l^2mn-54lm^2n+48lmn^2-18l^2m^2n^2$.

87. $16a^4-(b-c)^4$.

88. $a^2-6a(c+d)+5(c+d)^2$.

89. $(1+t-5t^2)^2-(6+t)^2$.

90. $9l^2+n^2-16k^2-d^2-6ln+8kd$.

What is c, if

91. $(x+3)$ is a factor of $x^2+cx-12$?

92. $(x-5)$ is a factor of x^2-7x+c?

93. $(2x-1)$ is a factor of $2x^2+5x+c$?

94. $(2x-3)$ is a factor of cx^2+x-6?

95. $(3x-2)$ is a factor of $3x^2+cx-4$?

96. $(5x+1)$ is a factor of $10x^2-13x+c$?

Find values of a for which the following expressions will factorise.

97. $5k^2+ak-3$.

98. $ax^2+23x+10$ (a positive).

99. $4x^2-9x+a$ (a positive).

100. $6x^2-ax-20$.

TEST PAPERS IV

A

1. (i) What is 7 per cent. of a?

(ii) What per cent. is x of y?

2. Solve (i) $\frac{1}{2}(x-1)-\frac{2}{3}(3x+8)=\frac{5x}{12}-\frac{x-2}{6}$,

(ii) $20x=9y$; $8+75x=63y$.

3. Simplify (i) $4l^2-9m^2-7m^2-7l^2$,

(ii) $\frac{15x^2}{y^3}\div\frac{5x}{3y^2}$,

(iii) $\frac{3(c-d)}{7}-\frac{5(d+c)}{7}$.

4. Factorise (i) $81p-54$, (ii) $x^2-2x-24$, (iii) $25c^2-16d^2$.

5. One of the sides of a right-angled triangle is 3 ft. 4 in., and the perimeter is 11 ft. 8 in. Find the hypotenuse.

B

1. (i) If K per cent. of a number is d, what is the number?

(ii) An article costs £A. The profit is x per cent. ; what is the selling price?

2. Solve (i) $0{\cdot}4(2t-1{\cdot}5)-0{\cdot}5(0{\cdot}3-t)=1{\cdot}2t$,

(ii) $5x+6y-21=4x-5y+56=7x+63$.

3. Simplify (i) $(l^2-3m^2)-(l^2-3m)$,

(ii) $\left(\frac{A^2}{P^3Q}\div\frac{5AX^2}{PQ^3}\right)\times\frac{10X^5}{P^2Q}$.

4. Factorise (i) $5l+lm+5m+m^2$,

(ii) $15x^2-32x-7$.

5. The distance of 85 miles between two towns is usually covered by a motor driver at a certain average speed. If this average speed were increased by 25 per cent. of its value, 30 minutes would be saved on the journey. Find the usual time and average speed.

6. A plane cuts off from a sphere of radius 10″ a segment of height h in. The fraction cut off is given by the function

$$\frac{1}{4}\left(\frac{h}{10}\right)^2\left(3-\frac{h}{10}\right).$$

Plot this function against h for $h=0$, 2, 4, 6, 8, 10. (Represent 2 by 1″ horizontally, and represent 0·5 by 5″ vertically.) What is the height of the segment, if its volume is $\frac{1}{4}$ of that of the whole sphere?

C

1. (i) Find b per cent. of c.

(ii) A trader sells a car for £x, making a profit of y per cent. How much did the car cost him?

2. Simplify (i) $\left\{\frac{(-x)^3}{5y^2}\div\frac{3(-y)^2}{(-x)^5}\right\}\div\frac{x^6}{y^6}$,

(ii) $\frac{2c+3}{12}-\frac{7c+5}{3}+\frac{3c-9}{18}$.

3. Solve (i) $\frac{1}{3}(2x+1)-\frac{1}{2}(6x-\frac{1}{3})+(2x+\frac{1}{2})=0$,

(ii) $\frac{2x-y}{2x+y}=\frac{13}{11}$, $3x+10y=4$.

4. Factorise (i) $2d^3-4d^2-2d$, (ii) c^3-c^2+c-1,

(iii) $3-31ab+10a^2b^2$.

5. Two men, X and Y, travel towards each other from towns 108 miles apart. X starts at 10 a.m. and walks at a uniform rate; Y starts at 1 p.m. and motors at a uniform rate. They meet at 3 p.m. If they had both started at 11 a.m., and travelled at the same rate as before, they would have met at 1.15 p.m. At what rate does Y travel?

6. Find (i) the H.C.F. of $7a^5bc^3$, $21a^2b^5c^4$ and $42a^2b^2c^7$,

(ii) the L.C.M. of $6a^2$, $5a^2$, $4a^2$, $3a^2$.

D

1. (i) If h cwt. of sugar costs £p and is sold for £q, what is (a) the profit, (b) the profit per cent.?

(ii) If a ship unloads x per cent. of its cargo, what percentage remains?

2. Solve (i) $x-\frac{1}{8}-\frac{1}{3}(4x+5)=3\left(\frac{x}{2}-\frac{1}{4}\right)$,

(ii) $\frac{4x+15y+4}{49}=\frac{2x-25y-6}{56}=8$.

3. Factorise (i) c^2z^2-3cz, (ii) $5x^2+7x-12$, (iii) $4x^2-(y-z)^2$.

4. Simplify (i) $(-5)^3$, (ii) $0-3(2a-3c)$,

(iii) $12\left[\frac{3s-5t}{36}-\frac{s-t}{18}\right]$.

5. A square plot of ground is surrounded by a gravel path 8 feet wide, the outside boundary of the gravel also forming a square. It is desired to double the width of the path, and it is found that $1\frac{2}{7}$ times as much gravel is required for the extension as for the original path. Find the length of the side of the plot of ground.

6. A train starts from rest at A and covers a distance of $\frac{t^2}{30}\left(1-\frac{t}{90}\right)$ miles from A in t min. Another train is moving towards the first with half its speed and is at a distance of $50-\frac{t^2}{60}\left(1-\frac{t}{90}\right)$ miles from A at the time t min. Find graphically when and how far from A the trains meet.

E

1. (i) If a ton of coal costs £c, and the dealer makes a profit of p shillings, what is (a) the selling price, (b) the gain per cent.?

(ii) A piece of elastic x feet long is stretched y inches. By how much per cent. is its length increased?

2. Simplify (i) $2x^2+3x+7-(-3x^2+15x+50)$,

(ii) $\frac{r-s}{rs}+\frac{s-t}{st}+\frac{t-r}{tr}$.

3. Solve (i) $1{\cdot}5(t-1)-\frac{t+2}{1{\cdot}5}+0{\cdot}25(t-3)=4$,

(ii) $x+y=9(x-y)$, $2x+y=3(2x-y)+1$.

4. The difference between a number consisting of two digits and the number formed by reversing the digits is 45. The sum of three times the tens digit and five times the units digit is 47. Find the number.

5. Factorise (i) $2cd - 3ac + 10ad - 15a^2$,
(ii) $1 + 5b - 24b^2$, (iii) $4x^2y^2 - 4xyz + z^2$.

6. Draw the graph of $y = -2 + 4x - x^2$ for values of x from -2 to 4. From your graph find (*a*) the greatest value of y, (*b*) the values of x between which $-2 + 4x - x^2$ is always positive.

F

1. A rectangular box is a ft. long, b ft. broad and c ft. high. It is enlarged by increasing its length by x per cent., its breadth by y per cent. and its height by z per cent. What per cent. is the new volume of the old volume?

2. Solve (i) $\frac{1}{4}(2x - 9) - \frac{3}{5}\left(2x + \frac{1}{2}\right) + \frac{1}{3}(4x + 1) = 0$,
(ii) $\frac{x}{4} - \frac{y}{6} = 8$, $\frac{x}{6} + \frac{y}{4} = 1$.

3. Factorise (i) $3x^5y - 6x^4y^2 + 9x^3y^3$, (ii) $10x^2 - 37x + 21$,
(iii) $2x^2 - 288$.

4. Simplify (i) $\left(\frac{3a^3}{7bc^2} \div \frac{9a}{28bc^7}\right) \times \frac{3a^2}{4c^2}$,
(ii) $15\left[\frac{a-2}{3} + \frac{3a-1}{5}\right]$.

5. A certain alloy contains 6 parts by weight of a metal A and 5 parts by weight of a metal B; another alloy contains 7 parts by weight of A and 13 parts by weight of B. If these alloys are melted and mixed together, how many pounds of the second alloy must be mixed with 11 pounds of the first alloy to make a mixture which contains 40 per cent. of A?

6. Find (i) the H.C.F. of $76r^2s^4$ and $95r^3s^2$,
(ii) the L.C.M. of $3xy^2z^2$, $2x^2yzt$ and $4xy^2zt^3$.

G

1. A square plate has a side of length a ft. When heated, each side expands $\frac{1}{e}$ inches. What is the new area of the plate, and by how much per cent. has the area increased?

2. (i) Expand and simplify $(2x - 1)^2 + (3x + 2)^2 + (4x - 3)^2$,
(ii) Divide $-35 + 22y + 23y^2 - 10y^3 - 3y^4$ by $7 - y^2 - 3y$.

3. Solve (i) $3(3x - 1{\cdot}5) + 4(6x - 1{\cdot}65) = 7(3x + 0{\cdot}9)$,
(ii) $\frac{2}{x} - 3y = 1$, $\frac{1}{3x} + 2y = 4$.

4. Simplify (i) $\frac{3}{5}\times\frac{2x^5y^2}{21z^4}\div\frac{11xy}{70z}$,

(ii) $\left(\frac{3}{x}\right)-\left(\frac{3}{x}\right)^2-\left(\frac{3}{x}\right)^3$.

5. Factorise (i) $an^2-5a-5b+bn^2$, (ii) $3p^2+10q^4-13pq^2$,

(iii) $4(p-q)^2-9(p+q)^2$.

6. A fraction is such that if 2 is added to the numerator and 5 to the denominator, the value of the fraction becomes $\frac{1}{2}$. If the numerator of the original fraction is trebled and the denominator increased by 15, the value of the resulting fraction is $\frac{1}{3}$. Find the original fraction.

H

1. Two kinds of tea are mixed in the ratio $l : m$. How much of each kind will there be in k lb. of the mixture?

2. Solve (i) $\frac{x-7}{2}-\frac{7-2x}{5}=6{\cdot}8$,

(ii) $2x=2{\cdot}1+3y$, $3(x+y-0{\cdot}2)=-y$.

3. (i) Expand and arrange in descending powers of x

$$(4x^2-2x-5)(3x^2+x-2)+(2x^2+1)^2,$$

(ii) Divide

$$18x^5-6x^4-121x^3-63x^2+141x+108 \text{ by } 3x^2-5x-9.$$

4. Factorise (i) $km+(l-m)^2-kl$, (ii) $2-x^3-2x^2+x$,

(iii) $4l^2m^2+9n^2+12lmn$.

5. Two men, X and Y, travel towards each other from towns 120 miles apart. X starts at 8 a.m. and cycles at a uniform rate; Y starts at 10.30 a.m. and motors at a uniform rate. They meet at 12.6 p.m. If they had both started at 9.45 a.m., and travelled at the same rate as before, they would have met at 12.9 p.m. At what rate did Y travel?

6. Draw a graph from the following table to show the relation between the total annual output of cars from a certain motor factory and the corresponding cost of production for each car:

Annual output of cars -	1000	2000	3000	4000	5000	6000
Cost per car in £ -	900	660	520	440	410	400

The manufacturer finds that on fixing the selling price of each car at £480 he can find a market for 3200 cars in the year. Find from your graph the profit or loss per car.

I

1. Superior sugar is mixed with inferior sugar in the ratio $p : q$. What weight of the mixture contains x lb. of superior sugar?

2. Solve (i) $\frac{2}{15}\left(12-\frac{x}{2}\right)+\frac{5}{3}\left(2\frac{1}{4}-\frac{x}{4}\right)=1$,

(ii) $\frac{1}{2}\left(\frac{3x}{2}-\frac{2y}{3}\right)=11,\quad x+\frac{y}{9}=6\frac{1}{3}$.

3. Factorise (i) l^3-l^2-42l, (ii) $20+7x-6x^2$,

(iii) $54a^4-24c^2$.

4. Simplify (i) $\sqrt[3]{K^{18}}-\sqrt[9]{-K^{54}}$,

(ii) $\frac{a^2}{b^2c^2}\div\frac{ax^3}{bc^3}\times\frac{b^4s^3}{ac}\times\frac{x^3}{b^3s^3}$.

5. A boy spent £1 4s. on rowing, always hiring a boat at 1s. 6d. per hour or a better one at 2s. per hour. Had he spent on the first what he actually spent on the second, and on the second what he actually spent on the first, he could have had two hours more on the water. How much did he spend on the better boat?

6. A party of tourists set out for a station 4 miles distant and go at the rate of 4 miles an hour. After going $\frac{2}{3}$ mile, one of them has to return to the starting point; at what rate must he now travel in order to reach the station at the same time as the others? Assume that all speeds are uniform.

J

1. A man can plough a field alone in A days; his son could plough it alone in B days. How long would it take them, working together, to plough the field?

2. Simplify (i) $8\left[\frac{3l-4m}{18}-1+\frac{2l+5m}{3}\right]$,

(ii) $\frac{r-s}{2rs}+\frac{s-t}{4st}+\frac{t-r}{3tr}$.

3. Solve (i) $1{\cdot}25\left(\frac{2z}{3}-1\right)+0{\cdot}75\left(\frac{z}{3}-0{\cdot}5\right)=\frac{z-9}{3}$.

Give the answer correct to two places of decimals.

(ii) $\frac{2x}{7}+y=\frac{4x}{3}+6y-1=-\frac{3y}{2}-2$.

4. Factorise (i) $(x^2+xy)^2-(xy+y^2)^2$, (ii) $10a^2-63ab+18b^2$.

5. The difference between a number consisting of two digits and the number formed by reversing the digits is 27. The sum of five times the tens digit and seven times the units digit is 51. Find the number.

6. Find (i) the H.C.F. of $14xy^3$, $22x^2y^2$ and $26x^3y^4$,

(ii) the L.C.M. of $12x^3y^2$, $3xy^5$ and $8x^2y^2z$.

K

1. The adult population of a town consists of a men and b women. The average age of the men is x years; the average age of the women is y years. What will be the average age of the whole adult population?

2. Solve (i) $\frac{2}{3}(2x-3)+\frac{3}{4}\left(2x+\frac{7}{3}\right)=\frac{5}{8}$,

(ii) $\frac{3(1-x)}{7}+\frac{2}{y}=-5$, $\frac{4(1+x)}{5}+\frac{5}{2y}+14=0$.

3. (i) Write down the square root of $x^{24}y^6$ and the cube root of the result. (ii) Find the L.C.M. of $8l^2m^3n^4$, $5l^4mn^4$ and $12lm^3n^2$.

4. Factorise (i) $x^4-2x^3-63x^2$, (ii) $lm(x^2+y^2)+xy(l^2+m^2)$, (iii) $81c^2-144(c-d)^2$.

5. In a race of 100 yd. when X and Y start level, X passes the winning post $\frac{2}{5}$ sec. before Y, but if X gives Y a start of 5 yd., Y can win by $\frac{1}{5}$ sec. Find the number of seconds each takes to run 100 yd.

6. Draw the graph of x^2-x+2 for values of x from -2 to 3. Find the minimum value of x^2-x+2 and solve $x^2-x-1=0$.

L

1. £A amounts to £B in C years at Simple Interest. What is the rate per cent. per annum?

2. Solve (i) $\frac{1}{10}(x-3)-\frac{1}{4}(x+7)=\frac{1}{4}\left(\frac{x}{5}+\frac{3}{4}\right)-\frac{1}{8}(3x-4)$,

(ii) $\frac{10}{x}-\frac{6}{y}-25=\frac{8}{x}+\frac{27}{y}=-6$.

3. (i) Multiply $3b^2-5ab+2a^2$ by $6a^2+7ab-5b^2$.

(ii) Divide $3x^7+20x^5-34x^3+9x^6-34x^4+41x+20-19x^2$ by $x^4+7x^2-9x+3x^3-5$.

4. Simplify (i) $\frac{3l^2m^2}{10r^2s^2}\div\frac{12ls}{5mr}\times\frac{4rs^5}{9lm^5}$,

(ii) $66\left\{\frac{6(K-5L)}{11}-\frac{5(3L-2K)}{3}\right\}$.

5. Factorise (i) a^4+2a^2-3, (ii) $alm-ak-xyk+xylm$, (iii) $(x+y-2z)^2-(x-y+3z)^2$.

6. A fraction is such that if 4 is added to the numerator and 3 to the denominator, the value of the fraction becomes $\frac{3}{4}$. If the numerator of the original fraction is doubled and the denominator increased by 16, the value of the resulting fraction is $\frac{2}{3}$. Find the original fraction.

CHAPTER XVII

QUADRATIC EQUATIONS. EASY PROBLEMS LEADING TO QUADRATIC EQUATIONS

104. The degree of an equation is the degree of the highest term that occurs in it when it is cleared of fractions.

Thus $2x^3 - 5x^4 = 7$ is of the fourth degree; $2x^2 - x - 5 = 0$ is of the second degree; $x - \frac{7}{x} = 3$ is of the second degree, for before we estimate its degree we must clear of fractions by multiplying each side by x. The equation is then written $x^2 - 7 = 3x$, and is clearly of the second degree.

An equation of the second degree is called a **quadratic equation,** and we have already seen in Chapter XIV how approximate solutions of such equations may be obtained graphically.

In general, there is no algebraic process which enables us to find the exact solution of an equation of any degree, although approximate solutions may be obtained graphically or otherwise.

There is, however, one general theorem which frequently enables us to solve equations of higher degree than the first:

If one of the factors of the product of a finite number of finite factors is zero, then the product is zero. Conversely, if the product of a finite number of finite factors is zero, then one of the factors of the product must be zero.

It follows that an equation can be solved, if it is possible to write it in the form

Product of factors of the first degree $= 0$.

Example 1. *Solve* $x^2 = x + 6$.

We may write this in the form $x^2 - x - 6 = 0$, i.e. $(x - 3)(x + 2) = 0$. A value of x which makes either of the factors zero will make the product $(x - 3)(x + 2)$ zero, i.e. such a value will satisfy the equation.

The values of x required are therefore those which make $x - 3 = 0$ and $x + 2 = 0$, i.e. $x = 3$ and $x = -2$.

The only roots of the equation are 3 and -2.

Example 2. *Solve* $(x^2-4)(x^2-2x-15)=0$.

We have $(x+2)(x-2)(x-5)(x+3)=0$,

$\therefore\ x+2=0$, or $x-2=0$, or $x-5=0$, or $x+3=0$;

$\therefore\ x=-2$, or 2, or 5, or -3. The only roots are $-2, 2, 5, -3$.

Note 1. This argument should be given in full. If the pupil leaves out the step "$\therefore\ x+2=0$, or $x-2=0$, or $x-5=0$, or $x+3=0$", he may make mistakes in sign.

Note 2. It should be particularly noted that if a product is equal to any number other than zero, we know nothing about the individual factors of the product.

Thus, if all we know about two numbers x and y is that $xy=12$, it is impossible to find the value either of x or of y. There is an unlimited number of possible pairs of values.

105. The converse of the principle of factors is also true. Thus, if $x=3$ is a solution of an equation, $x-3$ must be a factor of the expression which is equal to zero.

Example 3. *Form the equation whose roots are p and q.*

The root p is derived from a factor $x-p$, and the root q from a factor $x-q$. The required equation is therefore $(x-p)(x-q)=0$, or

$$x^2-(p+q)x+pq=0. \qquad \text{.........................(i)}$$

Any quadratic equation may be written in the above form; we first bring all the terms to the left hand side, and then divide both sides by the coefficient of x^2, e.g. if the equation is

$$-2x^2+4=7x,$$

we first write it in the form

$$-2x^2-7x+4=0,$$

and then divide both sides by -2, getting

$$x^2+\tfrac{7}{2}x-2=0.$$

When the quadratic is in this form we see from (i) that

(*a*) **the sum of the roots, $\mathbf{p+q}$, = the coefficient of x with the sign changed ;**

(*b*) **the product of the roots, $\mathbf{pq}$, = the constant term.**

This gives a useful method for checking answers.

If the general quadratic equation is taken to be $\mathbf{ax^2+bx+c=0}$, it becomes $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, when put in the above form.

We therefore have as a general rule:

The sum of the roots $= -\dfrac{b}{a}$;

The product of the roots $= \dfrac{c}{a}$.

EXERCISE 50. a (*Oral*)

1. If $x=3$, what is the value of:

(i) $(x-2)(x+7)$, (ii) $(x-3)(5x+4)$, (iii) $(2x-7)(x-3)$?

2. What can you say about the value of y, if $xy=0$ and

(i) $x=5$, (ii) $x=0$, (iii) $x=-4$?

3. What can you say about the value of x, if

(i) $xy=10$, (ii) $(a-5)x=0$, (iii) $(x-3)(y-4)=0$?

Solve the equations:

4. $(x-2)(x-5)=0$. **5.** $(x+3)(x-6)=0$. **6.** $(x+6)(x+7)=0$.

7. $5x(x-5)=0$. **8.** $8y(y+2)=0$.

9. $(2x+3)(3x+5)=0$. **10.** $(5t-7)^2=0$.

11. $6(4a-3)(a+11)=0$. **12.** $9(6p-5)(8p+1)=0$.

13. $(x-1)(x-2)(x-3)=0$. **14.** $9z^2=0$.

15. $(x+2)(x+3)(x+4)=0$. **16.** $x(x+7)(3x-1)(4x-3)=0$.

17. $5(2z-11)(3z+8)(5z-1)=0$. **18.** $5(4x+13)^2=0$.

EXERCISE 50. b (*Oral*)

1. If $c=-4$, what is the value of:

(i) $(c-5)(c-3)$, (ii) $(c+4)(7c-3)$, (iii) $5(3c+7)(c+4)$?

2. What can you say about the value of x if $xy=0$ and

(i) $y=-3$, (ii) $y=0$, (iii) $y=1\frac{1}{2}$?

3. What can you say about the value of y, if

(i) $xy=42$, (ii) $(c+6)y=0$, (iii) $(x+2)(y+7)=0$?

Solve the equations:

4. $(x-3)(x-8)=0$. **5.** $(x+9)(x+5)=0$.

6. $4(y-1)(y+2)=0$. **7.** $6t(t-5)=0$.

8. $(4x+17)(9x-2)=0$. **9.** $3z(z+4)=0$.

10. $8(5z+6)(11z-3)=0$. **11.** $(2c-25)^2=0$.

12. $(3p-2)(11p-5)=0$. **13.** $(x+1)(x-2)(x+3)=0$.

14. $(x-3)(x-4)(x+5)=0$. **15.** $12(3x-13)^2=0$.

16. $4x^3=0$. **17.** $3x(x+9)(2x-11)(3x+5)=0$.

18. $(3y-11)(2y+5)(17y-3)=0$.

EXERCISE 51. a

Solve the equations :

1. $y^2+5y+6=0$. **2.** $x^2+x-12=0$. **3.** $x^2-16x+64=0$.
4. $3x^2+4x+1=0$. **5.** $a^2-9a+20=0$. **6.** $2y^2-5y+3=0$.
7. $2z^2+7z+6=0$. **8.** $3x^2-5x-2=0$. **9.** $x^2-14x+40=0$.
10. $2t^2+9t-5=0$. **11.** $3x^2-11x+6=0$. **12.** $6-7x-3x^2=0$.
13. $5z^2+7z-6=0$. **14.** $15c^2+7c-2=0$. **15.** $21=z^2-4z$.
16. $4x^2+20x+25=0$. **17.** $x=2(3x^2-1)$. **18.** $x(x+1)=20$.
19. $40=x(x-3)$. **20.** $-28=2x^2+15x$. **21.** $-81=30a+a^2$.
22. $4+3z=10z^2$. **23.** $x^2+2x=35$. **24.** $2(a^2+5)=9a$.
25. $6x^2+7=17x$. **26.** $-3=14x+15x^2$. **27.** $5=4x(x-2)$.
28. $2(c^2+1)=5c$. **29.** $x=2(x^2-14)$. **30.** $10a^2=23a+21$.
31. $14=t(4t+1)$. **32.** $37x=5(3x^2+4)$. **33.** $5-x=4x^2$.
34. $-5x^2=6+11x$. **35.** $3x^2+x=10$. **36.** $21x^2=23x-6$.

Form the equations whose roots are :

37. $3, 7$. **38.** $4, -8$. **39.** $-\frac{2}{3}, 9$.
40. $-2, -\frac{3}{4}$. **41.** $0, 2, -11$. **42.** $2, -2, 5$.

EXERCISE 51. b

Solve the equations :

1. $x^2+5x+4=0$. **2.** $x^2-x-2=0$. **3.** $x^2+3x-10=0$.
4. $2x^2-8x+8=0$. **5.** $3z^2+23z+30=0$. **6.** $t^2-16t+15=0$.
7. $7l^2+10l-8=0$. **8.** $5x^2-8x-21=0$. **9.** $5x^2-13x+6=0$.
10. $x^2+2x-24=0$. **11.** $90-x-x^2=0$. **12.** $4a^2-23a+30=0$.
13. $9x^2+42x+49=0$. **14.** $8-18y+7y^2=0$.
15. $21-19z-2z^2=0$. **16.** $21x^2+17x+2=0$.
17. $x(x-9)=36$. **18.** $2(6a^2+5)=23a$. **19.** $14x^2+47x=7$.
20. $5x^2-104x=21$. **21.** $2=23t-21t^2$. **22.** $-42=13z+z^2$.
23. $50t=24t^2+21$. **24.** $72=c^2-14c$. **25.** $x(5x+37)=-14$.
26. $30=a(4a+37)$. **27.** $16+43z=15z^2$. **28.** $35=2x(19-4x)$.
29. $5y^2+21+22y=0$. **30.** $15(x^2+1)=34x$.
31. $5=4x(3-x)$. **32.** $12z^2-10=7z$. **33.** $3-29x=10x^2$.
34. $5t=3(7t^2-2)$. **35.** $10x^2=18-57x$. **36.** $35=2x(4x+9)$.

Form the equations whose roots are :

37. $1, 9$. **38.** $5, -4$. **39.** $-\frac{1}{3}, 6$.
40. $-\frac{2}{3}, -\frac{1}{2}$. **41.** $2, 0, -2$. **42.** $0, 0, 6$.

SOLUTION BY COMPLETING THE SQUARE

106. Rational and irrational numbers. Positive and negative integers, and positive and negative fractions are called **rational numbers.** All other numbers, i.e. all numbers which cannot be expressed in an exact numerical form, are called **irrational numbers.** Thus $\sqrt{7}$, $\sqrt[3]{11}$ are irrational numbers.

It is not possible to resolve all quadratic expressions into factors of the first degree with rational coefficients. It is therefore necessary to consider a more general method than the method of solving by factors.

The general method is based on the fact that any quadratic equation may be put into the form

$$(x+l)^2=m,$$

where l and m are rational numbers, either positive or negative.

When an equation has been written in this form, we may either obtain the solutions by taking the square root of each side, or use factors which may have irrational coefficients.

Example 4. *Solve* (i) $(x+\frac{2}{3})^2=9$, (ii) $(x-4)^2=7$.

(i) Taking the square root of each side, we get

$$x+\tfrac{2}{3}=3 \text{ or } -3, \quad \text{.........................(i)}$$

$$\therefore\ x=2\tfrac{1}{3} \text{ or } -3\tfrac{2}{3}.$$

It should be carefully noted that the square root of $(x+\frac{2}{3})^2$ is $(x+\frac{2}{3})$ or $-(x+\frac{2}{3})$, and the square root of 9 is 3 or -3. It would at first sight appear that we should consider the four possibilities $x+\frac{2}{3}=3$, $x+\frac{2}{3}=-3$, $-(x+\frac{2}{3})=3$, $-(x+\frac{2}{3})=-3$, but it is easily seen that $x+\frac{2}{3}=3$, and $-(x+\frac{2}{3})=-3$ are equivalent statements. Similarly with $x+\frac{2}{3}=-3$ and $-(x+\frac{2}{3})=3$. Thus it is only necessary to write down the two forms given above in (i).

OR

$$(x+\tfrac{2}{3})^2-9=0, \quad \therefore\ (x+\tfrac{2}{3})^2-(3)^2=0,$$

$$\therefore\ (x+\tfrac{2}{3}+3)(x+\tfrac{2}{3}-3)=0,$$

$$\therefore\ x+\tfrac{2}{3}+3=0, \text{ or } x+\tfrac{2}{3}-3=0,$$

$$\therefore\ x=-3\tfrac{2}{3} \text{ or } 2\tfrac{1}{3}.$$

(ii) Taking the square root of each side, we get

$$x-4=\sqrt{7} \text{ or } x-4=-\sqrt{7},$$

$$\therefore\ x=4+\sqrt{7}=4+2{\cdot}646=6{\cdot}646 \text{ approximately},$$

$$\text{or } x=4-\sqrt{7}=4-2{\cdot}646=1{\cdot}354 \text{ approximately}.$$

OR

$$(x-4)^2-(\sqrt{7})^2=0, \quad \therefore (x-4+\sqrt{7})(x-4-\sqrt{7})=0,$$

$$\therefore x-4+\sqrt{7}=0, \text{ or } x-4-\sqrt{7}=0,$$

$$\therefore x=4-\sqrt{7}=1{\cdot}354 \text{ approx., or } x=4+\sqrt{7}=6{\cdot}646 \text{ approx.}$$

Note 1. If the numerical values of x are not required, the solution may be left in the form $4+\sqrt{7}$ or $4-\sqrt{7}$, usually written for brevity $4 \pm \sqrt{7}$.

Note 2. The value $2{\cdot}646$ of $\sqrt{7}$ may be obtained either by the usual arithmetical rule or from the square root tables, which are given at the end of the book.

107. In order to put a quadratic equation in the form $(x+l)^2=m$, we make use of the identities obtained in Ch. XV:

$$(x+a)^2=x^2+2ax+a^2, \quad (x-a)^2=x^2-2ax+a^2.$$

If therefore we have an expression x^2+2ax or x^2-2ax (i.e. **a quadratic expression in which the coefficient of x^2 is $+1$ and the constant term zero**), we may make the expression a perfect square by adding a^2, i.e. the square of half the coefficient of x.

Example 5. *Solve* $2x^2-2x-3=0$.

First obtain the equation in the form $x^2+2ax=$ a number, by dividing by the coefficient of x^2 and transposing the constant term.

We obtain $\qquad x^2-x-\frac{3}{2}=0, \text{ or } x^2-x=\frac{3}{2}.$

We now add $\left(-\frac{1}{2}\right)^2$, i.e. (half the coefficient of x)^2, to each side, in order to make the left-hand side a perfect square;

$$\therefore x^2-x+\left(-\frac{1}{2}\right)^2=\frac{3}{2}+\frac{1}{4},$$

$$\therefore \left(x-\frac{1}{2}\right)^2=\frac{7}{4}.$$

We proceed:

EITHER $\qquad \left(x-\frac{1}{2}\right)=\frac{\sqrt{7}}{2} \text{ or } \frac{-\sqrt{7}}{2},$

$$\therefore x=\frac{1}{2}+\frac{\sqrt{7}}{2} \text{ or } \frac{1}{2}-\frac{\sqrt{7}}{2},$$

$$\therefore x=\frac{3{\cdot}646}{2} \text{ or } \frac{-1{\cdot}646}{2}, \text{ i.e. } 1{\cdot}823 \text{ or } -0{\cdot}823 \text{ approx.}$$

OR $\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{7}}{2}\right)^2=0, \quad \therefore \left(x-\frac{1}{2}+\frac{\sqrt{7}}{2}\right)\left(x-\frac{1}{2}-\frac{\sqrt{7}}{2}\right)=0,$

$$\therefore x-\frac{1}{2}+\frac{\sqrt{7}}{2}=0, \text{ or } x-\frac{1}{2}-\frac{\sqrt{7}}{2}=0;$$

$$\therefore x=\frac{1}{2}-\frac{\sqrt{7}}{2}=-0{\cdot}823, \text{ or } x=\frac{1}{2}+\frac{\sqrt{7}}{2}=1{\cdot}823 \text{ approx.}$$

Note. The first arrangement is the shorter, and is usually preferred. If he uses this, the pupil must take care to write down both values of the square root.

The second arrangement reveals that the general method is essentially the same as the factor method. It is, in fact, the factor method when the factors may contain irrational coefficients.

Example 6. *Solve* $6x^2+3x-2=0$.

This may be written $x^2+\frac{x}{2}-\frac{1}{3}=0$, or $x^2+\frac{x}{2}=\frac{1}{3}$.

Add $(\frac{1}{4})^2$ to each side, then

$$x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2=\frac{1}{3}+\frac{1}{16}=\frac{19}{48}=\frac{57^*}{144}.$$

(* Notice this step very carefully. The object is to make the denominator a perfect square, so that the square root may be more easily calculated. The pupil must resist the temptation to find the square roots of 19 and 48 and then divide. This greatly increases the amount of working, and usually leads to a less accurate result.)

$$\therefore \left(x+\frac{1}{4}\right)^2=\frac{57}{144},$$

$$\therefore x+\frac{1}{4}=\frac{\sqrt{57}}{12} \text{ or } \frac{-\sqrt{57}}{12},$$

$$\therefore x=-\frac{1}{4}+\frac{\sqrt{57}}{12} \text{ or } -\frac{1}{4}-\frac{\sqrt{57}}{12},$$

$$\therefore x=\frac{-3+\sqrt{57}}{12}=\frac{4{\cdot}550}{12}=0{\cdot}379,$$

or
$$x=\frac{-3-\sqrt{57}}{12}=-\frac{10{\cdot}550}{12}=-0{\cdot}879,$$

in each case correct to 3 decimal places.

Example 7. *Solve* $2x^2+x+1=0$.

This may be written $x^2+\frac{x}{2}+\frac{1}{2}=0$, or $x^2+\frac{x}{2}=-\frac{1}{2}$.

Add $(\frac{1}{4})^2$ to each side, then

$$x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2=-\frac{1}{2}+\frac{1}{16}=-\frac{7}{16},$$

$$\therefore\ \left(x+\frac{1}{4}\right)^2=-\frac{7}{16},$$

$$\therefore\ x+\frac{1}{4}=\sqrt{-\frac{7}{16}}\ \text{or}\ -\sqrt{-\frac{7}{16}}.$$

But we cannot find any number the square of which is $-\frac{7}{16}$, or any other negative number,

$$\therefore\ x+\frac{1}{4}\ \text{cannot be calculated,}$$

$$\therefore\ x\ \text{cannot be calculated.}$$

It is therefore impossible to find any number which satisfies the given equation.

Note. If the equation is set with decimal coefficients, these may be replaced by vulgar fractions. Thus, to solve $0{\cdot}32x^2+2x=0{\cdot}4$, we may first replace the equation by the equivalent form

$$\frac{8x^2}{25}+2x=\frac{2}{5},\ \text{i.e.}\ 4x^2+25x=5.$$

We then proceed as above.

108. During the first reading of the book the pupil should be content to stop at this stage. At present the square root of a negative number is unintelligible to him ; it cannot be calculated ; it has not even been defined.

It may, however, be of interest to state that it is possible to invent a new class of numbers having the property that their squares are negative numbers. It is also possible to invent operations corresponding to addition, subtraction, multiplication and division, and these operations obey the fundamental laws of algebra.

These numbers are of great importance in higher mathematics ; they are called **imaginary** or **unreal numbers** and an expression such as $-\frac{1}{4}+\sqrt{\frac{-7}{16}}$ is called a complex number.

109. Numbers which are not imaginary are called **real** numbers. Thus $\sqrt{16}=4$ is real and rational, $\sqrt{7}$ is real but irrational, $\sqrt{-7}$ is unreal and irrational.

It will be shown in a later chapter that $\sqrt{a \times b}=\sqrt{a} \times \sqrt{b}$, and that $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$. If we assume that these results are true even when either a or b is negative, $\sqrt{\frac{-7}{16}}$ may be written $\frac{\sqrt{-7}}{4}$, $\sqrt{-9a^2}$ may be written $3a\sqrt{-1}$, etc.

The solution to Ex. 7 is usually written $x=\frac{-1 \pm \sqrt{-7}}{4}$.

Note. In the exercises which follow, equations with imaginary roots occur in Ex. 52 *c*, Nos. 31–48, only.

Solution by Formula

110. Another method of solution, by means of a formula, is given in Chapter XXI.

EXERCISE 52. a

Solve, giving the answers as whole numbers or fractions :

1. $(x-2)^2=25$. **2.** $(z+8)^2=49$. **3.** $(3x-2)^2=16$.

4. $(2x+1)^2=9$. **5.** $9(3x+5)^2=16$. **6.** $49(2t-7)^2=81$.

7. $(a-\frac{1}{2})^2=4$. **8.** $(x+\frac{2}{3})^2=64$. **9.** $(x+\frac{5}{3})^2=\frac{16}{9}$.

10. $(y-\frac{3}{5})^2=\frac{4}{25}$. **11.** $(x-\frac{3}{4})^2=\frac{25}{64}$. **12.** $(c+\frac{3}{2})^2=\frac{1}{16}$

Evaluate, correct to two places of decimals :

13. $2 \pm 2{\cdot}449$. **14.** $-3 \pm 4{\cdot}472$. **15.** $-5 \pm 7{\cdot}874$.

16. $9 \pm 5{\cdot}831$. **17.** $6 \pm 8{\cdot}888$. **18.** $-7 \pm 3{\cdot}317$.

19. $\frac{4 \pm 5{\cdot}831}{2}$. **20.** $\frac{-2 \pm 7{\cdot}211}{3}$. **21.** $\frac{8 \pm 5{\cdot}745}{7}$.

22. $\frac{7 \pm 9{\cdot}798}{5}$. **23.** $\frac{-4 \pm 7{\cdot}937}{6}$. **24.** $\frac{-9 \pm 4{\cdot}583}{8}$.

What numbers must be added to the expressions in Nos. 25–36 to make the result a perfect square? Of what is it then the square?

25. x^2-8x. **26.** x^2+6x. **27.** c^2-5c. **28.** y^2+7y.

29. $x^2-\frac{8x}{9}$. **30.** $x^2+\frac{5x}{6}$. **31.** $y^2-\frac{4y}{7}$. **32.** $y^2+\frac{5y}{2}$.

33. $z^2-\frac{7z}{3}$ **34.** $x^2+\frac{8x}{11}$. **35.** $x^2+\frac{4ax}{5}$. **36.** $x^2-\frac{2cx}{3}$.

Solve by completing the square, leaving the answers as whole numbers or fractions :

37. $x^2-12x=45$. **38.** $x^2+8x+12=0$. **39.** $x^2+3x-18=0$.
40. $y^2-7y-98=0$. **41.** $2y^2+3y-2=0$. **42.** $3z^2-7z-6=0$.
43. $2x^2-x=21$. **44.** $4x^2+11x-3=0$. **45.** $7z=3(1-2z^2)$.
46. $12(1-y^2)=7y$. **47.** $2x^2-ax=6a^2$. **48.** $3y^2+2cy-8c^2=0$.

Solve by completing the square, giving the answers correct to two places of decimals :

49. $x^2-6x+7=0$. **50.** $y^2+8y=10$. **51.** $a^2=5a-3$.
52. $x^2+3x=1$. **53.** $2x^2=5x+1$. **54.** $5c^2=4-2c$.
55. $4x^2-3x=3$. **56.** $3a^2=10a-4$. **57.** $6x^2=7x+2$.
58. $10x^2=3x+2$. **59.** $7y^2=12y-4$. **60.** $9x^2=2-5x$.

EXERCISE 52. b

Solve, giving the answers as whole numbers or fractions :

1. $(x-5)^2=4$. **2.** $(z+7)^2=64$. **3.** $(3x+4)^2=4$.
4. $(2x-3)^2=1$. **5.** $25(3x-4)^2=9$. **6.** $16(2c-9)^2=49$.
7. $(b-\frac{2}{3})^2=9$. **8.** $(x+\frac{3}{4})^2=81$. **9.** $(x-\frac{3}{2})^2=\frac{25}{16}$.
10. $(z+\frac{2}{3})^2=\frac{1}{4}$. **11.** $(t-\frac{4}{5})^2=\frac{4}{9}$. **12.** $(x+\frac{4}{3})^2=\frac{4}{25}$.

Evaluate, correct to two places of decimals :

13. $3\pm9{\cdot}086$. **14.** $-1\pm6{\cdot}633$. **15.** $-4\pm5{\cdot}916$.
16. $8\pm2{\cdot}449$. **17.** $-9\pm5{\cdot}292$. **18.** $2\pm8{\cdot}944$.
19. $\dfrac{5\pm7{\cdot}348}{6}$. **20.** $\dfrac{-3\pm8{\cdot}185}{5}$. **21.** $\dfrac{-7\pm3{\cdot}873}{9}$.
22. $\dfrac{6\pm9{\cdot}849}{7}$. **23.** $\dfrac{4\pm3{\cdot}317}{11}$. **24.** $\dfrac{-2\pm7{\cdot}141}{4}$.

What number must be added to the expressions in Nos. 25–36 to make the result a perfect square? Of what is it then the square?

25. x^2+10x. **26.** x^2-4x. **27.** y^2-3y. **28.** z^2+9z.
29. $x^2+\dfrac{5x}{8}$. **30.** $x^2-\dfrac{6x}{7}$. **31.** $y^2+\dfrac{8y}{11}$. **32.** $t^2-\dfrac{3t}{2}$.
33. $c^2+\dfrac{9c}{5}$. **34.** $z^2-\dfrac{12z}{13}$. **35.** $x^2-\dfrac{6ax}{7}$. **36.** $x^2+\dfrac{3cx}{5}$.

Solve by completing the square, leaving the answers as whole numbers or fractions :

37. $x^2-8x=48$. **38.** $y^2+12y+27=0$. **39.** $x^2-5x-84=0$.
40. $z^2+9z-52=0$. **41.** $2x^2-5x+2=0$. **42.** $3x^2+8x+4=0$.
43. $4x^2-3x=52$. **44.** $5x^2-9x=18$. **45.** $3(1-z^2)=8z$.

46. $76x = 5(1 + 3x^2)$. **47.** $22ax - 3x^2 = 19a^2$. **48.** $2z^2 + dz - 36d^2 = 0$.

Solve by completing the square, giving the answers correct to two places of decimals :

49. $x^2 + 4x + 1 = 0$. **50.** $y^2 - 6y - 11 = 0$. **51.** $z^2 = 7z + 4$.
52. $c^2 + 9c + 2 = 0$. **53.** $3x^2 = 3 - 7x$. **54.** $6c^2 = 7c + 1$.
55. $2c^2 = 5c - 1$. **56.** $7t^2 = 12t - 2$. **57.** $8x^2 = 2 - x$.
58. $9y^2 = 16y - 5$. **59.** $4x^2 + 7x + 2 = 0$. **60.** $10z^2 = 1 - 7z$.

EXERCISE 52. c

Solve the following equations by factors, if possible, otherwise by completing the square. If the roots are irrational, give the answers correct to two places of decimals, except in Nos. 31–48, in which the answer may be left in a form containing the square root, e.g. $\frac{3 \pm \sqrt{7}}{5}$.

1. $18x^2 = 5 - 9x$. **2.** $x^2 - 4x + 1 = 0$. **3.** $(x + 17)^2 = 16x^2$.
4. $3t^2 + 5t = 5$. **5.** $2x^2 + 4x = 3$. **6.** $2x^2 = 9x + 12$.
7. $(2x - 3)^2 = 9x^2$. **8.** $5x^2 = 3 + 4x$. **9.** $48x^2 = 22x + 15$.
10. $10x^2 = 11x + 2$. **11.** $(4x + 7)^2 = 4x^2$. **12.** $8x^2 - 18x + 6 = 0$.
13. $(x + 2)^3 = 117 + (x - 1)^3$. **14.** $(5x - 1)(10x + 7) = 5(3x + 1)$.
15. $11 = 20x^2 + 14x$. **16.** $10x + 8 = 75x^2$. **17.** $6x^2 + 3x = 34$.
18. $3(4x - 7)^2 - 4(2x - 3)^2 = 2(2x - 3)(x - 3) + 3$.
19. $(3x + 5)^2 = (5x - 3)^2$. **20.** $9x^2 - 39x + 35 = 0$.
21. $21(x + 3)^2 - 40(x + 3) = 21$. **22.** $(5x - 1)(10x + 3) = 2$.
23. $7x^2 + 26 = 34x$. **24.** $(4x - 3)(8x + 5) = 13$.
25. $2x(2x + 1) - 1 = 3(2x + 1)$. **26.** $(7x - 3)^2 = (5x + 1)^2$.
27. $2(5x - 2)(3 - 5x) = 15(7 - 5x)(x + 1)$.
28. $2{\cdot}72x^2 + 4x + 1 = 0$. **29.** $63x^2 - 48x + 8 = 0$.
30. $(4x - 3)^2 = 16x$. **31.** $x^2 + 7ax + a^2 = 0$.
32. $x^2 + 4x + 10 = 0$. **33.** $x^2 + ax + 7a^2 = 0$. **34.** $4x^2 - 3x = 6$.
35. $3x^2 + 2x + 5 = 0$. **36.** $40x^2 + 8x = 15$. **37.** $4x^2 + x + 3 = 0$.
38. $5x^2 - x + 2 = 0$. **39.** $2x^2 - 3ax = 3a^2$. **40.** $2x^2 - 3x + 5 = 0$.
41. $16x^2 + 3 = 25x$. **42.** $x^2 + 14 = 7x$. **43.** $2x^2 = 3x + 7$.
44. $8x^2 - 4x + 1 = 0$. **45.** $41x^2 = 5a^2 + ax$. **46.** $24x^2 + 12x + 1 = 0$.
47. $6x^2 + 3x + 1 = 0$. **48.** $1 - 3(x - 1)(x - 2) = 11(x - 1)$.

111. Easy problems leading to quadratic equations. We shall now give some problems in which the solution depends on a quadratic equation. It will be seen that each solution of the quadratic equation does not necessarily give a solution of the problem. The

quadratic equation may have no real roots, in which case there is no solution to the problem ; or one solution of the equation may be negative, and a negative number may be inadmissible as a solution of the problem ; and so on.

In every such case the solutions of the equation give us the only possible values of the unknown and we must then decide which of these, if any, give solutions of the problem.

Example 8. *A number of two digits is less than three times the product of its digits by* 8, *and the digit in the tens' place exceeds the digit in the units' place by two. Find the number.*

Let x be the digit in the units' place, then $x+2$ is the digit in the tens' place and the number is $10(x+2)+x=11x+20$.

Three times the product of the digits is $3x(x+2)$;

$$\therefore\ 3x(x+2)=11x+20+8,$$
$$\therefore\ 3x^2+6x=11x+20+8,$$
$$\therefore\ 3x^2-5x-28=0,\quad \therefore\ (x-4)(3x+7)=0,$$
$$\therefore\ x-4=0 \text{ or } 3x+7=0,\quad \therefore\ x=4 \text{ or } -\tfrac{7}{3}.$$

The solution $-\frac{7}{3}$ is inadmissible because a digit must be a positive integer, $\therefore\ x=4$ is the only solution.

If $x=4$, $x+2=6$ and the number is 64. The check is left to the pupil.

Example 9. *A man has* 30 *yd. of fencing. With it he encloses* 100 *sq. yd. of his garden, the boundary fence forming one side of the enclosure. What are the possible dimensions of the enclosure?*

Let x yd. be the breadth of the enclosure.

Then $(30-2x)$ yd. is its length.

We then have

$$100=x(30-2x)=30x-2x^2,$$
$$\therefore\ 2x^2-30x+100=0,$$
$$\therefore\quad x^2-15x+50=0.$$

(Note this step. Pupils often make their work unnecessarily heavy by omitting to divide through by a common numerical factor.)

$$\therefore\ (x-5)(x-10)=0,$$
$$\therefore\ x-5=0 \text{ or } x-10=0,$$
$$\therefore\ x=5 \text{ or } 10.$$

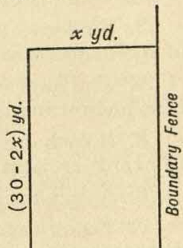


FIG. 16.

If $x=10$, the breadth is 10 yd., the length is 10 yd., and the area is 100 sq. yd.

If $x=5$, the breadth is 5 yd., the length is 20 yd., and the area is 100 sq. yd.

Both solutions are valid, and there are two possible ways of forming the enclosure.

Important note. The answers to most problems are represented by rational numbers. It is possible that the solution may be represented by an irrational number, but this does not occur very frequently. It is probable, therefore, that the quadratic equation which arises out of the problem can be solved by factors.

The pupil should always satisfy himself by checking his work, or otherwise, that a solution by factors is impossible, before he proceeds to solve by completing the square. The solution of the equation is more easily found, if x is a small number than if x is a large number, and it is well to bear this in mind when choosing which unknown is to be represented by x.

112. Many problems leading to quadratic equations introduce fractions, and the work is too difficult to be given at this stage. Such problems are considered in Chapter XXVI.

EXERCISE 53. a

1. Find two numbers differing by 4, such that the sum of their squares is 170.

2. Find two numbers differing by 2, such that twice the square of the smaller exceeds the square of the larger by 73.

3. Find two consecutive positive integers, such that the sum of their squares is 265.

4. The sum of a number and its square is twelve times the next highest number. Find it.

5. If each one of a family sends a card to each of the rest, and 182 cards are sent, how many are there in the family?

6. A number plus 5 times its square is 616. Find it.

7. Twenty-two years hence a man's age will be the square of what it was 34 years ago. Find his present age.

8. A house bought for £$100x$ is sold for £672 at a profit of $2x$ per cent. Find x.

9. A number of two digits is less by 155 than the square of the number formed by reversing the digits, and the digit in the tens' place exceeds the other digit by 3. Find the number.

10. X sold goods which cost £5 to Y at a gain of c per cent., Y sold them back to X at a gain of c per cent. As a result X lost 11s. Find c.

11. The adjacent sides of a rectangular plot differ in length by 4 yd.; the area is 780 sq. yd. Find its dimensions.

12. The perimeter of one square exceeds that of another by 60 yd.; the area of the larger square is less by 61 sq. yd. than 5 times the area of the smaller. Find the lengths of their sides.

13. A stone is projected vertically upwards, so that its height above the ground after t sec. is $(72t - 16t^2)$ ft. After what time is it 81 ft. above the ground? When does it strike the ground?

14. A man is $18x$ years old and his son is $2x^2$ years old. When he was $3x^2$ years old, his son was $x+4$ years old. How old is he now?

15. Of 6 consecutive positive integers, the product of the three largest exceeds the product of the others by 1644. Find them.

16. Divide a line 10 cm. long internally into 2 parts, so that the square on one part may be 3 times the square on the other part.

17. A man has 120 yd. of fencing. With it he encloses 1152 sq. yd. of his garden, the boundary fence forming one side of the enclosure. What are the possible dimensions of the enclosure?

18. A polygon of x sides has $\frac{1}{2}x(x-3)$ diagonals. How many sides has a polygon with 135 diagonals?

19. A rectangular grass plot is 90 ft. long and 84 ft. wide. It is surrounded by a walk of uniform width. The area of the plot is equal to the area of the walk. Find the width of the walk.

20. The perimeter of a rectangle is 30 in., and the sum of the two squares described on adjacent sides exceeds twice the area of the rectangle by 9 sq. in. Find the length of the rectangle.

21. The total external surface area of the sides and base of an open rectangular tank is 203 sq. ft. The base of the tank is square, and its edge is $1\frac{1}{2}$ ft. greater than the height. Find the dimensions of the tank. Neglect the thickness of the material.

22. A walks at 5 ft. per sec. and takes 102 sec. to go from one corner of a rectangular field to the opposite corner along two sides. B walks at the same rate along the diagonal of the field and takes 24 sec. less to reach the opposite corner. Find the length of the field.

23. A square plot of land is bought at 8d. a sq. yd., and a fence is placed all round it costing 10d. a yd. If the total expense is £35, find in yards the length of a side of the plot.

24. The sum of the first x whole numbers is $\frac{1}{2}x(x+1)$. How many must be taken to give 351 as the sum?

25. Two straight roads cross at right angles at O. Two men, A and B (one on each road) approach O at constant speeds, B walking and A cycling 3 times as fast. When A is 25 miles from O, B is 10 miles from O, and 2 hours later they are 5 miles from each other measured in a straight line, neither of them having reached O. Find their speeds.

26. $ABCD$ is a square of side 6 in.; L, M are points on the sides DC, CB respectively, such that $DL = BM = x$ in. If the area of the $\triangle ADL$ is three-quarters of the area of the $\triangle CLM$, find x.

EXERCISE 53. b

1. Find two consecutive positive integers, such that the sum of their squares is 145.

2. Find two consecutive odd numbers whose product is 323.

3. Find two numbers differing by 9, such that the sum of their squares is 185.

4. The sum of a positive integer plus its square is 7 times the next highest number. Find it.

5. Find two numbers differing by 3, such that 3 times the square of the smaller exceeds the square of the larger by 47.

6. Seven times the square of a number minus 3 times the number equals 54. Find the number.

7. The adjacent sides of a rectangle differ by 7 in.; if the area is 638 sq. in., find the dimensions.

8. Three years hence a boy's age will be 4 times the square of what it was 11 years ago. Find his present age.

9. The perimeter of one square exceeds that of another by 44 yd.; the area of the larger square exceeds 8 times the area of the smaller by 1 sq. yd. Find the lengths of their sides.

10. A diamond ring bought for £x is sold for £31 5s. at a profit of x per cent. Find x.

11. A number of two digits is less by 699 than 5 times the square of the number formed by reversing the digits, and the digit in the tens' place exceeds the other digit by one. Find the number.

12. X sold goods which cost £10 to Y at a gain of c per cent.; Y sold them back to X at a loss of c per cent. As a result Y lost 10s. 6d. Find c.

13. If $7x(x+1)^\circ$ W. of N. is the same as $2x^\circ$ N. of W., find x.

14. A stone is projected vertically upwards so that its height above the ground after t sec. is $(108t - 16t^2)$ ft. After what times is it 126 ft. above the ground?

15. A square floor is covered with carpet, except a border 18″ wide, round the carpet, which is covered with linoleum. The carpet and linoleum cost respectively 9s. and 5s. per sq. yd. The whole cost is £9 9s. Find the length of the floor.

16. Divide a line 8 in. long internally into two parts, so that the rectangle contained by the whole line and one part may be twice the square on the other part.

17. The sum of the first x whole numbers is $\frac{1}{2}x(x+1)$. How many must be taken to give 253 as the sum?

18. A path $2\frac{1}{2}$ ft. wide surrounds a square plot of grass and its area is $1\frac{1}{4}$ times that of the grass. Find the length of the plot.

19. A piece of wire 52 cm. long is cut into two parts, each of which is bent into the form of a square. The total area enclosed by the two squares is 97 sq. cm. Find the sides of the two squares.

20. The length of the diagonal of a rectangle is 25 ft. and the difference of the lengths of the sides is 3 ft. Find the length of each side in feet, to one decimal place.

21. A square plot of land is bought at 4d. a sq. yd., and a fence is placed all round it costing 9d. a yd. If the total expense is £69, find in yards the length of a side of the plot.

22. The price of petrol is reduced by x per cent., and a man uses x per cent. more. As a result his petrol bill is reduced from £25 to £24 15s. 0d. Find x.

23. If the edges of a rectangular box were increased by 2 in., 3 in., and 5 in. respectively, the box would become a cube and its capacity would be increased by 720 cu. in. Find its dimensions.

24. A carpet, whose length is $1\frac{1}{6}$ times its width, is laid on the floor of a rectangular room, with a margin of 1 ft. all round. The area of the floor is 4 times that of the margin. Find the width of the room.

25. A square floor is covered with carpet, except a border 12″ wide round the carpet, which is covered with linoleum. The carpet and linoleum cost respectively 10s. and 4s. 6d. per sq. yd. The whole cost is £19 18s. Find the length of the floor.

26. Two straight roads cross at right angles at O. Two men, A and B (one on each road), approach O at constant speeds, A walking and B cycling 4 times as fast. When A is 20 miles from O, B is 64 miles from O, and 4 hours later they are 10 miles from each other measured in a straight line, neither of them having reached O. Find their speeds.

CHAPTER XVIII

GRAPHS (*Continued*)

113. Graphical solution of equations. In Chapter XIV, it was shown that by drawing the graph of $y=6x^2-7x-11$, it was possible to obtain approximate values of the roots of any equation of the type $6x^2-7x-11=a$, a being a constant. This is a particular case of a more general theorem, which we now proceed to discuss.

If we have a pair of simultaneous equations in x and y, and if the graphs corresponding to the equations are drawn with the same axes and with the same scales, then, at the points of intersection of the graphs,

1. **The coordinates are roots of the simultaneous equations ;**
2. **The x-coordinates are roots of the equation in x obtained by eliminating y from the two equations ;**
3. **The y-coordinates are roots of the equation in y obtained by eliminating x from the two equations.**

114. We shall prove these statements for a particular pair of equations, but it is clear that the method is quite general, provided that the eliminations can be performed.

Let us consider the equations $y=3x^2-6x+3$, $2x=3y-5$. The graphs corresponding to the equations are drawn in Fig. 17 on p. 223.

The graphs meet at P and Q, and PN, QM are the perpendiculars drawn from P and Q respectively to the axis Ox.

P lies on the curve, $\therefore\ NP=3\,.\,ON^2-6\,.\,ON+3$. ....(i)

P lies on the st. line, $\therefore\ 2\,.\,ON=3\,.\,NP-5$. ....(ii)

Thus, $x=ON$, $y=NP$ satisfy both the equations $y=3x^2-6x+3$ and $2x=3y-5$. It may similarly be shown that $x=OM$, $y=MQ$ satisfy these equations. This is the first result given above.

Also from (ii), $$NP=\frac{2\,.\,ON+5}{3}.$$

Substituting this value of NP in (i), we have

$$\frac{2 \,.\, ON + 5}{3} = 3 \,.\, ON^2 - 6 \,.\, ON + 3,$$

i.e. $x = ON$ satisfies the equation

$$\frac{2x + 5}{3} = 3x^2 - 6x + 3. \quad \text{.......................(iii)}$$

It may similarly be shown that $x = OM$ satisfies (iii).

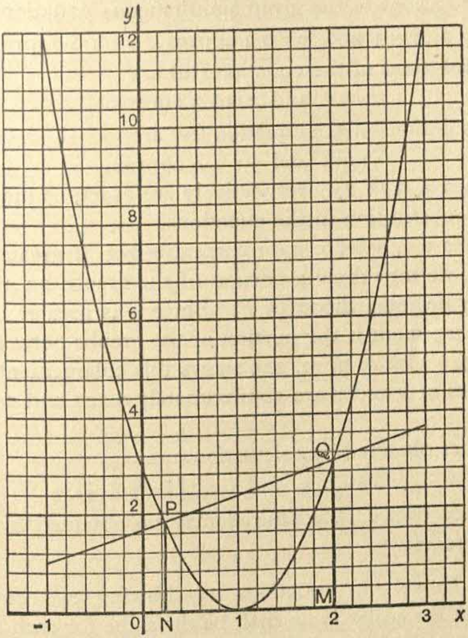


FIG. 17.

But (iii) is the equation obtained by eliminating y from the given equations. This is the second result given above.

Again, from (ii) $\quad ON = \dfrac{3 \,.\, NP - 5}{2}.$

Substituting this value of ON in (i), we have

$$NP = 3\left(\frac{3 \,.\, NP - 5}{2}\right)^2 - 6\left(\frac{3 \,.\, NP - 5}{2}\right) + 3,$$

i.e. $y=NP$ satisfies the equation

$$y=3\left(\frac{3y-5}{2}\right)^2-6\left(\frac{3y-5}{2}\right)+3. \qquad \text{...............(iv)}$$

It may similarly be shown that $y=MQ$ satisfies (iv).

But (iv) is the equation obtained by eliminating x from the given equations. This is the third result given above.

From Fig. 17 it is easily seen that

(1) The solutions of the given simultaneous equations are

$$x=2,\ y=3 \text{ and } x=0{\cdot}2 \text{ approx., } y=1{\cdot}8 \text{ approx.}$$

(2) The solutions of the equation (iii) are

$$x=2 \text{ and } x=0{\cdot}2 \text{ approx.}$$

(3) The solutions of the equation (iv) are

$$y=3 \text{ and } y=1{\cdot}8 \text{ approx.}$$

The values 2, 3 of x, y respectively are exact, as may easily be verified by substitution in the equations.

The values 0·2, 1·8 are approximate only ; if greater accuracy is required, we may draw a portion of the graphs on a very large scale in the neighbourhood of P. Since x is greater than 0·2, a suitable enlargement is the portion of the graphs between $x=0{\cdot}21$ and $x=0{\cdot}24$. Fig. 18 on p. 225 shows this enlargement.

The values of x and y are approximately 0·222 and 1·81 respectively.

(The values obtained by calculation are

$$x=\tfrac{2}{9}=0{\cdot}\dot{2}, \text{ and } y=1\tfrac{22}{27}=1{\cdot}\dot{8}1\dot{4}).$$

Any degree of accuracy desired may be obtained by repeating the above process.

Note 1. In solving equations by drawing two graphs, it is essential that the scales for x shall be the same for each graph, and also that the scales for y shall be the same for each graph. But it is not necessary that the scale for x should be the same as the scale for y.

Note 2. When we speak of graphs "on the same diagram", it is implied that the same axes are used, and also that the same scales have been used for x for each graph, and also for y.

One important result which follows from the general theorem is that graphical solutions of any quadratic equation may be

obtained by drawing the graph of $y=x^2$ and the graph of a straight line. For the equation $ax^2+bx+c=0$ may be written

$$x^2=-\left(\frac{bx+c}{a}\right).$$

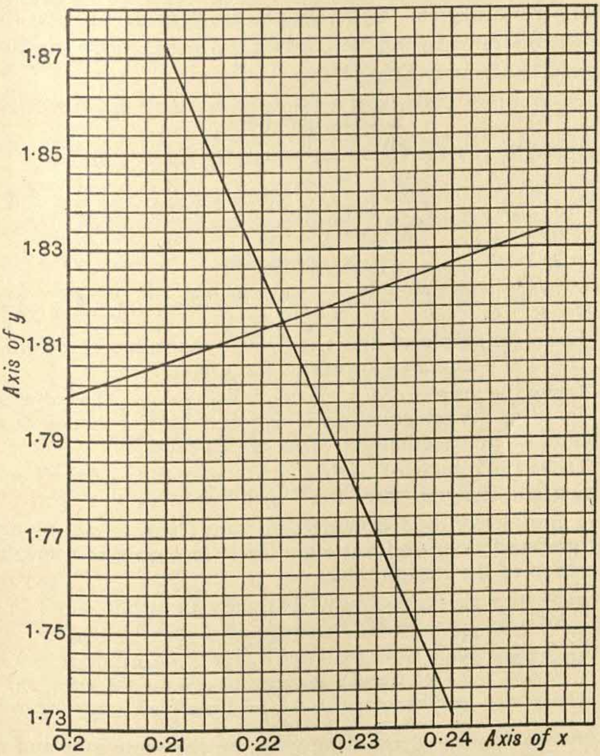


FIG. 18.

This is the equation obtained by eliminating y from the equations $y=x^2$ and $y=-\left(\frac{bx+c}{a}\right)$. It is therefore satisfied by the x-coordinates of the points of intersection of the graphs representing $y=x^2$ and $y=-\left(\frac{bx+c}{a}\right)$.

The advantage of this method is that the graph of $y=x^2$ may be accurately drawn on a large scale and kept for regular use. The graph of the straight line required in any instance may be quickly drawn on the same figure as the graph of $y=x^2$.

In this chapter we consider mainly graphs of expressions of the first and second degree. Other graphs are considered only if they can be easily drawn by simple plotting. More difficult graphs are considered in Chapter XXXIII.

EXERCISE 54. a

Solve graphically the following equations (Nos. 1–9):

1. $3y=2x-1$, $x+3y=5$.

2. $8x=3y$, $4x-y=2$.

3. $2x+6y=1$, $x-4y=4$.

4. $2x+9y=14$, $3x-3y=10$.

5. $4x-5y=28$, $x-y=6{\cdot}1$.

6. $3y-4x=9$, $4x+7y=11$.

7. $10y=7x^2-22x+18$, $10y=3x-4$.

8. $y=2x^2+2x-4$, $x+y+4=0$.

9. $y=2x^2-6x+3$, $y=x^2-2x-1$.

10. Draw the graphs of $2x+1$ and x^2-1 for values of x from -1 to 3. Solve $x^2-2x=2$ by means of the graph.

11. Draw the graphs of $y=x^2-x-6$ and $y=-4$ for values of x from -3 to 3. What equation is satisfied by the values of x at their points of intersection and what are its roots?

12. Draw the graphs of $y=(x-2)(6-x)$ and $7x-6y=7$ on the same diagram. Hence solve $6(x-2)(6-x)-7(x-1)=0$.

Solve graphically the following equations (Nos. 13–18) by drawing the graph of $y=x^2$ and a straight line. In each case write down the equation of the straight line.

13. $2x^2-5x=12$.

14. $2x^2-3x=1$.

15. $5x^2+2x=24$.

16. $5x^2+15x+11=0$.

17. $10x^2+x=27{\cdot}2$.

18. $11x^2-18x+8=0$.

19. Draw the graphs of $y=x+\dfrac{9}{x}$ and $y=\dfrac{3}{8}(x^2-25)$ for values of x between 7 and -7, omitting $x=0$, on the same axes and with the same scales. Show that the two curves intersect where

$$3x^3-8x^2-75x-72=0.$$

Solve this equation.

20. Draw the graph of $y=x^2-4x+5$ for values of x from -1 to 5. Use your graph to solve the equation $2x^2-8x+1=0$. By drawing a line through the origin, find the range of values of x within which the values of x are less than the corresponding values of y.

21. Draw the graph of $10y = 3x^2 - 7x$ for values of x from -2 to 4. Draw on the same diagram the graph of $2y = 1 - x$ and use your figure to determine (*a*) the values of x for which $3x^2 - 7x$ assumes the value of 7·5, (*b*) the values of x for which $3x^2 - 7x = 5 - 5x$.

22. Draw the graph of $y = \dfrac{4 - x^2}{5 + x}$ for values of x from -3 to 3. Use this graph to solve the equation $2x^2 + x - 3 = 0$. Explain your method.

23. Draw the graph of $y = \dfrac{3x^2 - 2}{x + 10}$ for values of x from -3 to 4. From the graph find solutions of the equation $6x^2 - 3x = 34$, indicating how you do it.

24. Draw the graphs of $5y = 7x - 4$ and $4y = x^3$ on the same diagram. Deduce the solutions of the equation $x^3 = 5{\cdot}6x - 3{\cdot}2$.

EXERCISE 54. b

Solve graphically the following equations (Nos. 1–9):

1. $5x = 2y - 3$, $y = 15x - 6$.   **2.** $4x - 5y = 16$, $8x + y = 10$.   **3.** $4y - x = 4$, $4x + 30y = 7$.

4. $15y - x = 9$, $3x + 5y = 13$.   **5.** $5x + 3y = 8{\cdot}8$, $7x + 5y = 10{\cdot}4$.   **6.** $10x = 3y + 14$, $5x + 2y = 0$.

7. $5y = -2x^2 + 6x - 4$, $10y = 5x - 11$.   **8.** $3y = x^2 - 16$, $x = y + 2$.   **9.** $y = 1 + x - 2x^2$, $x = 1 + y - 2y^2$.

10. Draw the graphs of x^2 and $3x + 1$, and hence solve

$$x^2 - 3x - 1 = 0.$$

11. Draw the graph of $8y = 7x^2 - 12x - 11$, and hence solve the equation $7x^2 = 12x + 11$.

12. Draw the graphs of $y = x^3$ and $y = 2x^2 + 3x$ for values of x from -2 to 3. Deduce the roots of the equation $x^3 - 2x^2 - 3x = 0$.

Solve graphically the following equations (Nos. 13–18) by drawing the graph of $y = x^2$ and a straight line. In each case write down the equation of the straight line.

13. $4x^2 + 6x + 1 = 0$.   **14.** $23x = 24 + 5x^2$.   **15.** $5x^2 - x = 4$.

16. $4x^2 = x - 1$.   **17.** $3x^2 + 2x = 8$.   **18.** $5x + 6 = 4x^2$.

19. Draw the graphs of $y = \dfrac{x^2}{900}$ and $y = \dfrac{261}{x + 3}$ for values of x between 40 and 80, on the same diagram. Solve $\dfrac{x^2}{900} = \dfrac{261}{x + 3}$.

20. Plot in one figure and with the same scales the graphs of $y = x^2 - 2$ and $2y = x + 8$ for values of x from -4 to 4. Find the equations whose roots are (1) the values of x, (2) the values of y, at the points of intersection of the graphs.

21. Draw the graph of $y=\dfrac{3x^2}{x^2+2}$ for values of x from -4 to 4. On the same diagram draw the graph of $4y=3x$. Read off from the figure the values of x at the points of intersection of the two graphs. What equation in x is satisfied by these values?

22. Draw the graph of $y=\dfrac{2x^2-3}{x+8}$ for values of x from -3 to 3. Indicate how to find approximately the roots of the equation $2x^2-x-11=0$ from this graph, and give the values you find.

23. Draw the graph of $y=2+3x-x^2$ for values of x from -1 to 4. By means of the graph obtain the values of x for which the expression $2+3x-x^2$ equals $1+\dfrac{x}{2}$.

24. Draw the graphs of $y=x^2$ and $5y=6x+4$ on the same diagram for values of x from -2 to 3. From the graphs solve $5x^2=6x+4$. Also find out roughly from the graphs, by drawing the appropriate parallel line, for what value of a the equation $5x^2=6x+a$ will have equal roots.

115. Derivation of laws from experimental data. When pairs of values of related quantities, x and y, have been obtained from given statistics or by experiment, it may be possible to deduce some algebraic law connecting them.

To do this, we first plot the graph of y against x. If the points lie on a straight line, it follows that x and y are connected by an equation of the first degree and we may assume that the law is $ax+by+c=0$, where a, b, c are constants, which may be determined either from the graph or by substituting pairs of values in the equation.

If the values of x and y are not exact, but only approximate, as is the case when they are obtained by experiment, the plotted points may not all lie exactly on a straight line. In such cases we allow for possible errors of observation, and (provided that the discrepancies are small) we draw the line that passes most nearly through the points, leaving some on one side and some on the other. We then find the equation represented by this line and we may regard this as expressing approximately the algebraic law connecting the quantities.

More difficult cases will be considered in Chapter XXVIII.

Example 1. *Plot the following values of x and y in a graph, and find the equation connecting them.*

x	2	4·5	6	7·2
y	8	15·5	20	23·6

The graph is shown in Fig. 19.

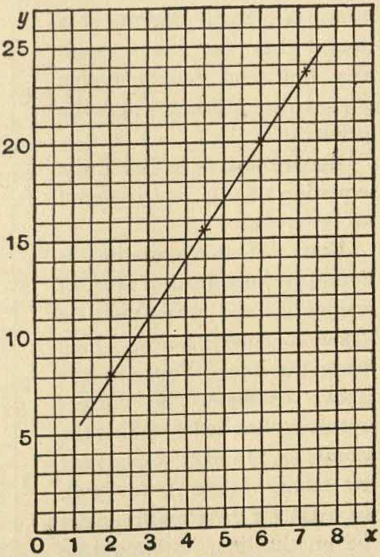


FIG. 19.

All the points lie on a straight line,

$\therefore$ the required equation is of the form

$$ax + by + c = 0.$$

This is satisfied by (2, 8) and by (6, 20),

$$\therefore\ 2a + 8b + c = 0,$$
$$6a + 20b + c = 0\ ;$$
$$\therefore\ 4a + 12b = 0,$$
$$\therefore\ a = -3b.$$

Also

$$-6b + 8b + c = 0,$$
$$\therefore\ c = -2b.$$

The equation is therefore

$$-3bx + by - 2b = 0,$$

i.e. $y = 3x + 2$.

Or, we notice from the graph that the gradient is 3 ;

$\therefore$ the equation is of the form $y = 3x + l$.

But this passes through (2, 8),

$$\therefore\ 8 = 6 + l, \qquad \therefore\ l = 2\ ;$$

$\therefore$ the equation is $y = 3x + 2$.

Example 2. *The following table of values of x and y was obtained by experiment. Plot the graph and obtain approximately the equation connecting them.*

x	2	3	4	5	6
y	1	2·4	3·5	5	6·42

The graph is shown in Fig. 20.

It is seen that the points lie nearly on a straight line. By trial—a convenient method is to use a piece of cotton—it is found that the line through the first and fourth points passes most nearly through the points.

By working as in Ex. 1, the equation of this line is found to be $3y=4x-5$.

Note. If this equation is found by substituting pairs of values of x and y, care must be taken to choose points which lie on the line. Thus, in the above example, the correct result is obtained by taking the points (2, 1) and (5, 5), but not by taking (3, 2·4), (4, 3·5), (6, 6·42), for these points do not lie on the line. It should be noted that the points used for the calculation need not be in the given table of values.

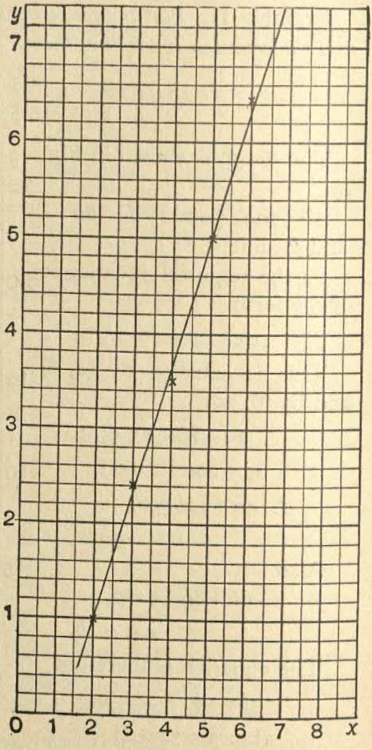


FIG. 20.

EXERCISE 55. a

In the following examples (Nos. 1–6), plot the points, and deduce as accurately as possible the equation connecting the variables :

1.	x	0	2	3	4	6
	y	3	7	9	11	15
2.	x	1	2·6	4	5·2	
	y	3	7	10·5	13·5	
3.	x	2	3	5	6	8
	y	−3	−4·6	−7·4	−9	−11·9
4.	x	1	2·4	4	5	5·6
	y	5	0·7	−4	−6·9	−8·8

5.

x	2	3	4	5	6	7
y	3	3·3	3·7	4	4·3	4·7

6.

x	−2	−1	0	1	2	3	4
y	6	5·4	5	4·6	3·9	3·6	3

7. The following observations obey a law $s=ut+\frac{1}{2}ft^2$. Plot $\frac{s}{t}$ against t and find the law.

s	75	84	91	96	99	100
t	1	1·2	1·4	1·6	1·8	2

8. The following observations are believed to obey the law $s=a+2t+bt^2$. Plot $s-2t$ against t^2 and find the law.

s	3	2	−1	−6
t	1	2	3	4

EXERCISE 55. b

In the following examples (Nos. 1–6), plot the points, and deduce as accurately as possible the equations connecting the variables:

1.

x	−1	2·2	3·4	4
y	−4	5·6	9·2	11

2.

x	−1	0·3	2	4	6
y	−1·3	0·4	2·7	5·3	8

3.

x	0	1	2·2	4	5
y	2·8	2·5	2·2	1·7	1·5

4.

x	−1	0	1	2·5	5
y	−2·2	−1·6	−1	−0·1	1·4

5.

x	−0·2	−0·6	0	1	1·6
y	1	0·1	1·4	4	5·5

6.

x	2	3	4	5	6	7
y	−1·1	−2	−2·9	−3·8	−4·6	−5·5

7. The following observations are believed to obey the law $y=kx^2$. Plot $\frac{y}{x}$ against x and find the law.

x	2	2·2	2·4	2·6	2·8	3
y	20	24·2	28·8	33·8	39·2	45

8. The following table shows corresponding values of x and y. Plot y against x^2. There is reason to suppose that one pair of the given values is wrong; find which it probably is, and determine the law connecting x and y.

x ...	1	1·6	3	3·7	4	5	5·7	6	6·3
y ...	5·25	6	7·9	8·5	9·4	11·25	12·5	13·6	16

CHAPTER XIX

FRACTIONS (*Continued*)

116. In Chapter XII we dealt with fractions with very simple denominators. The same principles apply to fractions with less simple denominators. For convenience, the fundamental principle is repeated here :

The value of a fraction is unaltered by multiplying (or dividing) both its numerator and denominator by the same expression. The expression must not, however, be zero.

As in Chapter XII, proficiency in dealing with fractions is best attained by considering worked examples.

117 Reduction of fractions to their lowest terms.

Example 1. *Simplify* $\frac{2a^2 - ab - b^2}{4a^2 - 3ab - b^2}$.

As in Arithmetic, to reduce a fraction to its lowest terms it is necessary to divide the numerator and denominator by any factors which may be common to both. A first essential step is to find the factors of the numerator and denominator. In this instance,

$$2a^2 - ab - b^2 = (2a + b)(a - b) \text{ ; } 4a^2 - 3ab - b^2 = (4a + b)(a - b) \text{ ;}$$

$$\therefore \text{ the given expression} = \frac{(2a + b)(a - b)}{(4a + b)(a - b)} = \frac{2a + b}{4a + b}.$$

Note 1. The beginner must be careful to reduce fractions to their lowest terms by dividing common factors into the whole of the numerator and the whole of the denominator. A common and serious error is to divide a factor into part of the numerator and part of the denominator. If this is done the value of the fraction is altered. It cannot be too strongly emphasised that $\frac{a^2 + b^2}{a + b}$ **is not equal to** $\frac{a + b^2}{1 + b}$. There will be less temptation to make this error if the following rule is observed :

Never cancel until both numerator and denominator have been factorised.

Note 2. In general, the value of a fraction is altered if equal quantities are added to or subtracted from the numerator and the denominator. Thus $\frac{a+5}{a+7}$ **is not equal to** $\frac{5}{7}$, unless $a=0$.

Example 2. *Simplify* $\dfrac{(2x^2y-4xy^2)^2}{2x^3y-8xy^3}$.

The expression equals $\dfrac{[2xy(x-2y)]^2}{2xy(x^2-4y^2)}$

$$=\frac{4x^2y^2(x-2y)^2}{2xy(x+2y)(x-2y)}=\frac{2xy(x-2y)}{(x+2y)}.$$

118. Multiplication and division of fractions.

Example 3. *Simplify* $\left(\dfrac{2x^2+x-1}{x^2-4x+3}\times\dfrac{2x^2-5x+3}{2x^2-7x+3}\right)\div\dfrac{6x^2+x-2}{3x^2-7x-6}$.

The expression equals

$$\frac{(2x-1)(x+1)}{(x-1)(x-3)}\times\frac{(x-1)(2x-3)}{(2x-1)(x-3)}\times\frac{(3x+2)(x-3)}{(3x+2)(2x-1)}$$

$$=\frac{(x+1)(2x-3)}{(x-3)(2x-1)},$$

for the factors $(2x-1)$, $(x-1)$, $(3x+2)$, $(x-3)$ are common to the numerator and denominator.

119. In simplifying fractions it is of the greatest importance to recognise the relation between expressions of the form $x-y$ and $y-x$, i.e. $x-y=-(y-x)$ or $y-x=-(x-y)$. The beginner will recognise these relations more easily, if he arranges the terms of each expression in some systematic order, e.g. in ascending or descending powers of some selected letter.

Example 4. *Simplify* $\dfrac{a^2-4b^2}{2b-a}\div\dfrac{a+2b}{3}$.

The expression equals $\dfrac{a^2-4b^2}{-a+2b}\times\dfrac{3}{a+2b}$

$$=\frac{(a+2b)(a-2b)}{-(a-2b)}\times\frac{3}{(a+2b)}=-3,$$ for the factors $(a+2b)$, $(a-2b)$ are common to the numerator and denominator.

EXERCISE 56. a

Simplify:

1. $\dfrac{2xy}{x^2y^2-xy}$.
2. $\dfrac{3a^2+2ab}{3ab+2b^2}$.
3. $\dfrac{2x^3+8x^2y}{3x^2y+12xy^2}$.
4. $\dfrac{6a^2-3ab}{4a^2-b^2}$.
5. $\dfrac{(4x+3y)^2}{32x^3-18xy^2}$.
6. $\dfrac{2x^2+3x-14}{2x^2-11x+14}$.
7. $\dfrac{a^2+3a+2}{a+4}\times\dfrac{a^2+3a-4}{2a^2+a-1}$.
8. $\dfrac{a^2-3a+2}{a^2+4a-12}\times\dfrac{a^2+7a+6}{a^2-4}$.
9. $\dfrac{y^2-49}{x^2-y^2}\div\dfrac{y-7}{x-y}$.
10. $\dfrac{6-4x}{2x-3}$.
11. $\dfrac{2a^2-a-1}{1+3a-4a^2}$.
12. $\dfrac{3x^2-6xy+3y^2}{(5y-5x)^2}$.
13. $\dfrac{x^2-2x-35}{30+11x+x^2}$.
14. $\dfrac{(a^2-4)(4+3a-a^2)}{(a^2+1)(-8+6a-a^2)}$.
15. $\dfrac{x^2-7x+12}{x^2-9x+20}\div\dfrac{x^2-5x+6}{x^2-3x+2}$.
16. $\dfrac{a^2+2a-24}{a^2-16}\times\dfrac{a^2+a-12}{a^2-6a+9}$.
17. $\dfrac{(r+2s)^2-t^2}{r^2+2rs-rt}\times\dfrac{(2s+t)^2-r^2}{(r+t)^2-4s^2}\div\dfrac{2st+t^2-rt}{r^2+rt-2rs}$.
18. $\dfrac{c^2+7c+12}{c^2-25}\div\dfrac{c+4}{c+5}$.
19. $\dfrac{(4c^2-2cd)^2}{8c^2-8cd+2d^2}$.
20. $\dfrac{s+t}{2s^2+3st-2t^2}\div\left(\dfrac{s^2-2st-3t^2}{s+2t}\times\dfrac{s-t}{s-3t}\right)$.
21. $\dfrac{x-y}{x^2-xy+xz-yz}\times\dfrac{x^2-z^2}{x^2+xy-xz-yz}\div\dfrac{1}{x^2+xy+xz+yz}$.

EXERCISE 56. b

Simplify:

1. $\dfrac{3cd}{4c^2d^2-5cd}$.
2. $\dfrac{2ab-5b^2}{2a^2-5ab}$.
3. $\dfrac{4c^2+3cd}{4cd+3d^2}$.
4. $\dfrac{x^2-9y^2}{x^2+3xy}$.
5. $\dfrac{(3a-2b)^2}{6a^3-4a^2b}$.
6. $\dfrac{a^2-4a-45}{3a^2+14a-5}$.
7. $\dfrac{x^2-1}{x^2+x-6}\times\dfrac{x^2+7x+12}{x^2+6x-7}$.
8. $\dfrac{x^2}{2y+3}\times\dfrac{6y^2+9y}{4x^4}$.
9. $\dfrac{y^2-16}{y^2+8y+16}\div\dfrac{2y-8}{3y+9}$.
10. $\dfrac{4x-14y}{35y-10x}$.
11. $\dfrac{2c^2-3cd-2d^2}{6d^2+cd-2c^2}$.
12. $\dfrac{3x-6}{2x-x^2}$.
13. $\dfrac{y^2+y-6}{12-2y-2y^2}$.
14. $\dfrac{(3x-4y)^2-(4x-3y)^2}{(3x+4y)^2-(4x+3y)^2}$.
15. $\dfrac{a^2-4a}{a^3-9a}\div\dfrac{a^2-2a-8}{a^2-a-6}$.
16. $\dfrac{a^2-9}{3ab}\div\left(\dfrac{a+3}{6a^2}\times\dfrac{2a-3}{ab^2}\right)$.

17. $\dfrac{(2x+5y)(4x^2-z^2)}{(4x^2+10xy-2xz-5yz)} \div \dfrac{(4x^2-10xy+2xz-5yz)}{(4x^2-10xy-2xz+5yz)}$.

18. $\dfrac{(3x+2y)^2-25z^2}{9x^2-(2y+5z)^2} \div \dfrac{(3x-2y)^2-25z^2}{9x^2-(2y-5z)^2}$. **19.** $\dfrac{(2y-z)^2}{z^2-4y^2} \div \dfrac{y^2}{2y+z}$.

20. $\dfrac{l^2-lm-6m^2}{l^2+lm-2m^2} \div \dfrac{l^2-3lm}{l^2-lm}$. **21.** $\dfrac{x^4-y^4}{c+d} \times \dfrac{c^2-d^2}{x+y} \div \dfrac{x-y}{x^2}$.

H.C.F. AND L.C.M. (*Continued*)

120. The work in this section involves no new principles. The method is that used in Chapter XI, but a knowledge of the factors discussed in Chapter XVI is required.

Example 5. *Find the H.C.F. and L.C.M. of* $3x^2+12x+12$, $4x^2-4ax+8x-8a$ *and* $6x^2-18x-60$.

As in Arithmetic, we must resolve into factors. Thus,

$$3x^2+12x+12=3(x^2+4x+4)=3(x+2)^2,$$
$$4x^2-4ax+8x-8a=4(x^2-ax+2x-2a)=4(x-a)(x+2),$$
$$6x^2-18x-60=6(x^2-3x-10)=6(x+2)(x-5).$$

Hence, the H.C.F. is $(x+2)$ and the L.C.M. is

$$12(x+2)^2(x-a)(x-5).$$

EXERCISE 57. a

Find, in factors, the L.C.M. of:

1. $(x-2)(x+5)$, $(x+5)(3x-1)$, $(3x-1)(x-2)$.

2. $3(x-2)^2$, $4(x-2)(x+2)$, $18(x+2)$.

3. a^2-4b^2, $a^2+ab-6b^2$. **4.** x^2-9, $3x+9$.

5. $16t^2-9k^2$, $8t^2-18kt+9k^2$. **6.** x^4-4x^2, $3x^2+6x$.

7. $x^2-8x-33$, $x^2-13x+22$, x^2+x-6.

8. $8x^2-52x-28$, $6x^2-24x-126$, $2x^2+7x+3$.

9. x^2-a^2, $x^2+ax-bx-ab$, $x^2-2bx+b^2$.

10. $30x^3+110x^2-40x$, $6x^3-2x^2$, $8x^2+20x-48$.

Find, in factors, the H.C.F. and L.C.M. of:

11. $2x^2-14x+20$, $30x^2-70x+20$, $4x^3-16x^2+16x$.

12. $3x^3+2x^2-x$, $20x^2+12x-8$, $4x^2+6x+2$, $6x^3+6x^2$.

EXERCISE 57. b

Find, in factors, the L.C.M. of:

1. $(2x+1)(x-4)$, $(x-4)(3x+2)$, $(3x+2)(2x+1)$.

2. $5(7x-2)^2$, $6(7x-2)(x+6)$, $10(x+6)^2$.

3. x^2+6x+9, $x^2+7x+12$. **4.** $4x^2-25$, $2x^2-x-15$.

5. $a^3 - a,\ a^2 - 2a + 1.$ **6.** $9c^2 - 16d^2,\ 6c - 8d.$

7. $3x^2 - 2x - 5,\ 7x^2 + x - 6,\ 21x^2 - 53x + 30.$

8. $x^3 - 4xy^2,\ 6x^2 - 24xy - 72y^2,\ 8x^2 - 40xy - 48y^2.$

9. $4x^2 - 12x + 9,\ 20x^2 + 70x - 150,\ 6x^2 - 150.$

10. $4a^2 - 10ab - 6b^2,\ 3a^3 - 10a^2b + 3ab^2,\ 12a^2 + 2ab - 2b^2.$

Find, in factors, the H.C.F. and L.C.M. of:

11. $3a^2 + ab - 10b^2,\ 6a^3 - a^2b - 15ab^2,\ 6a^2 - 19ab + 15b^2.$

12. $2x^3 - x^2 - x,\ 4x^4 - 10x^3 - 6x^2,\ 8x^2 + 4x.$

121. Addition and subtraction of fractions.

Example 6. *Simplify* $\dfrac{a^2}{a^2 - 4b^2} - \dfrac{a - 3b}{a + 2b}.$

As in Arithmetic, we first find the L.C.M. of the denominators, i.e. of $a^2 - 4b^2$, or $(a + 2b)(a - 2b)$, and $a + 2b$.

The L.C.M. is $(a + 2b)(a - 2b)$ and the expression equals

$$\frac{a^2}{(a+2b)(a-2b)} - \frac{(a-3b)(a-2b)}{(a+2b)(a-2b)}$$

$$= \frac{a^2 - (a-3b)(a-2b)}{(a+2b)(a-2b)} = \frac{a^2 - (a^2 - 5ab + 6b^2)}{(a+2b)(a-2b)}$$

$$= \frac{5ab - 6b^2}{(a+2b)(a-2b)}.$$

This is in its lowest terms; it is easily seen that $a + 2b$ and $a - 2b$ are not factors of the numerator. It should be noted that there is no need to factorise the numerator; we have merely to make sure that the numerator and denominator have no common factors.

Note. In finding the value of $-(a - 3b)(a - 2b)$, we first express the product in brackets and then remove the brackets. It is unwise for the beginner to attempt both operations at once.

Example 7. *Simplify* $\dfrac{x+3}{x(x^2-9)} + \dfrac{4}{3x} + \dfrac{1}{3(3-x)}$ $(=E).$

$$E = \frac{(x+3)}{x(x+3)(x-3)} + \frac{4}{3x} - \frac{1}{3(x-3)}$$

$$= \frac{1}{x(x-3)} + \frac{4}{3x} - \frac{1}{3(x-3)}$$

$$= \frac{3 + 4(x-3) - x}{3x(x-3)} = \frac{3 + 4x - 12 - x}{3x(x-3)}$$

$$= \frac{3x - 9}{3x(x-3)} = \frac{3(x-3)}{3x(x-3)} = \frac{1}{x}.$$

Note 1. The working was very much simplified by reducing the first fraction to its lowest terms before adding the fractions. It is most important that this should not be overlooked.

Note 2. This example again illustrates the importance of Art. 119.

EXERCISE 58. a

Express the following in their simplest form :

1. $\frac{1}{x+2}+\frac{1}{x-2}$. **2.** $\frac{1}{x+7}-\frac{1}{x+9}$. **3.** $\frac{5}{x-3}-\frac{3}{x+1}$.

4. $\frac{5}{a+6b}-\frac{3}{a+3b}$. **5.** $\frac{7}{4x-3y}-\frac{2}{3x+2y}$. **6.** $\frac{2b}{a-2b}-\frac{3c}{a-3c}$.

7. $\frac{2}{(a+3)}+\frac{5}{(a+3)(a-3)}$. **8.** $\frac{a}{(a-2b)^2}-\frac{2b}{(a+2b)(a-2b)}$.

9. $\frac{x^2}{(x+3y)(x-2y)}-\frac{x}{(x+3y)}$. **10.** $\frac{8}{y^2+2y-15}-\frac{7}{y^2+3y-10}$.

11. $\frac{3}{a^2+13a+40}+\frac{1}{a^2+9a+20}$. **12.** $\frac{x+2y}{x^2-2xy-3y^2}-\frac{x+3y}{x^2-xy-2y^2}$.

13. $\frac{2x+3}{6}+\frac{6}{2x-3}$. **14.** $\frac{3}{x+3}+\frac{4}{x+4}-\frac{7}{x+7}$.

15. $\frac{4}{(3x+4)^2}+\frac{1}{3(x-2)}-\frac{1}{3x+4}$. **16.** $\frac{1}{1-3a}-\frac{3a}{(1-3a)^2}-1$.

17. $\frac{1}{2(3x-8)}-\frac{2}{3(x-3)}+\frac{9x+2}{6x(3x-8)}$.

18. $1+\frac{2x}{2x-1}-\frac{8x^2}{4x^2-1}$. **19.** $\frac{1}{2t^2+18t}+\frac{1}{2t^2-18t}-\frac{1}{t^2-81}$.

20. $\frac{x}{x^2-5x+6}-\frac{2x+3}{x^2+x-6}+\frac{x-5}{x^2-9}$.

21. $\frac{3x-7}{x^2-5x+6}-\frac{5}{x^2+x-6}-\frac{x+9}{x^2-9}$.

22. $-\frac{1}{x-3}-\frac{1}{2(x+3)^2}+\frac{6}{x^2-9}$.

23. $\frac{2x+4}{x^2+2x-3}+\frac{1}{x^2+4x+3}+\frac{x+2}{x^2-1}$.

24. $\frac{13}{6(a^2-b^2)-5ab}-\frac{5}{6(a^2+b^2)-13ab}+\frac{8}{9a^2-4b^2}$.

EXERCISE 58. b

Express the following in their simplest form :

1. $\frac{1}{3x+5}-\frac{1}{3x+8}$.
2. $\frac{1}{2x+7}+\frac{1}{2x-7}$.
3. $\frac{7}{a+2b}-\frac{4}{a+3b}$.
4. $\frac{8}{c-2}-\frac{5}{c+3}$.
5. $\frac{3y}{x-6y}-\frac{5z}{x-10z}$.
6. $\frac{3}{5l-2m}-\frac{4}{7l+3m}$.
7. $\frac{7}{(2x-3)(2x+3)}-\frac{3}{x(2x-3)}$.
8. $\frac{3}{(3x-7y)}+\frac{4y}{(3x-7y)(3x+2y)}$.
9. $\frac{3}{(a+b)(a-b)}+\frac{2}{(a-b)^2}$.
10. $\frac{3}{a^2-5a+4}-\frac{2}{a^2-4a+3}$.
11. $\frac{2}{5x-2}-\frac{5x}{25x^2-15x+2}$.
12. $\frac{2}{x^2-x}-\frac{1}{x^3-x}$.
13. $3x-2+\frac{6x+1}{3x}$.
14. $\frac{4}{7x+2}-\frac{3}{7x+3}-\frac{1}{7x}$.
15. $\frac{3}{4(x-1)}-\frac{3}{4(x+1)}+\frac{3}{2(x+1)^2}$.
16. $\frac{2x-1}{2(x-1)(2x-3)}-\frac{1}{2(x-1)}-\frac{2}{(2x-3)}$.
17. $1-\frac{a-2b}{2b}-\frac{2a}{a-2b}+\frac{a^2}{2b(a-2b)}$.
18. $\frac{4}{x+5}+\frac{1}{x-2}-\frac{7}{x^2+3x-10}$.
19. $\frac{2}{a^2-1}-\frac{1}{(a+1)^2}-\frac{1}{(a-1)^2}$.
20. $\frac{3}{(x-2)^2}-\frac{2}{x^2-5x+6}-\frac{1}{x^2-4}$.
21. $\frac{3c+2d}{3c-2d}-\frac{16d^2}{9c^2-4d^2}-\frac{3c-2d}{3c+2d}$.
22. $\frac{6x-1}{9x^2-3x-6}-\frac{3x+2}{18x^2-21x+3}+\frac{3x-3}{18x^2+9x-2}$.
23. $\frac{1}{4c^2-9y^2}-\frac{4}{6cy-9y^2}+\frac{4}{6cy+9y^2}$.
24. $\frac{3a}{9a^2-16b^2}-\frac{6a+b}{9a^2-6ab-8b^2}+\frac{3a-b}{9a^2+18ab+8b^2}$.

*EXERCISE 58. c

Express the following in their simplest form :

1. $\frac{2x-1}{2x^2+x-1}+\frac{3x-4}{3x^2-7x+4}$.
2. $\frac{a^2-9b^2}{a^2+3ab}-\frac{a-3b}{a}$.

3. $\dfrac{x+3}{x^2-4x-21}-\dfrac{x-6}{x^2+x-42}$.

4. $\dfrac{10x+5y}{4}-\dfrac{10x^2+13xy-3y^2}{4x+6y}$.

5. $\dfrac{1}{1+x}-\dfrac{1}{1-x}-\dfrac{3x}{x^2-1}$.

6. $1-\dfrac{2}{x+2}-\dfrac{2x}{4-x^2}$.

7. $\dfrac{1}{3c(3c-2)}+\dfrac{1}{2c}+\dfrac{1}{2(2-3c)}$.

8. $\dfrac{1}{(1-x)(x-y)}+\dfrac{1}{(1-y)(y-x)}$.

9. $\dfrac{3}{3-2x}-\dfrac{7}{3+2x}-\dfrac{4(3-5x)}{4x^2-9}$.

10. $\dfrac{5}{1+a}-\dfrac{2}{1-a}+\dfrac{2(1-3a)}{a^2-1}$.

11. $\dfrac{x^2-(y-2z)^2}{(2z+x)^2-y^2}+\dfrac{y^2-(2z-x)^2}{(x+y)^2-4z^2}+\dfrac{4z^2-(x-y)^2}{(y+2z)^2-x^2}$.

12. $\dfrac{x}{(y-x)^2}-\dfrac{3x}{2x^2-xy-y^2}+\dfrac{1}{2x+y}$.

13. $\dfrac{(1-x)}{(x-1)^2}+\dfrac{1-2x}{4x^2-1}$.

14. $\dfrac{3b}{a^2+3ab+2b^2}-\dfrac{b}{(a+b)^2}-\dfrac{(2a-3b)}{2a^2+ab-6b^2}$.

Further examples involving more difficult factors are given in Chapter XXV.

122. Miscellaneous fractions.

Example 8. *Simplify*

$$\left(3a-5b-\frac{2b^2}{a}\right)\left(3a+5b-\frac{2b^2}{a}\right)\div\left(a-\frac{4b^2}{a}\right).$$

The expression

$$=\left(\frac{3a^2-5ab-2b^2}{a}\right)\left(\frac{3a^2+5ab-2b^2}{a}\right)\div\left(\frac{a^2-4b^2}{a}\right)$$

$$=\frac{(3a+b)(a-2b)(3a-b)(a+2b)}{a\,.\,a}\times\frac{a}{(a-2b)(a+2b)}$$

$$=\frac{(3a+b)(3a-b)}{a}.$$

Example 9. *Simplify* $\dfrac{\dfrac{8}{a}+\dfrac{a}{4}-3}{\dfrac{a}{12}-\dfrac{1}{3}-\dfrac{8}{3a}}\ (=E)$.

The value of the fraction will not be altered, if we multiply numerator and denominator by the same quantity. If we multiply by $12a$ (i.e. the L.C.M. of a, 4, 12, 3, $3a$), we have

$$E=\frac{96+3a^2-36a}{a^2-4a-32}=\frac{3(a-4)(a-8)}{(a+4)(a-8)}=\frac{3(a-4)}{(a+4)}.$$

EXERCISE 59. a

Simplify :

1. $\dfrac{1}{2x+\dfrac{3y}{5x}}$.

2. $\dfrac{\dfrac{x}{2y}-\dfrac{2y}{x}}{\dfrac{1}{2y}+\dfrac{1}{x}}$.

3. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{a}{c}+\dfrac{b}{d}}$.

4. $\dfrac{1-\dfrac{1}{a}-\dfrac{6}{a^2}}{\dfrac{2}{a}-\dfrac{15}{a^2}+1}$.

5. $\left(x-\dfrac{1}{9x}\right)\div\left(1+\dfrac{1}{3x}\right)$.

6. $\left(3x-\dfrac{1}{3x}\right)\div\left(3x-2+\dfrac{1}{3x}\right)$.

7. $\dfrac{2x}{x-\frac{1}{2}}-\dfrac{3x-1}{2x+1}$.

8. $\dfrac{\dfrac{1}{2a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{2}{b}}-\dfrac{\dfrac{1}{2a}-\dfrac{1}{b}}{\dfrac{1}{a}+\dfrac{2}{b}}$.

9. $\dfrac{2-\dfrac{7}{x+3}}{1-\dfrac{5}{x+3}}\times\dfrac{1-\dfrac{1}{x-1}}{1-\dfrac{1}{2(x-1)}}$.

10. $\left(x+4+\dfrac{8}{x-2}\right)\left(x+2-\dfrac{8}{x}\right)$.

11. $\left(1-\dfrac{28}{3y+2}+\dfrac{14}{2y+1}\right)\left(1+\dfrac{28}{3y-2}-\dfrac{14}{2y-1}\right)$.

12. $\left\{\dfrac{1}{x}-\dfrac{3}{x^2(2x+1)}\right\}\div\left\{\dfrac{2}{x}+\dfrac{3(2x-1)}{2x^2(2x+1)}\right\}$.

13. $\left\{\dfrac{1}{9x^2-9x+2}+\dfrac{1}{9x^2-15x+6}\right\}\times\left\{3x-\dfrac{3}{4-3x}\right\}$.

14. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)\left(\dfrac{1}{x+y}-\dfrac{1}{x-y}\right)\left\{\dfrac{x^2-y^2+y(x+y)}{(2x+y)^2-y^2}\right\}$.

15. $\left(\dfrac{\dfrac{x}{a}-\dfrac{y}{b}}{\dfrac{a}{x}-\dfrac{b}{y}}+\dfrac{\dfrac{x}{b}+\dfrac{a}{y}}{\dfrac{b}{x}+\dfrac{y}{a}}\right)\div\left(\dfrac{a}{y}-\dfrac{y}{a}\right)$.

16. $\left(3m-11-\dfrac{4}{m}\right)\left(3m+11-\dfrac{4}{m}\right)\div\left(\dfrac{m}{4}-\dfrac{4}{m}\right)$.

17. $\dfrac{1}{x+10}-\dfrac{3}{x+6-\dfrac{24}{x+4}}$.

18. $5-\dfrac{5}{1-\dfrac{3}{2-\dfrac{3}{1+y}}}$.

EXERCISE 59. b

Simplify :

1. $\dfrac{\dfrac{3a}{2b}}{3-c}$.

2. $\dfrac{5-\dfrac{10y}{7x}}{\dfrac{7x}{y}-2}$.

3. $\dfrac{\dfrac{a}{b}-\dfrac{c}{d}}{\dfrac{b}{a}-\dfrac{d}{c}}$.

4. $\dfrac{\dfrac{12}{x}-\dfrac{11}{2}+\dfrac{x}{2}}{\dfrac{1}{3}+\dfrac{4}{3x}-\dfrac{7}{x^2}}$.

5. $\left(x^2-\dfrac{1}{x^2}\right)\div\left(x+\dfrac{1}{x}\right)$.

6. $\left(1-\dfrac{3b}{2a}\right)\div\left(\dfrac{2a}{3b}-\dfrac{3b}{2a}\right)$.

7. $\left(1+\dfrac{3}{x}-\dfrac{18}{x^2}\right)\div\left(1+\dfrac{18}{x}+\dfrac{72}{x^2}\right)$.

8. $\left(\dfrac{3x}{3x+y}+\dfrac{y}{3x-y}\right)\div\left(\dfrac{3x}{3x-y}-\dfrac{y}{3x+y}\right)$.

9. $\left(\dfrac{2a}{2a+3b}+\dfrac{3b}{2a-3b}\right)\times\left(1+\dfrac{12ab}{4a^2+9b^2}\right)$.

10. $\left(4-\dfrac{5b}{x+3b}+\dfrac{3b}{x-b}\right)\left(1-\dfrac{5b}{2x+b}+\dfrac{4b}{x+b}\right)$.

11. $\left\{\dfrac{2}{3}-\dfrac{4}{x(x+1)}\right\}\div\left\{2+\dfrac{3(x-1)}{x(x+1)}\right\}$.

12. $\dfrac{1+\left(\dfrac{1-x}{1+x}\right)^2}{1-\left(\dfrac{1-x}{1+x}\right)^2}\times\dfrac{2x}{x+\dfrac{1}{x}}$.

13. $\left(\dfrac{5}{14}+\dfrac{1}{6x-1}-\dfrac{3}{3x+2}\right)\left(\dfrac{5}{14}+\dfrac{3}{3x-4}-\dfrac{1}{3(2x-1)}\right)$.

14. $\dfrac{x^2}{\dfrac{x^2+y^2}{x+y}-y}+\dfrac{y^2}{\dfrac{x^2+y^2}{x+y}-x}$.

15. $\left\{\dfrac{3x}{3x-2y}+\dfrac{2y}{3x+2y}-\dfrac{8y^2}{4y^2-9x^2}\right\}\div\left\{1+\dfrac{4y}{3x-2y}\right\}^2$.

16. $\dfrac{2}{7}-\dfrac{2}{7+\dfrac{1}{2a-3}}$.

17. $2-\dfrac{3}{1+\dfrac{3a}{2+\dfrac{36a^2-25}{5(3a+5)}}}$.

18. $\left\{2-\dfrac{b^2+c^2-a^2}{bc}\right\}\div\left\{2+\dfrac{a^2+b^2-c^2}{ab}\right\}$.

Further examples involving more difficult factors are given in Chapter XXV.

CHAPTER XX

FORMULAE. THEIR CONSTRUCTION AND USE. CHANGE OF SUBJECT OF A FORMULA

123. In Chapter I we constructed various simple formulae, e.g. in Ex. 2 of that chapter we obtained a formula connecting shillings and pence; we also considered a number of arithmetical problems with letters instead of numbers, i.e. we obtained formulae expressing a general result from which any number of particular results may be obtained by giving special values to the letters. In the early part of this chapter we shall consider harder questions of the type discussed in Chapter I; in the latter part we shall consider the use of formulae and the transformation of formulae.

Example 1. *What fraction is $15a$ shillings of $4b$ pounds?*

Expressing both sums of money in shillings, the required fraction is

$$\frac{15a}{4b \times 20} = \frac{3a}{16b}.$$

Example 2. *If a car, travelling at a uniform speed, covers s ft. in t sec., find its speed in miles per hour.*

The car covers s ft. in t sec.,

$\therefore$ it ,, $\frac{s}{t}$ ft. in 1 sec.,

$\therefore$ it ,, $\frac{s \times 60 \times 60}{t}$ ft. in 1 hr.,

$\therefore$ it ,, $\frac{s \times 60 \times 60}{t \times 5280}$ miles in 1 hr.

After reduction $\frac{s \times 60 \times 60}{t \times 5280} = \frac{15s}{22t}$,

$\therefore$ the required rate is $\frac{15s}{22t}$ miles per hour.

If difficulty is found with this type of question, it is a good plan to work a numerical case before doing the general case.

EXERCISE 60. a

1. What fraction is $7\frac{1}{2}$d. of t shillings?

2. A sheet of paper is $\frac{3}{a}$ in. thick. How many sheets are there in a pile b in. high?

3. A tap can fill a bath in c minutes. What fraction of the bath is filled in $40d$ seconds?

4. A man smokes $\frac{3}{16}$ lb. of tobacco a week. How long does X lb. of tobacco last him?

5. It takes n men $8c$ days to repair a certain road. How long should it take $12n$ men?

6. A man cycles at $\frac{2}{3}u$ m.p.h. How far does he go in 35 minutes?

7. What is the area of the four walls of a room l ft. long, d ft. wide, h ft. high?

8. Postage to France is $2\frac{1}{2}$d. for the first ounce, and $1\frac{1}{2}$d. for each additional ounce or part of an ounce. What is the cost of posting $(n+\frac{3}{4})$ ounces, n being a positive integer?

9. A cyclist travels at u m.p.h., if there is no wind. If there is a wind blowing at v m.p.h., find the speed of the cyclist, if he travels (i) with the wind, (ii) against the wind. What time will the cyclist take to travel n miles against the wind?

10. A rectangular room is x ft. long and y ft. wide and is carpeted so as to leave a border z in. wide all round the room. Find expressions for: (i) the area of the carpet in sq. ft., (ii) the area of the border in sq. ft., (iii) the cost in pence of staining the border at n shillings per sq. yd.

11. The quarterly rental of a private telephone is £1, and for each local call after the fiftieth made by the subscriber there is a charge of 1d. If n local calls and no trunk calls were made during one quarter, find the average cost in pence of each call, (i) if $n < 50$, (ii) if $n > 50$.

12. A tank is $24a$ in. long, $18a$ in. wide and contains water to a depth of $15a$ in. Find in sq. ft. the area of the wetted surface.

13. An article is sold for £y, and there is a profit of z pence; what is (a) the cost price, (b) the profit per cent.?

14. From a rod x ft. long, y in. are cut off. What per cent. of the original length is left?

15. Three kinds of coffee are mixed in the ratio $x : y : z$. How much of each kind is there in A lb. of the mixture?

16. Gunpowder is made by mixing a parts of charcoal, b parts of sulphur, and c parts of saltpetre. How much charcoal will there be in A cwt. of gunpowder?

17. A batsman has made x runs in a completed innings. In the next innings he gets out after making y runs. What is his new average?

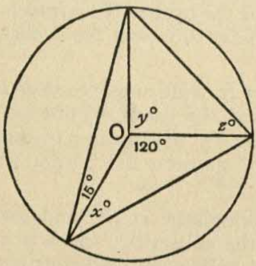
18. Find, in *shillings*, the simple interest on £x for y years at z per cent. per annum.

19. Of a regiment a per cent. are killed, b per cent. are wounded and c per cent. are taken prisoners. What per cent. are left? If X men are taken prisoners, how many were there in the regiment originally, and how many were killed?

20. The price of soap is increased by a per cent. Later the new price is decreased by a per cent. What per cent. is the final price of the original price?

21. A bath is filled by one tap alone in x seconds, by another alone in y seconds. How long will it take to fill the bath when both taps are running?

22. If O is the centre of the circle, find x, y, z.



23. A man buys £A of stock which pays b per cent. interest. What will be his income?

24. £P is invested in a stock whose price is A. If the stock is sold when the price becomes B, what will be the proceeds?

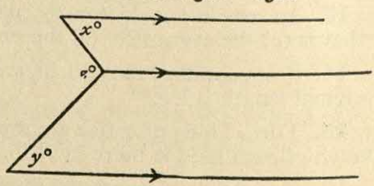
25. A car travels l miles in m hours. How many miles will it travel in p hours at one-third of this speed?

26. A man buys £B of stock when the price is C. He sells out when the price has fallen to D. What does he lose?

27. Eggs bought at a shillings a hundred are sold at b for a shilling, making 20 per cent. profit. Prove that $3ab = 250$.

28. In how many years will a sum of money double itself at x per cent. per annum simple interest?

29. Find a relation between x, y, and z.



30. In a cyclic quadrilateral $ABCD$, the diagonals meet at E. If $AE = BE$, $\angle DAC = y°$, $\angle BEC = x°$, find $\angle$s BCA, BCD.

EXERCISE 60. b

1. A waste pipe can empty a bath in $3t$ minutes. What fraction of the bath is emptied in $4k$ seconds?

2. What fraction is $9l$ in. of $5l$ ft.?

3. A man smokes $\frac{1}{x}$ lb. of tobacco a week. How long does 8 oz. of tobacco last him?

4. A sheet of paper is $\frac{a}{b}$ in. thick. How many sheets are there in a pile $\frac{a}{6}$ ft. high?

5. Find a formula for the time a train takes to go a given distance, d miles, at u m.p.h.

6. It costs a shillings to cover with linoleum a room x ft. long and y ft. wide. What will it cost to cover an area of z sq. yd. with linoleum of the same quality?

7. After the $3n$th day of April, what fraction of April is left?

8. Find in cu. ft. the volume of water in a tank x yd. long, y ft. wide and z in. deep.

9. The scale for Inland Parcel Post at the end of 1936 was as follows: up to 3 lb.—6d. ; 3 to 4 lb.—7d. ; 4 to 5 lb.—8d. ; 5 to 6 lb.—9d. ; 6 to 7 lb.—10d. ; 7 to 8 lb.—11d. ; 8 to 15 lb.—1s.

What was the cost of sending a parcel weighing n lb. 7 oz., if n is a positive integer, (i) < 3, (ii) greater than 2 and less than 8, (iii) greater than 7 and less than 15? What must be done if $n > 15$?

10. A book is x in. thick, each cover is y in. thick, and there are z sheets. What is the thickness of each sheet?

11. A clerk types n words an hour; how many minutes does she take to type $\frac{k}{400}$ words?

12. How many tiles, measuring 6″ by 4″, are required for the floor of a hall $4x$ ft. long and $\frac{10x}{3}$ ft. broad?

13. If goods are marked at k shillings, and a customer is allowed a discount of c per cent., how much will the customer pay and what is the cash value of his discount?

14. In a forest there are X trees; y per cent. are blown down. How many remain?

15. Write down the average of $6p$, $9q$, $15r$.

16. A bowler has taken a wickets for x runs. His next wicket costs him c runs. What will then be his average of runs per wicket?

17. A blend of butter is made by mixing three different kinds in the ratio $x : y : z$. What weight of blended butter contains p lb. of the first kind?

18. The simple interest on £P for y years is £Q. What is the rate per cent. per annum?

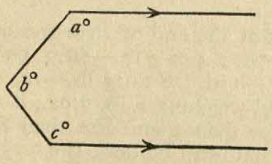
19. Sugar at x shillings per lb. is mixed with sugar at y shillings per lb. in the ratio $l : m$. What should be the price of a pound of the mixed sugar?

20. A per cent. of the trees in a wood are blown down. X per cent of the remainder are hewn down. Y trees are left standing. How many trees were there in the wood?

21. An x per cent. stock at Q brings in £P income. What sum of money is invested in it? How much stock is held?

22. A man has £B of stock. He sells it and receives £C. At what price did he sell?

23. A bath is filled by a tap in a minutes, whilst the waste pipe could empty it in b minutes. If $a < b$, and the waste pipe is open when the tap is running, how long will it take to fill the bath?

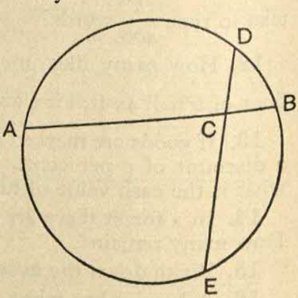


24. If x men can do the same work as y women, and a women can do the same work as b children, how many children are required to do the same work as c men?

25. Find a relation between a, b and c in the above figure.

26. If x cows give y gallons of milk in z weeks, how many cows must be kept to supply g gallons a day?

27. A train whose normal speed is u m.p.h. is c minutes late The speed is now increased to v m.p.h. Prove that the lost time is made up in $\dfrac{cuv}{60(v-u)}$ miles.



28. A man has £X of p per cent. stock. He sold out when the price was Y and re-invested in q per cent. stock at Z. What was the change in his income?

29. If $AB = c$ in., $AC = x$ in., $DE = a$ in., $DC = b$ in., find a relation between a, b, c and x.

30. In a quadrilateral $ABCD$,

$$\angle DAB = x^\circ,\ \angle DCB = y^\circ,\ \angle DBC = z^\circ.$$

If $x+y=180^\circ$, find $\angle$s DAC and BDC.

EXERCISE 60. c

1. A rectangular tank, x ft. long, y ft. wide and z ft. deep, is full of water. If a gallons of water are taken out, what depth (in feet) of water is left, given that b gallons occupy 1 cu. ft.?

2. A man's gross income is £x. Find in pounds his net income after paying income-tax at y shillings in the £. What percentage of his gross income is tax?

3. A man rows from A to B and back, a distance of a miles. He rows in still water at the rate of x m.p.h. Find the difference between the times taken for two such journeys, the first taken when there was a current flowing at $\frac{1}{2}$ mile per hour and the second when there was no current.

4. A man bought p bananas at a shillings a score, and then q bananas at b pence per dozen. How much did he pay altogether, and what was the average price (in shillings per score) of bananas?

5. Taking 1 cu. ft. of water to weigh x lb., find in tons the weight of water which falls on a y-acre field in a rainfall of z in.

6. A cyclist cycles from A to B at an average speed of x m.p.h. and returns from B to A at an average speed of y m.p.h. What will be his average speed (in m.p.h.) for the whole journey?

7. On a non-stop train between two towns one-fourth of the passengers went first class, and the rest went third class. The first and third class single fares were in the ratio $a:b$. There were N passengers and the third class single fare was £F. Find the total sum paid by the N passengers for a single journey.

8. Two men are walking along a straight road in the same direction. The faster man is walking at x m.p.h., the other at y m.p.h. At the start the faster man is a miles behind, but after z min. he is only b miles behind. Find an expression for b in terms of a, x, y, z. Find also an expression for the time (in min.) at which the faster man will be c miles behind the other ($c < a$).

9. If a packet of $2x$ plain envelopes costs $3y$ pence, how many similar envelopes, each with a $1\frac{1}{2}$d. stamp, should there be in a packet which is sold for $5z$ pence?

10. A shopkeeper spends £x in the purchase of oranges at y shillings per score, and sells them at the rate of z oranges for a shilling. Find expressions to represent (i) his gross profit, (ii) his profit per cent., on the transaction.

11. What length (in ft.) of lead pipe, of cross-section x sq in., weighs n cwt., if p c. ft. of lead weigh z lb.?

12. The external measurements of a wooden box without a lid are length, a ft.; breadth, b ft.; depth, c ft. The thickness of the material is d in. Find the weight of the box in lb., if the specific gravity of the wood is 0·8 [1 cu. ft. of water weighs 62·5 lb.].

13. A sheet of cardboard is x ft. by y ft.; z of these sheets are cut up to make boxes. If k per cent. of the cardboard is wasted and the area of cardboard in each finished box is p sq. in., write down an expression for the number of boxes made.

14. The income-tax paid by a bachelor whose income is £x is reckoned as follows: deduct one-fifth of the income, and from what remains deduct "personal allowance" of £y. This gives the "taxable income". On the first £z of "taxable income", income-tax is charged at p shillings in the £; on the rest of the "taxable income", it is charged at $3p$ shillings in the £. Give the total tax (in shillings), in a simplified form, free from brackets, (i) when $4x>5(y+z)$, (ii) when $5(y+z)>4x>5y$, (iii) when $4x<5y$.

USE OF FORMULAE

124. Consider the following well-known formulae:

(1) If a pyramid of height h ft. stands on a base whose area is A sq. ft., its volume (V cu. ft.) is given by the formula $V=\frac{1}{3}Ah$.

(2) The distance (D miles) of the horizon from a point h ft. above the surface of the earth is approximately given by the formula $D=\sqrt{1 \cdot 5h}$.

In each of these formulae we have one letter (on the L.H.S. of the formula) expressed in terms of one or more letters (on the R.H.S. of the formula). When we are given the numerical values of the various letters on the R.H.S. of a formula we can calculate the value of the single letter on the left. Thus,

(1) If a pyramid of height 8 ft. stands on a square base each side of which is 12 ft., its volume (V cu. ft.) is given by

$$V=\tfrac{1}{3}\times 12\times 12\times 8=384,$$

i.e. the volume is 384 cu. ft.

(2) From a point 216 ft. above the surface of the earth the distance (D miles) of the horizon is given by $D=\sqrt{1 \cdot 5\times 216}=\sqrt{324}=18$, i.e. the distance of the horizon is 18 miles.

EXERCISE 61. a

1. If $f=\dfrac{22u}{15}$, find (i) f when $u=60$, (ii) u when $f=33$.

2. If $A=2h(l+b)$, find (i) A when $h=6$, $l=6{\cdot}5$, $b=3{\cdot}5$, (ii) l when $A=260$, $h=6{\cdot}5$, $b=7{\cdot}7$.

3. If $2S=n[2a+(n-1)d]$, find (i) S when $n=40$, $a=7$, $d=2$, (ii) a when $S=460$, $n=20$, $d=-2$.

4. If $S=\dfrac{uvt}{60(v-u)}$, find (i) S when $u=30$, $v=45$, $t=6$, (ii) t when $S=96$, $u=60$, $v=80$.

5. If $A=\sqrt{s(s-a)(s-b)(s-c)}$, find A when $a=13$, $b=14$, $c=15$, $s=21$.

6. If $H=0{\cdot}4nd^2$, find (i) H when $n=6$, $d=0{\cdot}5$, (ii) d when $H=0{\cdot}1$, $n=4$.

7. If $R=\dfrac{h}{2}+\dfrac{x^2}{6h}$, find (i) R when $h=4$, $x=6$, (ii) h when $R=5\frac{1}{4}$, $x=9$.

8. If $c=\sqrt{a^2+b^2-2ap}$, find c when $a=21$, $b=13$, $p=5$.

9. If $n=5$, $a=3$, $b=-\frac{2}{3}$, find the values of

(i) $\left(\dfrac{a}{3}+3b\right)^{n+3}$, (ii) $(a^2+2ab+b^2)^{n-3}$, (iii) $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b$.

10. If $d=\dfrac{3c}{88}$, c yards a minute is the same as d miles an hour. Express (i) in m.p.h. (*a*) 44 yd. per min., (*b*) 660 yd. per min.; (ii) in yd. per min., (*a*) 6 m.p.h., (*b*) 21 m.p.h.

11. By means of the formula $V=\frac{1}{3}Ah$. find (i) the volume of a pyramid of height 12 ft. on a base whose area is 25 sq. ft.; (ii) the height of a pyramid whose volume is 128 cu. in. and whose base has an area of 32 sq. in.

12. C° Centigrade is the same as F° Fahrenheit, if $5F=160+9C$. Express (i) in degrees Fahrenheit (*a*) 40° C., (*b*) 0° C., (*c*) 75° C.; (ii) in degrees Centigrade (*a*) 41° F., (*b*) −40° F., (*c*) 212° F.

13. The sum of the first n integers is $\dfrac{n(n+1)}{2}$. Find (i) the sum of the first 500 integers, (ii) the sum of all the integers from 401 to 800 inclusive, (iii) how many consecutive integers starting 1, 2, 3, ⋯ must be taken to add up to 5050.

14. The sum of the cubes of the first n integers is $\dfrac{n^2(n+1)^2}{4}$. Find (i) the sum of the cubes of the first 16 integers, (ii) the sum of the cubes of the integers from 11 to 20 inclusive.

15. If the simple interest on a sum of money £P for t years at r per cent. is £I, it is known that $r=\frac{100I}{Pt}$. If the interest on £300 for 2 years is £24, find the rate per cent.

16. If a bath can be filled by one tap in a minutes and by another tap in b minutes, it can be filled by both taps together in c minutes, where $\frac{1}{c}=\frac{1}{a}+\frac{1}{b}$. Find b, if $c=9$, $a=15$.

17. A sum of money £P amounts at r per cent. compound interest to £$P\left(1+\frac{r}{100}\right)^n$ in n years. Find (i) to what £800 amounts in 2 years at 5 per cent., (ii) what sum amounts to £530 9s. in 2 years at 3 per cent.

18. The radius (R cm.) of an arc of a circle is given by the formula $2R=h+\frac{k^2}{h}$, where k cm. is the length of half the chord and h cm. the height of the segment cut off by the chord. Find the radius of the arc of a circle cut off by a chord 20 cm. long, if the height of the segment cut off is 2 cm.

EXERCISE 61. b

1. If $S=90+10n$, find (i) S when $n=-7{\cdot}5$, (ii) n when $S=55$.

2. If $V=\frac{4}{3}\pi r^3$, find (i) V when $r=2{\cdot}1$, (ii) r when $V=11\frac{187}{375}$. Take $\pi=\frac{22}{7}$.

3. If $2A=h(a+b)$, find (i) A when $h=11$, $a=3{\cdot}7$, $b=9{\cdot}2$, (ii) a when $A=108$, $h=9$, $b=5$.

4. If $F=32+1{\cdot}8C$, find (i) F when $C=50$, (ii) C when $F=41$.

5. If $R=\frac{ab}{a+b}$, find (i) R when $a=7$, $b=3$, (ii) b when $R=1{\cdot}6$, $a=8$.

6. If $A=P+\frac{Prt}{100}$, find (i) A when $P=300$, $r=3{\cdot}5$, $t=3$, (ii) r when $P=400$, $A=490$, $t=5$.

7. If $A=\frac{180(n-2)}{n}$, find (i) A when $n=15$, (ii) n when $A=150$.

8. If $D=\frac{1}{2}n(n-3)$, find (i) D when $n=10$, (ii) n when $D=54$.

9. If $a=2$, $b=-1$, $c=-4$, $f=\frac{5}{2}$, $g=-\frac{3}{2}$, $h=\frac{1}{2}$, find the value of $abc+2fgh-af^2-bg^2-ch^2$. If a, b, f, g, h have the above values, and $c=-4+\lambda$, find what value must be given to λ in order that the above expression may be zero.

10. The surface (S sq. in.) of a sphere of radius r in. is given by the formula $S=4\pi r^2$. Find (i) the surface of a sphere whose radius is 2·8 in.; (ii) the radius of a sphere whose surface is 154 sq. cm. Take $\pi=3\frac{1}{7}$.

11. A train travelling at the rate of u ft. per sec. covers s ft. in t sec., where $s=ut$. Find (i) how many miles a train will run in 36 min. at 44 ft. per sec.; (ii) the speed in m..p.h. of a train which runs 3960 ft. in 1 minute.

12. A stone falling from rest under gravity falls a distance s ft. in t sec., where $s=\frac{1}{2}gt^2$ and $g=32$. Find (i) the height of a tower, if a stone dropped from the top takes 5 sec. to reach the ground, (ii) how long it would take a stone to drop from an aeroplane whose height above the ground is 4800 yd.

13. A beam l in. long, b in. wide, d in. thick, is built into a wall at one end and carries a load of W tons at the other end. It will break if $W>\dfrac{3bd^2}{4l}$. Will such a beam 3 ft. long, 5 in wide, 4 in thick, break under a load of 2 tons? What load can it support?

14. The sum of the squares of the first n integers is $\dfrac{n(n+1)(2n+1)}{6}$. Find (i) the sum of the squares of the first 20 integers, (ii) the sum of the squares of the integers from 31 to 40 inclusive.

15. If a bath can be filled by one tap in a minutes and emptied by a waste pipe in b minutes ($a<b$), it can be filled by the tap when the waste pipe is open in c minutes, where $\dfrac{1}{c}=\dfrac{1}{a}-\dfrac{1}{b}$. Find a if $c=12$, $b=18$.

16. If the simple interest on a sum of money £P for t years at r per cent. is £I, it is known that $P=\dfrac{100I}{rt}$. If the interest for 5 years at 4 per cent. is £42, find the principal.

17. A wind blowing at u m.p.h. exerts a direct pressure of P lb. per sq. ft. of surface it strikes, if $200P=u^2$. What pressure must a hoarding 12 ft. high, 25 ft. wide be able to withstand against a wind blowing at (i) 5 m.p.h., (ii) 80 m.p.h.?

18. At a point h ft. above the surface of the sea the distance in miles to the horizon is $1{\cdot}231\sqrt{h}$. Find the greatest distance to the nearest half-mile at which a light 145 ft. above the surface of the sea can be seen from a point on the surface of the earth.

Further examples suitable for work with logarithms are given in Chapter XXVIII.

Transformation of Formulae, or Change of Subject of a Formula

125. If we have a formula connecting several letters, it is usually possible to find the numerical value of any one of these letters, if the numerical values of all the other letters are given. It is not necessary that the unknown letter should be isolated on one side of the formula, but in practice it is often desirable that this should be done—it will be seen later that it is frequently necessary, in order to make possible the use of logarithms. When one of the letters appears only in a single term on one side of the formula, and all the other letters are on the other side, it is called the **subject of the formula.**

126. If £I is the simple interest on £P for t years at r per cent. per annum, it is well known that $I=\frac{Ptr}{100}$. In this case I is the subject of the formula. If it is required to find the length of time for which £300 should be lent at 4 per cent. per annum simple interest to yield as interest a sum of £66, we have to find t when I, P, r are given. We can obtain t in terms of I, P, r from the equation $I=\frac{Ptr}{100}$ by using precisely the same methods as are used in solving equations.

Thus, $I=\frac{Ptr}{100}$, $\therefore\ 100I=Ptr$ (multiplying each side by 100),

$$\therefore\ \frac{100I}{Pr}=t \text{ (dividing each side by } Pr\text{)}.$$

This is usually written $t=\frac{100I}{Pr}$; t is now the subject of the formula and the above process is called **changing the subject of the formula from I to t.** The worked examples which follow will show that the subject of a formula may usually be changed by applying the ordinary rules for the manipulation of equations, treating the various letters as if they were numbers. It is assumed that the values of the letters are such that no denominator will be zero.

Example 3. *From the formula* $K+\frac{3ln}{17}=l-H$, *find* l *in terms of the other letters.*

We have $H+K=l\left(1-\frac{3n}{17}\right)$, $\therefore \frac{H+K}{1-\frac{3n}{17}}=l$,

$$\therefore l=\frac{H+K}{1-\frac{3n}{17}}, \text{ or } \frac{17(H+K)}{17-3n}.$$

It should be noted that to make l the subject of the formula, it is essential that l should appear **only** as a single **term** on one side of the equation. We therefore first rearrange the equation, so that **all** the terms containing l are on one side and **all** the remaining terms on the other side of the equation. We then group together into a single term the terms containing l. The remainder of the work is clear. Beginners often make the mistake of writing $l=H+K+\frac{3ln}{17}$ as the answer to the question. This is wrong, because l does not appear **only** as a single term on one side of the equation. The whole object of the transformation is to obtain l as the equivalent of an expression containing **the other letters only,** so that, if the numerical values of the other letters are known, the value of l may be obtained **immediately** by simple substitution. Thus, in the above example, if $H=1$, $K=-3$, $n=8\frac{1}{2}$, from

$$l=\frac{17(H+K)}{17-3n} \text{ we obtain } l=\frac{17(-2)}{17-25\frac{1}{2}}=4.$$

The pupil will find little difficulty in the transformation of formulae, if he grasps this point clearly ; the process described in Example 3 is quite general. If difficulty is found with any transformation, it is a good plan to substitute numbers for all the letters except the one required and to solve that example first. The solution will reveal the order of the operations in the general case.

Example 4. *Change the subject of the formula* $S=n^2+\frac{2an^2}{7}$ *from* S *to* n.

We have $$S=n^2+\frac{2an^2}{7}=n^2\left(1+\frac{2a}{7}\right),$$

$$\therefore n^2=\frac{S}{1+\frac{2a}{7}}, \quad \therefore n=\pm\sqrt{\frac{S}{1+\frac{2a}{7}}}, \text{ or } \pm\sqrt{\frac{7S}{7+2a}}.$$

If in any particular question it is clear that n must be a positive number, the $-$ sign should be omitted.

EXERCISE 62. a

In the following, change the subject of the formula :

1. From A to n, $A=2n-14$. **2.** From C to k, $C=5k^2$.

3. From L to n, $L=\dfrac{3nx^2}{5}$. **4.** From y to x, $y=7x+z$.

5. From t to l, $t=2\pi\sqrt{\dfrac{l}{32\cdot 2}}$. **6.** From A to r, $A=\pi r^2$.

7. If $PV(1+aT)=QR(1+aS)$, express a in terms of the other letters.

8. If $9k=\sqrt{3l(l-x)}$, express x in terms of k and l.

9. If $L=-\dfrac{2mlx}{x+y}$, express x in terms of the other letters.

10. If $a=\dfrac{3-2b}{b+3}$, express b in terms of a.

11. If $S=\pi r\sqrt{h^2+r^2}$, express h in terms of S, r, π.

12. If $2x=3y\left(\dfrac{2t}{p+q}-1\right)$, (i) express t in terms of the other letters, (ii) express p in terms of the other letters.

13. If $\dfrac{m-p}{m}=\dfrac{1+xs}{1+xt}$, express x in terms of the other letters. Find x when $m=35$, $p=9$, $t=100$, $s=10$.

14. If $b^2-p^2=\dfrac{c^2x}{a-x}$, find x in terms of a, b, c, p; and find the value of $\dfrac{a}{x}$ when $p=12$, $b=6$, $c=9$.

15. If $x=\dfrac{y}{\sqrt{a^2+b^2c^2}}$, and all the letters denote positive quantities, find (a) c in terms of a, b, x, y, (b) b when $x=3$, $y=15$, $a=4$, $c=6$, (c) x when $y=75\times 10^8$, $a=6\times 10^9$, $b=40$, $c=0\cdot 2\times 10^9$.

16. If $pv=cd\left(1+\dfrac{t}{273}\right)$, find t in terms of the other letters.

17. If $\dfrac{1}{v}-\dfrac{1}{u}=\dfrac{1}{f}$, express u in terms of v and f. If $u=3v$, find the ratio of u to f.

18. If $3l=2m\sqrt{1-\dfrac{5+n}{5-n}}$, express n in terms of l and m.

19. From the formulae $s=ut+\frac{1}{2}ft^2$ and $v=u+ft$, express u in terms of the other quantities in each case. Equate these values of u

and obtain an expression giving v in terms of s, f and t. If $v=66$, $f=33$, $t=2$, find s.

20. The tonnage (T) of a vessel is given by the formula

$$188T=B^2(L-0{\cdot}6B),$$

where L is the length of a keel in feet and B is the breadth of the vessel in feet. Find (i) the tonnage of a vessel with a keel $355'\ 3''$ and breadth $58'\ 9''$, (ii) L in terms of B and T, (iii) the length of the keel for tonnage 6000 and breadth $60'$.

EXERCISE 62. b

In the following, change the subject of the formula :

1. From A to h, $A=\frac{1}{2}bh$.

2. From L to x, $L=\frac{3nx^2}{7}$.

3. From s to c, $s=\frac{1}{2}(a+b+c)$.

4. From t to l, $t=3\sqrt{l}$.

5. From F to C, $F=32+1{\cdot}8C$.

6. From I to P, $I=\frac{Prt}{100}$.

7. If $K(3b-s)=2as$, express s in terms of K, a, b.

8. If $3a=2+\frac{5}{2-b}$, express b in terms of a.

9. If $U=3c-2dV^5$, express V in terms of c, d, U.

10. If $v=l\left(1-\frac{H}{h}\right)\sqrt{k}$, express H in terms of the other letters.

11. Change the subject of the following formula from x to h,

$$x=\frac{k(1+3h)}{1-h}.$$

12. If $c=\frac{ax-2by}{ax+2by}$, (i) express x in terms of a, b, c and y, (ii) express b in terms of a, c, x and y.

13. If $a=1-\frac{2b}{ct-b}$, express t in terms of the other letters. Find t if $a=4$ and $b=2c$.

14. Use the formula $S=\frac{n}{2}\left\{2a+(n-1)d\right\}$ to find

(i) an expression for a in terms of S, n, d ;
(ii) the value of a when $S=546$, $n=12$, $d=4$.

15. If $p=q+b\sqrt{\frac{x-a}{x+a}}$, find x in terms of p, q, a, b. Also find the value of a to two decimal places when $p=1{\cdot}3\times10^4$, $q=0{\cdot}9\times10^4$, $x=10$, $b=0{\cdot}3\times10^5$.

16. If $l=\sqrt{\dfrac{6m-n}{1-6mn}}$, find m in terms of l and n.

17. If $y=\sqrt{\dfrac{2x+25}{x-5}}$, express x in terms of y.

18. The volume of metal in a tube is given by the formula $V=\pi l\{R^2-(R-t)^2\}$, where $l=$ the length, R the radius of the outside surface and t the thickness of the material. Find R in terms of the other letters, and hence find R, correct to 2 places of decimals, when $V=200$, $l=7$, $\pi=\frac{22}{7}$, $t=2$.

19. From the formula $t=2\pi\sqrt{\dfrac{l}{g}}$ find l, correct to the nearest unit, being given $t=1{\cdot}20$, $\pi=3{\cdot}14$, $g=981$.

20. In each of the following, find x in terms of y:

(i) $y=mx+c$, (ii) $y=\dfrac{2-3x}{x+3}$, (iii) $y=\dfrac{-1\pm\sqrt{3x-2}}{3}$.

TEST PAPERS V

A

1. A closed rectangular box measures inside p ft. long, q ft. wide, and r ft. deep. What is the cost in shillings of lining it with lead at x pence per sq. ft.? How many cwt. of tea will it hold, if n lb. of tea occupies y cubic inches?

2. Factorise (i) $6x^3-38x^2-144x$, (ii) $t^3-9s+st^2-9t$.

3. Simplify (i) $\dfrac{x^2-2x-8}{x+2}$, (ii) $\dfrac{17}{12(x+17)}-\dfrac{5}{12(x+5)}$.

4. (i) If $5x=\dfrac{15-3y}{2}$ and $3y=\dfrac{15-20x}{3}$, find the numerical value of $5x^2-3y^2$.

(ii) Solve $3x(6x-1)+3(3x-8)=0$.

5. Find three consecutive positive integers such that the square of their sum exceeds the sum of their squares by 214.

6. Draw the graphs of x^2+x-2 and $3x+6$. Show that they meet on the axis of x and find the other point of intersection. Find the least value of x^2+x-2 and write down the equation in x which is satisfied by the values of x at the points of intersection of the graphs.

B

1. Write down a sum of money in which the number of pounds is greater than the number of pence. Reverse this sum of money and subtract the new sum from the old. Prove that the difference is always a multiple of 19s. 11d.

2. Simplify (i) $\dfrac{a^2+5a-14}{ab+7b}$, (ii) $\dfrac{x+6}{x-3}-\dfrac{2x+9}{x+12}$.

3. Solve (i) $6x-2y-7=\dfrac{y}{2}+20x-10=8x-\dfrac{7y}{2}$,

(ii) $8x^2+11=4+40x+20x^2$.

4. I have a certain number of marbles to divide equally amongst 18 boys; if the number of marbles and the number of boys were each increased by two, each boy would receive 5 marbles less. How many marbles have I to distribute?

5. Find the L.C.M. of $2x^2+x-1$, $6x^2+21x+15$ and $4x^2+8x-5$. Leave the answer in factors.

6. Draw the graphs of $2x+3y=5$, $y=3x+1$, $\dfrac{x}{2}+\dfrac{y}{8}=1$, using the same axes and the same scales. What values, if any, of x and y satisfy (i) the last two equations, (ii) all three equations?

C

1. In a $\triangle ABC$, $AB=AC$, $\angle A=x°$, $\angle C=y°$. Find x in terms of y.

2. Factorise (i) $(x-y)^3-x+y$, (ii) $35-2t^3-t^6$,

(iii) $98a^2-140a+50$.

3. Simplify (i) $\left(\dfrac{25-c^2}{c}\right)\times\dfrac{c^2}{3c^2+17c+10}$, (ii) $\dfrac{2+3t}{2-9t}-\dfrac{1}{(1-3t)^2}$.

4. Solve (i) $0{\cdot}4\left(\dfrac{x}{3}+1{\cdot}2\right)-1=0{\cdot}8$,

(ii) $x^2-7x=2$, correct to two places of decimals.

5. The side of one square exceeds twice the side of a second square by 6 inches; the sum of the areas of the two squares is 26·5 square feet. Find the lengths of the sides.

6. A goods train starts at noon from a terminus at A on a single line, travels at 20 m.p.h., goes into a siding for 10 minutes to allow an express to pass, and then proceeds. The express at noon is 30 miles from A and travels towards A at 40 m.p.h. without stopping, Find, graphically, or otherwise, the limits between which must be the distance of the siding from A. Assume that all speeds are uniform.

D

1. A sum of money, £A, is to be divided into two parts, one of which is c times the other. What are the parts?

2. Solve (i) $\dfrac{1}{3}\left(8x-\dfrac{5y}{3}\right)=3$, $6x+\dfrac{13y}{3}=3\dfrac{2}{5}$,

(ii) $3x^2+3x=5$, correct to one place of decimals.

3. Simplify $\dfrac{x-2}{2x^2+x-1}-\dfrac{x-1}{2x^2-x}+\dfrac{5}{2x^2-x-3}$.

4. Simplify (i) $\dfrac{a-\dfrac{1}{4a}}{(1+2a)^2}$,

(ii) $\left(\dfrac{x}{2y-x}-\dfrac{x}{2y+x}\right)\div\left(\dfrac{x^2}{x^2+4y^2}-\dfrac{x^2}{x^2-4y^2}\right)$.

5. Find the L.C.M. of $16x^2-48x+36$ and $12x^2-6x-18$. Leave the answer in factors.

6. A train running between two towns arrives at its destination 2 minutes early when it runs at 40 m.p.h. and 48 minutes late when it runs at 30 m.p.h. Find the distance between the towns and the scheduled time for the journey.

E

1. Factorise

(i) $9x^3-4xy^2$, (ii) $(x+y)^2+2(x+y)(x-y)-8(x-y)^2$.

2. A rectangular sheet of paper has sides of length a in. and b in. Equal square pieces of side x in. are cut out of the corners. The remainder is then folded up to form a box. What will be the volume and surface area of the box?

3. Simplify (i) $\dfrac{a+b}{a^2-2ab-3b^2}-\dfrac{a-b}{2(a^2-4ab+3b^2)}$,

(ii) $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)$.

4. Solve (i) $\frac{1}{2}(3x-2y)=6y-8\frac{1}{2}$, $\frac{1}{3}(3x+2y)=6x-17$;

(ii) $0=3x^2+1{\cdot}7x-2{\cdot}6$, correct to two places of decimals.

5. The table below gives the number, N, of gallons of water discharged per minute from a reservoir through a pipe of diameter D inches.

D -	2	4	6	8	9	10	11	12
N -	5	37	117	262	364	484	624	805

Plot a graph to show the relation between D and N, and use the graph to find (a) the number of gallons discharged per minute from a pipe of diameter 5″, (b) the diameter in cm. of a pipe which would discharge 1·25 cu. m. per minute, taking 1 cu. m. = 220 gallons, and 1″ = 2·54 cm.

6. Find two whole numbers such that, in each case, five times the square of the next whole number above exceeds four times the square of the next whole number below by 545.

F

1. A cricketer has an average of x runs for y completed innings. How many runs must he make in his next innings to get an average of $x+Z$, assuming that he is out?

2. Simplify $\left(\frac{1}{2x-1}-\frac{3x}{2x^2+x-1}\right)\div\frac{2}{x+1}$.

Find also the value of the expression when $x=-2$.

3. Solve (i) $\frac{0\cdot 6x+0\cdot 5}{1\cdot 4}+20(0\cdot 75-0\cdot 25x)=3+\frac{0\cdot 25x}{4}$,

correct to two places of decimals;

(ii) $(x-2)(x+8)+(x+2)(x-6)=12$.

4. Find the L.C.M. of $4x^2-4x+1$, $8x^2+4x-4$ and $6x^2+12x+6$. Leave the answer in factors.

5. I cycle against the wind at 8 m.p.h. to a town, wait 30 minutes to have my hair cut, and ride back with the wind at 12 m.p.h., the whole journey occupying 3 hours. How far off is the town?

6. Draw the graph of $y=\frac{2(36-x^2)}{18-x}$ for values of x from -4 to 4. Explain how the roots of the equation $2x^2-3x-18=0$ can be found from the graph, and read off approximate values for them.

G

1. Factorise (i) z^4-81, (ii) $a(a+b)(a-b)-6b^2(a+b)$.

2. One man alone can plough a field in a hours and another alone can plough it in b hours. What fraction of the field will be done in $\frac{1}{x}$ of an hour by both men working together?

3. Solve (i) $\frac{4x+3y}{2x+9y}=-\frac{4}{3}$, $14x+108y=-8\frac{1}{2}$;

(ii) $8x^2=11x+17$, correct to one place of decimals.

4. Simplify (i) $\left(2a+1-\frac{3}{a}\right)\div(3-4a-4a^2)$,

(ii) $\frac{3b}{(a+b)(a-2b)}-\frac{2b}{a(a-2b)}+\frac{2a+b}{a(a+b)}$.

5. The hypotenuse of a right-angled triangle is 5 in. long; of the remaining sides, one is 3 in. longer than the other. Calculate the lengths of the sides correct to three significant figures.

6. Solve graphically $\frac{x}{3}+\frac{y}{4}+2=0$, $y=3x+10$.

H

1. The price of 100 apples is a shillings and b pence. How many should a purchaser get for y shillings?

2. Simplify (i) $\dfrac{10x+1}{(5x-2)^2}-\dfrac{1}{5x-2}$,

(ii) $\dfrac{1-ab}{2a(1+ab)}-\dfrac{1+ab}{2a(1-ab)}$.

3. Solve (i) $28x-6\cdot5y=12$, $31\cdot5x-5y=19$, correct to two places of decimals;

(ii) $6(x-3)(x-4)-x(x-2)=3(x-2)(x-4)$.

4. A train starts in half an hour and the station is $3\frac{1}{2}$ miles away. I cycle part of the way at 8 m.p.h. and complete the journey on foot at 4 m.p.h. I just catch the train. How far do I walk?

5. Find the L.C.M. of

$$9a^2-4b^2,\quad 9a^2-12ab+4b^2,\quad \text{and}\quad 6a^2-11ab-10b^2.$$

Leave the answer in factors.

6. It is conjectured that the following values of x and y are connected by an equation of the form $y=ax+b$, where a, b are fixed numbers.

x	18	32	48	66
y	108	129	153	180

Show that this is the case, and find the values of a and b. Plot the graph of the function, and from the graph deduce for what values of x the values of y are positive, and for what values of y the values of x are positive.

I

1. A man has a salary of £A a year of which he spends £B. The next year he receives x per cent. more salary, and he saves y per cent. more than in the previous year. Find the alteration in his expenditure.

2. Factorise

(i) $(c+2d)(c+3d)(c+4d)^2-(c+d)(c+2d)(c+3d)^2$,

(ii) $15(c-d)(c+d)-16cd$.

3. Solve

(i) $\dfrac{0\cdot5x-0\cdot25}{8\cdot75}-\dfrac{0\cdot25x-0\cdot125}{3\cdot75}=0\cdot02x-0\cdot68$, correct to two places of decimals;

(ii) $7(3x+1)^2-16(3x+1)+8=0$, correct to two places of decimals.

4. A rectangular sheet of paper is 12″ long and 10″ wide. A rectangle is described on the paper with its sides parallel to and at the same distance from the corresponding edges of the paper. The area of the rectangle is two-thirds of the area of the paper. Find the distance of each side from the nearest parallel edge of the paper.

5. Simplify $\dfrac{3(1-x^2)}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

6. During certain years, income-tax on unearned income was levied as follows: no tax was charged on the first £150 of a man's income, on the next £250 the tax was 2s. in the £, and on any income beyond this the tax was 4s. in the £. Calculate the tax (if any) on incomes of £50, £100, £150, £200, and so on, up to £500, and draw a graph to show the tax payable on any income up to £500. From the graph find the tax payable on an income of £480 and check your result by calculation.

J

1. The price of coal is reduced x per cent. and a man uses x per cent. more. What percentage of his original expenditure does he gain or lose?

2. A father is four times as old as his son. In 8 years' time the sum of their ages will be 71 years. What was the father's age (in years) when the son was born?

3. Simplify (i) $\dfrac{7x-5}{3x^2+2x-1}-\dfrac{7x-2}{4x^2+5x+1}$,

(ii) $\dfrac{a-6x+4y}{(a+2x)(a+y)}+\dfrac{a+3x-2y}{(a+x)(a+y)}$.

4. Solve (i) $\dfrac{1}{3x+5y}+\dfrac{1}{3x-5y}=3,\ \dfrac{1}{3x+5y}-\dfrac{1}{3x-5y}=5$;

(ii) $10-15x^2=12x$, correct to two decimal places.

5. A sold some property to B at a profit of 20 per cent. B subsequently sold it back to A at the same rate of profit. A lost £12 by the double transaction. What was the original value of A's property?

6. Draw the graph of $y=\dfrac{4x}{1+4x^2}$ for values of x from -2 to 2. Use the graph to find approximately the value of x for which

$$(1+4x^2)(1-2x)=4x.$$

K

1. Factorise (i) $2x^3 + x^2 - 18x - 9$, (ii) $(dl - cm)^2 - (cl - dm)^2$.

2. If x men build a wall in y days, how long would z men take to build k such walls, at the same rate of working?

3. Solve (i) $\dfrac{5x - 5y - 4}{3} = \dfrac{3x - 10y - 7}{2} = 7x + 20y$,

(ii) $(5x - 7)^2 = 9(2x + 5)^2$.

4. Simplify (i) $\dfrac{8}{4 - 8x + 3x^2} - \dfrac{10}{2 - x - 3x^2} + \dfrac{2}{x + 1}$,

(ii) $\dfrac{6a^2 - 5ab + b^2}{6a^2 + ab - b^2} \div \dfrac{2a^2 + 11ab - 6b^2}{2a^2 + 11ab + 5b^2}$.

5. In a right-angled triangle ABC, in which the right-angle is at B, the side AB is 2 inches longer than the side BC, and the side BC is 18 inches shorter than the side AC. Find the lengths of the three sides of the triangle.

6. Draw the graphs of $\frac{1}{4}(3x + 2)$ and $\frac{1}{4}x^3$ for values of x from -3 to 3. Hence solve the equation $x^3 = 3x + 2$.

L

1. In an examination a candidates were boys and b were girls; x per cent. of the boys passed and y per cent. of the girls. What percentage of the whole number of candidates failed?

2. A certain sum of money is made up of an equal number of half-crowns and florins. If 24 half-crowns are added and the number of florins is decreased by 25 per cent., the sum is decreased by £1. Find the original sum.

3. Simplify $\left[x + \dfrac{y^2}{x + y} + \dfrac{y^2}{x - y}\right] \div \left[\dfrac{1}{y(x - y)} - \dfrac{1}{y(x + y)} - \dfrac{1}{x^2}\right]$.

4. (i) From the statement

$$\frac{3x + a}{2} - \frac{2a^2 - x}{3} - \frac{x}{a} = \frac{6 - 2a - 15a^2}{6}$$

find x when $a = -2$.

(ii) Solve the equation $2 - x - \dfrac{1}{2 - x} = 5 - \dfrac{1}{5}$.

5. The price of eggs having risen $\frac{1}{4}$d. each, it costs 5d. more to buy 40 eggs than it cost to buy 42 eggs before the increase. What was the original price of an egg?

6. Find the L.C.M. of $4x^2 - 81y^2$, $8x^2 + 72xy + 162y^2$ and $6x^2 + 21xy - 27y^2$. Leave the answer in factors.

CHAPTER XXI

HARDER EQUATIONS. LITERAL EQUATIONS. QUADRATIC EQUATIONS BY FORMULA

HARDER EQUATIONS

127. We now consider a number of equations which contain fractions whose denominators involve the unknown quantity. In Chapter IV we explained how to solve equations by means of certain fundamental axioms, and stated that, later, it would be necessary to discuss the result of multiplying or dividing both sides of an equation by an expression containing the unknown. We now proceed to deal with these points.

(A) **Multiplication.** Our third axiom states that if equals are multiplied by equals the products are equal, i.e. if $x=y$, then $x \times a = y \times a$. This result is true for all values of a, but, if $a=0$, the result is of no value to us, for we merely arrive at the result $0=0$. If the equation contains fractions, a further complication may arise. Consider the equation

$$1+\frac{3}{x-2}=\frac{2x+7}{x-2}.$$

If each side is multiplied by $(x-2)$, we obtain

$$(x-2)+\frac{3}{x-2}\times(x-2)=\frac{2x+7}{x-2}\times(x-2),$$

which **in general** reduces to $(x-2)+3=2x+7$.

But, if $x=2$, the expressions $\frac{3}{x-2}\times(x-2)$ and $\frac{x+1}{x-2}\times(x-2)$ become $\frac{3}{0}\times 0$ and $\frac{3}{0}\times 0$ respectively. These expressions have no meaning; they have not yet been defined.

Thus, the result of multiplying each side of an equation by an expression containing the unknown may only be used if the unknown has a value which does not make the expression zero. For example, if in solving an equation we multiply both sides by $(x+4)(x-1)$ the roots obtained are the required roots of the

original equation, only if they do not make $(x+4)(x-1)=0$; if either -4 or 1 is obtained as a root, it may have to be rejected. The only way to decide whether one of these is a solution of the original equation is to test by substitution. It cannot be too strongly emphasised that the ultimate test of every solution is that the values obtained for the variable (or variables) shall satisfy the original equation (or equations).

Note. The steps of an equation are said to be reversible, if the statements are true when read in the reverse order.

Thus, tc solve $3x+2=11$, we write

$$3x+2=11, \quad \therefore 3x=9, \quad \therefore x=3.$$

It would be equally true to write

$$x=3, \quad \therefore 3x=9, \quad \therefore 3x+2=11;$$

i.e. the steps of the equation $3x+2=11$ are reversible. Whenever the steps are reversible, it is not strictly necessary (although always advisable) to check the solution.

(B) **Division.** Our fourth axiom states that if equals are divided by equals, the quotients are equal, i.e. if $x=y$, then $x \div a = y \div a$, with the proviso that $a \neq 0$. Caution must be exercised whenever each side of an equation is divided by an expression containing the unknown; in general this leads to the loss of one or more roots.

Thus, we cannot say $7x^3-6x^2-x=0$, ...(i)

$\therefore 7x^2-6x-1=0$, etc. ....(ii)

The operation of dividing each side of (i) by x has no meaning if $x=0$. We cannot therefore state unconditionally that (ii) is a consequence of (i), but we may say that if x is not equal to 0, then $7x^2-6x-1=0$; or, in other words, *either* $x=0$ *or* $7x^2-6x-1=0$.

It follows that if, in solving an equation, we divide each side by an expression containing the unknown, we must examine values which make the expression zero and test whether they are solutions of the original equation. Division by an expression containing the unknown is a non-reversible step.

It may also be noted that squaring is another non-reversible step. Thus, if $x=2$, we may say that $x^2=4$, but it is not true to say that if $x^2=4$, then $x=2$. There is another possible consequence, viz. $x=-2$. We shall return to this in a later chapter.

128. The following examples illustrate the need for caution whenever each side of the equation is multiplied by an expression containing the unknown.

Example 1. *Solve* $\dfrac{5}{(x+3)}=\dfrac{3x}{2(x-2)}-\dfrac{15}{(x+3)(x-2)}$. ....(i)

Multiplying each side by $2(x+3)(x-2)$, we have

$$\frac{5}{(x+3)}\times 2(x+3)(x-2)$$

$$=\frac{3x}{2(x-2)}\times 2(x+3)(x-2)-\frac{15}{(x+3)(x-2)}\times 2(x+3)(x-2). \quad \text{......(ii)}$$

Provided that $(x+3)\neq 0$ and $(x-2)\neq 0$, this reduces to

$$10(x-2)=3x(x+3)-30. \quad \text{.....................(iii)}$$

[If either $x+3=0$ or $x-2=0$, lines (i) and (ii) have no meaning, for each then contains two undefined terms.]

We may therefore use (iii) to obtain solutions of (i) on the understanding that if we obtain either $x=-3$ or $x=2$ as a solution, it must be checked by substitution in (i).

From (iii) we obtain $\quad 0=3x^2-x-10,$

$$\therefore\ 0=(x-2)(3x+5),$$

$$\therefore\ x=2 \text{ or } -1\tfrac{2}{3}.$$

Check. When $x=-1\frac{2}{3}$, $\quad$ L.H.S. $=\dfrac{5}{1\frac{1}{3}}=\dfrac{15}{4}$.

$$\text{R.H.S.}=-\frac{5}{2(-3\frac{2}{3})}-\frac{15}{(1\frac{1}{3})(-3\frac{2}{3})}=\frac{15}{22}+\frac{135}{44}=\frac{15}{4},$$

$$\therefore\ x=-1\tfrac{2}{3} \text{ is a solution.}$$

[Instead of checking, we might state that since $x=-1\frac{2}{3}$ does not make $2(x+3)(x-2)$ zero, every step is reversible, $\therefore$ $x=-1\frac{2}{3}$ is a solution.]

When $x=2$,

L.H.S. $=\frac{5}{5}=1$.

R.H.S. $=\frac{6}{0}-\frac{15}{0}$, which is meaningless,

$\therefore$ we cannot say that $x=2$ is a solution. It was introduced by multiplying by $2(x+3)(x-2)$, and the step from (i) to (ii) is not reversible.

The presence of a non-reversible step makes it essential to check by substituting in the original equation.

Example 2. *Solve* $\frac{1}{x-8}+\frac{1}{x-4}=\frac{1}{x-5}+\frac{1}{x-7}$.

We have $$\frac{2x-12}{(x-8)(x-4)}=\frac{2x-12}{(x-5)(x-7)},$$

$$\therefore\ (2x-12)(x-5)(x-7)=(2x-12)(x-8)(x-4),$$

provided that $$(x-8)(x-4)(x-5)(x-7)\neq 0;$$

$$\therefore\ (2x-12)[x^2-12x+35-(x^2-12x+32)]=0,$$

$$\therefore\ 2\,.\,(x-6)\,.\,3=0,$$

$$\therefore\ x=6.$$

Since $x=6$ does not make $(x-8)(x-4)(x-5)(x-7)$ zero, every step is reversible, $\therefore\ x=6$ is a solution.

Or, we may check as usual by substitution.

The following exercise contains harder revision examples of the type given in Exercises 29 and 32, in addition to examples involving fractional equations.

EXERCISE 63. a

Solve the following equations. If the roots are irrational, and also in Nos. 6–8, give the answers correct to two decimal places.

1. $\frac{2}{3}\left(2x-\frac{1}{2}\right)-\frac{6}{7}(27-4x)=5-4x$. **2.** $\frac{x-2}{7}+\frac{25-x}{5}=3-\frac{8+x}{4}$.

3. $\frac{0{\cdot}75+2t}{3}-\frac{4t-0{\cdot}47}{5}=-\frac{8{\cdot}8t}{1{\cdot}5}$.

4. $\frac{1}{2}(3-5x)-\frac{1}{6}(5-8x)=\frac{4}{5}(1-3x)$.

5. $0{\cdot}06x-0{\cdot}05=0{\cdot}19$. **6.** $5(t-1{\cdot}5)-2(1{\cdot}3-t)=2{\cdot}2$.

7. $\frac{x}{0{\cdot}4}-\frac{x+5}{0{\cdot}6}=30-\frac{3x-2}{0{\cdot}5}$.

8. $0{\cdot}25(2y-0{\cdot}5)+0{\cdot}75(4y-1{\cdot}2)=2$.

9. $x(x+4)(x-6)=(x-2)^2(x+2)$.

10. $(x-1)(2x-7)=(x-2)(x-4)$.

11. $2(2x-3)^2+3(2x+3)^2=5(2x+2)^2$.

12. $(x-2)(x-4)(x-6)=(x+2)(x-5)(x-9)$.

13. $\frac{3}{x}+\frac{2}{y}=14$; $\frac{5}{x}-\frac{3}{y}=17$. **14.** $\frac{4}{x}-\frac{2}{5y}=3$; $\frac{6}{x}-\frac{1}{y}=5$.

[*Hint.* Take $\frac{1}{x}, \frac{1}{y}$ as the unknowns.]

15. $\frac{3y-5x}{2}+\frac{3x+5y}{3}=0,$
$1\cdot3x+5\cdot7y+19=0.$

16. $\frac{1}{8}(3x+y)+\frac{1}{6}(x-2y)+\frac{7}{10}=0,$
$5x-2y+6\frac{2}{5}=0.$

17. $\frac{3x}{2}-\frac{7}{3y}+2=2x-\frac{3(7-y)}{4y}=3(x+1).$

18. $-\frac{9}{x}-5y+3=\frac{1}{2}\left(5y-\frac{2}{x}-4\right)=\frac{11}{2}+y.$

19. $\frac{1}{7}(2x-y+7)=\frac{1}{2}(y+2),$
$\frac{3}{11}(4y-5x-13)=\frac{1}{3}(x+3).$

20. $\frac{4x-3y+1}{3}-\frac{2x+y+2}{5}+\frac{5}{7}=0,$
$\frac{7(2x-y)}{5}-\frac{4x-y+1}{2}+1=0.$

21. $10(x+1)+\frac{10}{x+1}=29.$

22. $\frac{(2-x)(3-x)}{(1-x)(5-x)}=1\frac{2}{3}.$

23. $\frac{2x}{2x+1}+\frac{2x+1}{2x}=3.$

24. $\frac{x+3}{2(x-5)}=\frac{x+7}{3(x-5)}+\frac{1}{4}.$

25. $\frac{2}{3x-1\cdot5}-\frac{2}{3x+2}=4.$

26. $\frac{x-2}{x-4}-\frac{x+2}{4x-11}=\frac{3}{4}.$

27. $\frac{3}{x-3}-\frac{4}{3}=\frac{9+4x-x^2}{x^2-2x-3}.$

28. $\frac{x+8}{2x}=\frac{x+12}{3x}+\frac{1}{6}.$

29. $\frac{1}{3x+3}-\frac{2x-1}{4x+3}=\frac{3}{4x+4}-\frac{6x+1}{12x+9}.$

30. $\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}=0.$

31. $\frac{1}{2(x-1)}+\frac{6x-7}{9(x^2-1)}+\frac{1}{12x}=0.$

32. $\frac{3x-1}{3x+1}+\frac{x+1}{x-1}-2\left(\frac{3x+2}{3x-2}\right)=0.$

33. $\frac{2}{x-5}-\frac{3}{x-6}=\frac{4}{x-7}-\frac{5}{x-8}.$

34. $\frac{1}{3x-10}+\frac{1}{3x-2}=\frac{1}{3x-5}+\frac{1}{3x-7}.$

35. $\frac{1}{3(x+1)}+\frac{2}{3(x-1)}=\frac{9x-4}{9x^2-4}.$

36. $\frac{2}{2x-5}+\frac{1}{x-3}=\frac{6}{3x-1}.$

37. $\frac{2}{3x}+\frac{3}{3x-1}=\frac{1}{3x+2}.$

38. $\frac{2x-1}{x-3}-\frac{2(x+1)}{x-2}=\frac{8}{x}.$

39. $x-1=\frac{x^2+2}{x-1}+\frac{x+2}{x-6}.$

40. $\frac{1}{2(x-4)(x+6)}+\frac{2}{3(x-4)^2}=\frac{7}{6x(x+6)}.$

EXERCISE 63. b

(*See note at head of Exercise 63 a*)

1. $2\left(\frac{x}{5}-\frac{1}{3}\right)+5\left(\frac{x}{10}-3\right)=4.$ **2.** $\frac{2x+7}{2\frac{5}{8}}-\frac{3+x}{\frac{1}{3}}=0.$

3. $\frac{2x-1}{2\cdot5}-\frac{10}{7}(12\cdot5-x)=5-2x.$ **4.** $\frac{4t-5}{3}-\frac{4}{5}=\frac{1}{10}\left(\frac{7t}{2}-5\right).$

5. $\frac{0\cdot3x-0\cdot2}{0\cdot5}-\frac{0\cdot5x+0\cdot1}{1\cdot6}=\frac{4x-1}{15}.$

6. $0\cdot2(x-0\cdot5)-0\cdot5(0\cdot25x-0\cdot125)=0\cdot35.$

7. $\frac{3\cdot5x+7}{0\cdot5}-1\cdot5(13-x)+17=0.$

8. $\frac{0\cdot3x+1\cdot8}{0\cdot07}-\frac{0\cdot9x-0\cdot3}{0\cdot09}=\frac{0\cdot3x}{0\cdot1}.$

9. $(2x-3)(x+1)(x+3)=(2x+1)(x^2+2x-2).$

10. $(x-4)(x-5)(x-6)=(x^2-4)(x-15)-24.$

11. $(2x-1)(3x+7)=(3x-2)^2.$

12. $(4x-3)(3x+1)=x(8x-3)+0\cdot5.$

13. $\frac{4}{x}+3y=-4;\ \frac{6}{x}-4y=11.$ **14.** $\frac{3}{2x}-2y=5,\ \frac{6}{x}+\frac{y}{7}=8\frac{1}{2}.$

[*Hint.* Take $\frac{1}{x}$, y as the unknowns.]

15. $\frac{5}{4+\frac{1}{2}y}-\frac{9}{20-\frac{1}{3}x}=\frac{9y}{2}-\frac{5x}{3}-26=0.$

16. $\frac{3x+4}{3}=\frac{2y-1}{2},\ \frac{4y-3x}{9x-4y}=1\frac{2}{3}.$

17. $\frac{1}{7}\left(\frac{7}{x}-\frac{3}{y}-1\right)=\frac{1}{4}\left(\frac{4}{x}-\frac{1}{y}-4\right)=\frac{3}{x}-\frac{2}{y}.$

18. $3y+4x=0,\ \frac{3}{4x}-\frac{4}{9y}=\frac{13}{36xy}.$

19. $\frac{x-2y}{3}=\frac{3x-y}{7},\ 3(x+y)=8.$

20. $3x-y-3\frac{7}{12}=\frac{1}{3}(2x+3y)+1\frac{1}{3}=\frac{1}{5}(10x+6y)-1\frac{7}{10}.$

21. $\frac{x+1}{x-1}=\frac{x+2}{x+6}$.

22. $\frac{2x+7}{2(2x-1)}-1=\frac{2x+11}{3(2x-1)}$.

23. $\frac{26x}{5}=1-\frac{1}{2x-3}$.

24. $\frac{1}{x-4}+\frac{1}{x-5}=\frac{2}{x-6}$.

25. $\frac{2}{3x-4}+\frac{1}{3x-2}+\frac{9}{9x^2-18x+8}=0$.

26. $\frac{2x-3}{x-1}-\frac{x-7}{x+2}=2\frac{1}{3}$.

27. $\frac{3}{5x-2}-\frac{2}{5x+3}=\frac{3}{4}$.

28. $\frac{2x+11}{2(2x+3)}-\frac{1}{6}=\frac{2x+15}{3(2x+3)}$.

29. $\frac{x^2-13}{x^2+2x-3}+\frac{3}{x-1}=1\frac{1}{3}$.

30. $\frac{1}{x+11}-\frac{1}{12x+54}=\frac{3}{2x+22}-\frac{1}{6x+27}$.

31. $\frac{2}{x-1}+\frac{3}{2(x+1)}=\frac{1}{2(x+2)}$.

32. $\frac{2x+2}{x-1}-\frac{x-1}{x+1}=\frac{x}{x-2}$.

33. $\frac{1}{2x-3}-\frac{3}{2x-15}=\frac{5}{2x-7}-\frac{7}{2x-11}$.

34. $\frac{1}{x+3}-\frac{2}{x+4}=\frac{3}{x+5}-\frac{4}{x+6}$.

35. $\frac{2x+4}{2x+1}+\frac{6x-20}{2x-7}=\frac{8(x-1)}{2x-3}$.

36. $\frac{x^2+2}{x-1}=x+1-\frac{x+2}{x-2}$.

37. $\frac{x-3}{x-5}+\frac{x-5}{x-7}-2=0$.

38. $\frac{1}{2x}+\frac{2}{2x-1}+\frac{3}{2x+1}=0$.

39. $\frac{15}{x-15}-\frac{9}{x-9}=\frac{6}{x+6}$.

40. $\frac{x+3}{x-1}+\frac{x-3}{x+2}=\frac{2(x-1)}{x+1}$.

LITERAL EQUATIONS

129. The change of subject of a formula which was considered in the last chapter is a particular case of a **literal equation**. Literal equations are equations in which some or all of the coefficients contain letters. In such equations, unless otherwise stated, it is understood that the letters at the end of the alphabet x, y, z, are regarded as the unknowns to be found in terms of the other letters.

No new principles are involved. The methods of solving literal equations are essentially the methods for solving equations with numerical coefficients. These methods have already been used in changing the subject of a formula, and others are illustrated by the worked examples which follow.

Example 3. *Solve* $c(x-5d)=d(2d-x)+3c^2$.

Removing brackets, $cx-5cd=2d^2-dx+3c^2$.

Rearranging to get **all** the terms containing x on one side of the equation and **all** the remaining terms on the other side,

$$cx+dx=3c^2+5cd+2d^2,$$

$$\text{i.e. } x(c+d)=(c+d)(3c+2d),$$

$$\therefore\ x=\frac{(c+d)(3c+2d)}{(c+d)}=3c+2d, \text{ provided that } c+d\neq 0.$$

If $c+d=0$, the equation is an identity and is satisfied by all values of x. The check is left to the pupil.

Note 1. It is essential that the pupil should realise the nature of the answer to be expected. In this question x is the unknown letter and c, d are the known letters.

It is required to express the unknown in terms of the known, i.e. x must be found in terms of c and d **only**.

Note 2. It is essential that x should be expressed in its simplest form. Thus $x=\dfrac{3c^2+5cd+2d^2}{c+d}$, although a true statement, is not the required answer, for the expression on the R.H.S. has not been reduced to its lowest terms.

EXERCISE 64. a

Solve the equations :

1. $3l-x=2m-5$.
2. $p(x-3q)=8a$.
3. $ax-3(x-2)=b$.
4. $l(x-4)-m=4(x-l)$.
5. $x(4c+3)+2c=4c(x+5)-3$.
6. $(x-1)(x-2)=x(x-a)$.
7. $\dfrac{x}{a}-2=\dfrac{x}{2}-a$.
8. $\dfrac{x}{c}+d=\dfrac{x}{d}+c$.
9. $\dfrac{2}{x}-\dfrac{a}{x}+\dfrac{b}{x}=3c$.
10. $2m-\dfrac{3l}{x}=5n$.
11. $\dfrac{1}{a}(2x+b)-\dfrac{1}{b}(3x-c)=d$.
12. $\dfrac{ax-b^2}{a}=\dfrac{a(b-x)}{b}+a-\dfrac{b^2}{a}$.
13. $\dfrac{1}{x}-\dfrac{a}{b}+\dfrac{b}{a}=0$.
14. $V=\pi l[x^2-(x-t)^2]$.
15. $\dfrac{x-a+b}{x+a-b}=\dfrac{1-a+b}{1+a-b}$.
16. $\dfrac{7p-x}{q-3p}+4=\dfrac{3x-5p}{3q-p}$.
17. $\dfrac{a-x}{b}+\dfrac{b+x}{a}=2$.
18. $\dfrac{5l-x}{4m}=\dfrac{4m+x}{5l}$.

EXERCISE 64. b

Solve the equations :

1. $l-2mx=3n$.
2. $px=2q(r-3x)$.
3. $3c-2x=6d-c$.
4. $a(nx-d)=b(mx+c)$.
5. $(c+2d)x=h-dx$.
6. $(x-a)(x+a)=x(x+b)$.
7. $\frac{(m-n)}{n}-\frac{x}{m}=\frac{(n+x)}{m}$.
8. $ax-\frac{x}{2b}=3cx$.
9. $\frac{2}{x}-\frac{1}{c}=\frac{3}{d}$.
10. $\frac{c}{x}-\frac{d}{x}=a-b$.
11. $\frac{1}{2}(2b-3ax)+\frac{1}{4}(ax+3b)=3\left(b-\frac{ax}{4}\right)$.
12. $k-\frac{1}{1-x}=\frac{1}{k}$.
13. $\frac{x}{x-a}+\frac{x}{x-b}=2$.
14. $\frac{x}{2}+\frac{7cx}{40b^2}+\frac{10b^2}{c}=\frac{cx}{24b^2}-\frac{8}{3}$.
15. $\frac{a-3x}{b}+\frac{b-3x}{a}+2=0$.
16. $\frac{x}{lm}+\frac{2x}{mn}+\frac{3x}{nl}=2l+3m+n$.
17. $m(b+x)-am=l(b+x)-al$.
18. $x^2-l^2=(2l-x)^2$.

EXERCISE 64. c

(*Examples marked * should be postponed until Chapter XXIV has been read*)

Solve the equations :

1. $\frac{x-2a}{2a+b}+\frac{x-b}{2a-b}=1$.
2. $\frac{x+3a}{3a+2b}+\frac{x+2b}{3a-2b}=\frac{(3a+2b)^2}{9a^2-4b^2}$.
3. $\frac{x-l}{l-3m}-\frac{x+3m}{l+3m}=\frac{2l(x-3m)}{l^2-9m^2}+\frac{l-3m}{l+3m}$.
4. $\frac{x-5a+b}{5a-b}-\frac{x}{5a+b}=\frac{10a(x+5a-b)}{25a^2-b^2}-1$.
5.* $\frac{a(2x-a)}{2x+a-2b}+\frac{b(2x+a-2b)}{2x-a}=a+b$.
6.* $\frac{l(l+2x)}{m}+\frac{m(m-2x)}{l}=l+m$.
7. $\frac{x+12a}{x-2a}+\frac{3x-8a}{x+4a}=\frac{2(2x+a)}{x+2a}$.
8. $\frac{x+5l}{x-10l}+\frac{1}{3}=\frac{2(2x+15l)}{3x+20l}$.
9.* $\frac{x-mn}{l}+\frac{x-nl}{m}+\frac{x-lm}{n}=2(l+m+n)$.
10. $\frac{b-c}{x-a}+\frac{c-a}{x}+\frac{a-b}{x-c}=0$.
11.* $\frac{m(m-x)}{n}-x=\frac{n(n+x)}{m}$.
12.* $x(a+b)^2+4a^2b^2=3abx+4(a^2+b^2)^2$.

SIMULTANEOUS EQUATIONS WITH LITERAL COEFFICIENTS

130. Example 4. *Solve the equations*

$$lx + my = n, \quad \ldots\ldots\ldots\ldots(1)$$

$$px + qy = r \quad \ldots\ldots\ldots\ldots(2)$$

[In this question x and y are the unknown and l, m, n, p, q, r the known letters. We have therefore to find x and y in terms of l, m, n, p, q, r **only.**]

Multiply (1) by q, and (2) by m; we then have

$$lqx + mqy = nq,$$

$$mpx + mqy = mr;$$

whence by subtraction

$$x(lq - mp) = nq - mr,$$

$$\therefore\ x = \frac{nq - mr}{lq - mp}, \text{ provided that } lq - mp \neq 0.$$

We might now obtain y by substituting in (1), but the work involved is awkward and it is easier to proceed thus:

Multiply (1) by p and (2) by l; we then have

$$lpx + mpy = np,$$

$$lpx + lqy = lr;$$

whence by subtraction

$$y(mp - lq) = np - lr,$$

$$\therefore\ y = \frac{np - lr}{mp - lq}, \text{ provided that } mp - lq \neq 0.$$

It is usual to write $y = \dfrac{lr - np}{lq - mp}$, so that the fractions which equal x and y have the same denominators.

Note. If $lq - mp = 0$, there are in general no finite values of x and y which satisfy both equations.

If, however, in addition $nq - mr = 0$ (or $np - lr = 0$), it is possible to find an unlimited number of solutions.

In this case we have $\dfrac{l}{p} = \dfrac{m}{q} = \dfrac{n}{r}$ and the second equation is equivalent to the first; any values of x and y which satisfy the first will also satisfy the second.

Thus, (1) the equations $2x - 3y = 6$, $4x - 6y = 11$ have no finite solutions; (2) the equations $2x - 3y = 6$, $4x - 6y = 12$ have an unlimited number of finite solutions.

In (1) the graphs of the equations are parallel straight lines: in (2) they are coincident straight lines.

EXERCISE 65. a

Solve the equations :

1. $lx+my=l^2+lm,$
 $x+y=2l.$

2. $x+y=p+2q,$
 $2qx+py=4pq.$

3. $x+y=6l,$
 $3lx-my=9l^2+m^2.$

4. $cx-dy=0,$
 $x+y=0.$

5. $ax+by=a^2+2ab-b^2,$
 $bx+ay=a^2+b^2.$

6. $l=7x+8y,$
 $m=4x+14y.$

7. $x-ay+b=1=ax-y+b.$

8. $\frac{ax}{2}-\frac{by}{3}=c=\frac{bx}{2}+\frac{ay}{3}.$

9. $\frac{x}{3}-\frac{2y}{5}=4l,$
 $\frac{x}{3}+\frac{y}{5}=l.$

10. $\frac{y}{a}-\frac{x}{4}=\frac{a-4}{2},$
 $\frac{x}{a}+\frac{y}{4}=\frac{-4-a}{2}.$

11. $(a-2b)x-(a+2b)y=-6ab,$
 $(a-2b)y-(a+2b)x=-2ab.$

12. $(5a-4b)x-(5a+4b)y=25a^2-40ab-16b^2,$
 $40ab=4bx+5ay.$

13. $(7a-11b)x+(7a+11b)y=2(49a^2+121b^2),$
 $7ay=11bx.$

14. $3ax-5by=6a+5b,$
 $9a^2x-25b^2y=9a^2+50b^2.$

15. Show that if $ax+by+c=0$ and $lx+my+n=0$, then
$$px+qy+r=0 \text{ cannot also be true unless}$$
$$p(bn-cm)+q(cl-an)+r(am-bl)=0.$$
Find the value of p if $2x+y-4=0$, $3x-y-1=0$, $px+2y-5=0$, are all satisfied by one pair of values of x and y.

EXERCISE 65. b

Solve the equations :

1. $2ax-3by=-12ab,$
 $3bx+2ay=4a^2-9b^2.$

2. $x+y=2l-m,$
 $2l(x+2l)=m(m-y).$

3. $cx-3dy=3cd,$
 $3dx+y=18d^2+c.$

4. $x-y=2a,$
 $ax+by=a^2+b^2.$

5. $ax+by=a-b,$
 $bx-ay=a+b.$

6. $7y-9x+60=u,$
 $4y-5x+30=v.$

7. $ax+by=bx-ay=a^2+b^2.$

8. $-(2mx+3ly)=3l-2m=x+y.$

9. $\dfrac{2x}{3}+\dfrac{3y}{4}=-8a,$
$x-\dfrac{y}{2}=53a.$

10. $\dfrac{x}{2l}+ym=m,$
$\dfrac{x}{m}-2yl=6l.$

11. $4a^2x+2y=(2a-1)^2,$
$x+2y=0.$

12. $(a-b)x+(a+b)y=a^2-2ab-b^2,$
$(a+b)(x+y)=a^2+b^2.$

13. $ax-by=a^3-3ab^2,$
$bx+ay=3a^2b-b^3.$

14. $5x+y=2a,$
$3y-10x=a-10b.$

15. If
$$2abu=(a^2+b^2)x-(a^2-b^2)y,$$
and
$$2abv=(a^2+b^2)y-(a^2-b^2)x,$$
prove that
$$\frac{u+v}{x+y}=\frac{x-y}{u-v}=\frac{b}{a}.$$

QUADRATIC LITERAL EQUATIONS

131. Example 5. *Solve* $\dfrac{x^2-x+1}{x^2+x+1}=\dfrac{a^2-a+1}{a^2+a+1}.$

[It is a help to notice that $x=a$ is obviously one solution.]

We have, provided that $(x^2+x+1)(a^2+a+1)\neq 0,$
$$(x^2-x+1)(a^2+a+1)=(x^2+x+1)(a^2-a+1),$$
$$\therefore\ 2ax^2-2x(a^2+1)+2a=0,$$
$$\therefore\ 2(x-a)(ax-1)=0,$$
$$\therefore\ x=a \text{ or } \frac{1}{a}.$$

132. Formula for the solution of quadratics.

Example 6. *Solve* $ax^2+bx+c=0.$

The expression on the L.H.S. has no simple factors, so that it is necessary to solve by completing the square.

Thus,
$$ax^2+bx=-c,$$
$$\therefore\ x^2+\frac{b}{a}x=-\frac{c}{a},\ \text{since } a\neq 0,$$
$$\therefore\ x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a},$$
$$\therefore\ \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},$$
$$\therefore\ x+\frac{b}{2a}=\frac{+\sqrt{b^2-4ac}}{2a} \text{ or } \frac{-\sqrt{b^2-4ac}}{2a}.$$

[If b^2-4ac is a negative number, the roots will be imaginary.]

$\therefore$ either $x = -\dfrac{b}{2a} + \dfrac{\sqrt{b^2-4ac}}{2a}$ or $x = -\dfrac{b}{2a} - \dfrac{\sqrt{b^2-4ac}}{2a}$.

This is usually written $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

This is the formula for solving any quadratic equation.

For all values of a, b, c the roots of the quadratic equation $ax^2+bx+c=0$ are $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Example 7. *Solve* $6x^2 = 2x + 1$.

We have $6x^2 - 2x - 1 = 0$. This is equivalent to $ax^2+bx+c=0$, if $a=6$, $b=-2$, $c=-1$;

$\therefore$ the roots are $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(-1)}}{2(6)} = \dfrac{2 \pm \sqrt{28}}{12}$;

$$\therefore\ x = \frac{2 \pm 5{\cdot}292}{12}, \quad \therefore\ x = \frac{7{\cdot}292}{12} \text{ or } -\frac{3{\cdot}292}{12}.$$

$\therefore$ $x = 0{\cdot}61$ or $-0{\cdot}27$, correct to 2 decimal places.

Note 1. The use of the formula for the solution of quadratic equations is not recommended for beginners. The factor method should always be tried first, and if that fails, it is better to solve by completing the square. If the formula is used, the pupil should substitute first and simplify afterwards. It is unwise to try to do both at once. If the pupil intends to use the formula, he must learn it by heart and he must know what the letters stand for.

Note 2. If the roots of a quadratic are rational, the equation may be solved by the factor method. The condition for this is that b^2-4ac should be a perfect square, i.e. **ax^2+bx+c may be resolved into simple factors, if b^2-4ac is a perfect square.**

EXERCISE 66. a

Solve the following equations :

1. $3x^2 + 10cx - 8c^2 = 0$.
2. $6x^2 - 2a^2 = ax$.
3. $2x^2 + 15l^2 = 13lx$.
4. $5x^2 + 9x = 5c^2 + 9c$.
5. $x^2 + 2ax + a^2 = k$.
6. $5x^2 - 4nx = 5l^2 - 4ln$.
7. $x^2 + 20cd = 4cx + 5dx$.
8. $2ax^2 + 2ax = 3x + 3$.
9. $x^2 - 9l^2 = m(2x - m)$.
10. $(4x-a)^2 = (2x+3a)^2$.
11. $4(x-a)^2 - 3(x-a) = 1$.
12. $156x^2 - ax = a^2$.

13. $x^2+4mx=k$.

14. $(x+l+1)(x+m+1)=lm$.

15. $x^2+2ax=c^2+2ac$.

16. $x+\dfrac{1}{x-k}=k-2$.

17. $\dfrac{a}{a-x}+\dfrac{b}{b-x}=2$.

18. $\dfrac{x^2+1}{a^2+1}=\dfrac{x+1}{a+1}$.

19. $\dfrac{x+3a}{x+a}+\dfrac{x-3a}{x-a}=\dfrac{a(x+2a)}{x^2-a^2}-\dfrac{2}{3}$.

20. $\dfrac{l+x}{m-x}+\dfrac{m-x}{l+x}=\dfrac{l}{m}+\dfrac{m}{l}$.

21. $x^2(a+2b)=a^2(x+2b)$.

22. $(3x-2c)^2+2(3x-2c)=24$.

23. $\dfrac{2}{x-l}=\dfrac{2}{x+m}+\dfrac{1}{l+m}$.

24. $\dfrac{a}{a-x}+\dfrac{a+b+x}{b}=\dfrac{3}{2}$.

Solve by means of the formula the equations :

25. $4x^2-10x+5=0$. 26. $9x^2=6x+5$. 27. $3+7x-3x^2=0$.

28. $5(x^2-1)=9x$. 29. $a^2+3a+3=0$. 30. $3x+2=x^2$.

EXERCISE 66. b

Solve the following equations :

1. $10x^2+3ax-a^2=0$.

2. $2x^2+21t^2=17tx$.

3. $12x^2=11lx+15l^2$.

4. $x^2-3l=4n$.

5. $3x^2+7x=3a^2+7a$.

6. $3x^2+2lx=3c^2+2cl$.

7. $4x^2-b^2=a(4x-a)$.

8. $2x^2+2bx=3ax+3ab$.

9. $5cx^2-4=2x-10cx$.

10. $3(ax-3b)^2-2(ax-3b)=16$.

11. $x^2-2mx+m^2=9$.

12. $2x^2+10b^2=21bx$.

13. $(x-2a)(x+3b-1)=2a+3b$.

14. $9x^2-12ax=b^2$.

15. $a(x^2+a-1)=x(a^2+x-1)$.

16. $x(3x-2)-p(3p-2)=0$.

17. $3bx\left(x-\dfrac{1}{3b}\right)=\dfrac{2}{3b}$.

18. $\dfrac{x^2+1}{b^2+1}=\dfrac{1+x}{1-b}$.

19. $\dfrac{x-a}{x-a+b}+\dfrac{x-2a}{x-2a-b}=\dfrac{a}{a-b}+\dfrac{2a}{2a+b}$.

20. $\dfrac{x^2+x+1}{3a^2+1}=\dfrac{x^2-x+1}{a^2+3}$.

21. $\left(\dfrac{2-x}{x-c}\right)^2+15=8\left(\dfrac{2-x}{x-c}\right)$.

22. $8x^2(a-3b)=8b^2(a+3x)$.

23. $\dfrac{l}{1+lx}+\dfrac{m}{1+mx}+l+m=0$.

24. $l(l-m)x^2+m(l+m)x=(l+m)^2$.

Solve by means of the formula the equations :

25. $3x^2-2x-3=0$. 26. $5-x=x^2$. 27. $5a^2+6a=10$.

28. $x-2x^2=1$. 29. $8c^2+c-3=0$. 30. $2t^2=7t-4$.

If further practice is required, Exs. 52 *a* and *b*, Nos. 37–60 in Chapter XVII may be done by the formula.

CHAPTER XXII

FURTHER SIMULTANEOUS EQUATIONS

133. (A) **Two equations with two unknowns, one of the first degree and the other of the second. Or, one linear and one quadratic.**

The method used is the method of substitution (see Chapter XIII, Method 2). We use the equation of the first degree to eliminate one of the unknowns from the equation of the second degree.

Example 1. *Solve the equations*

$$4x - 3y = 4. \quad \dots\dots\dots\dots\dots\dots(1)$$

$$4x^2 - 3xy - y^2 = 6. \quad \dots\dots\dots\dots\dots\dots(2)$$

From (1) $4x = 3y + 4$, $\quad \therefore\ x = \dfrac{3y+4}{4}. \quad \dots\dots\dots\dots(3)$

Using (3), substitute for x in (2),

$$\therefore\ 4\left(\frac{3y+4}{4}\right)^2 - \frac{3(3y+4)y}{4} - y^2 = 6,$$

$$\therefore\ \frac{\cancel{4}(9y^2+24y+16)}{\cancel{16}\ 4} - \frac{3(3y^2+4y)}{4} - y^2 = 6,$$

$$\therefore\ 9y^2 + 24y + 16 - 9y^2 - 12y - 4y^2 = 24,$$

$$\therefore\ -4y^2 + 12y - 8 = 0, \quad \therefore\ -4(y-1)(y-2) = 0,$$

$$\therefore\ y - 1 = 0 \text{ or } y - 2 = 0, \quad \therefore\ y = 1 \text{ or } 2.$$

Substituting in (3), if $y = 1$, $x = \dfrac{3+4}{4} = 1\dfrac{3}{4}$;

if $y = 2$, $x = \dfrac{6+4}{4} = 2\dfrac{1}{2}$.

The check is left to the pupil; the solutions must be substituted in (1) and (2).

Note 1. The pupil should always consider whether to substitute for x or for y. If possible, a substitution should be chosen which does not introduce fractions. The resulting quadratic should be solved by factors, if possible.

Note 2. When the value of one unknown has been obtained, the value of the other must be found by substituting in the linear

equation, i.e. **in the equation which has been used for elimination.** If, in Ex. 1, after obtaining $y=1$ or 2, we had substituted in (2), we should have obtained $y=1,\ x=1\frac{3}{4}$ or -1,

$$y=2,\ x=2\tfrac{1}{2} \text{ or } -1.$$

But $x=-1,\ y=1$; $x=-1,\ y=2$ do not satisfy (1) and are not solutions of the pair of equations (1) and (2).

Note 3. In writing down the solutions care must be taken to pair the values correctly. The answer may be written $x=1\frac{3}{4}, y=1$; $x=2\frac{1}{2}, y=2$; or $(1\frac{3}{4}, 1), (2\frac{1}{2}, 2)$. It should **not** be written $x=1\frac{3}{4}$ or $2\frac{1}{2}$, $y=1$ or 2.

Note 4. It should be carefully noted that

$$\left(\frac{3y+4}{4}\right)^2=\frac{9y^2+24y+16}{16}.$$

A very common error is to write $4\left(\frac{3y+4}{4}\right)^2=(3y+4)^2$; this must be avoided. Care with signs is necessary when removing brackets.

The work may be much simplified, if the quadratic has factors.

Example 2. *Solve the equations*

$$10x^2-9xy=7y^2. \quad \text{..........................(1)}$$
$$3x-y=10. \quad \text{..........................(2)}$$

From (1) $10x^2-9xy-7y^2=0$, $\therefore\ (5x-7y)(2x+y)=0$,

$\therefore$ either $5x-7y=0$ or $2x+y=0$.

Hence (1) and (2) are equivalent to the following pairs of equations:

(i) $5x-7y=0,\ 3x-y=10$, giving $x=4\frac{3}{8},\ y=3\frac{1}{8}$;

(ii) $2x+y=0,\ 3x-y=10$, giving $x=2,\ y=-4$;

$\therefore$ the required solutions are $(4\frac{3}{8}, 3\frac{1}{8}), (2, -4)$.

The check is left to the pupil.

EXERCISE 67. a

Solve the following equations (Nos. 28–30, correct to two places of decimals).

1. $4x+y=5,$
$4x^2-y^2=3.$

2. $xy=8+7y,$
$x+2y+3=0.$

3. $x+y=1,$
$x^2-xy=15.$

4. $x^2-4xy=3,$
$3x-4y=5.$

5. $x+2y+1=0,$
$x^2-2xy=3-4x.$

6. $8y=7x^2-12x-11,$
$3x-4y=1.$

7. $3x-10y=4,$
$6y-5x=34xy.$

8. $10x-3y=25,$
$xy-2x-2y=3.$

9. $2x-3y-3=0,$
$x^2-y^2+4=4x.$

10. $27x^2-12xy+5y^2=12,$
$9x-5y=8.$

11. $24y=12x+1,$
$6y=3x+xy.$

12. $3x-2y=1,$
$2y^2-3x^2=5.$

13. $x^2-5xy+3y^2+3=0,$
$3x+1=2y.$

14. $4x-3y=1,$
$8x^2-4xy-3y^2=1.$

15. $x+2y=7,$
$x^2+2y^2=17.$

16. $4(x-y)^2-3(x-y)=1,$
$x+2y=7.$

17. $2y=3+2x,$
$7x^2=2-y^2.$

18. $4x^2+y^2=200,$
$4x-3y=22.$

19. $2x-6y=1,$
$6x^2=6xy+5.$

20. $3x^2-4xy+5y^2=12,$
$5y-3x=8.$

21. $x(4x-3y)=y^2,$
$2x+y=6.$

22. $4y-x=2xy,$
$3x+2y=10.$

23. $2(x^2+y^2)=5xy,$
$x+y=6.$

24. $9x^2+2=9xy+6y^2,$
$3x+2y=2.$

25. $11x^2-20xy+9y^2=16x-14y,$
$6x-5y=0.$

26. $12x^2+6xy-8y^2-35y=87,$
$2y+3=2x.$

27. $15x^2=6y^2-xy,$
$3y-4x=3.$

28. $6x-3y=1,$
$6xy-y^2+12x=y.$

29. $4x-3y=4,$
$\frac{3}{2x}+\frac{2}{y}=6.$

30. $\frac{2y}{3}-\frac{x}{2}=1,$
$8xy+3=0$

EXERCISE 67. b

Solve the following equations (Nos. 24, 29, 30, correct to two places of decimals) :

1. $2x-y=3,$
$4x^2-3y^2=13.$

2. $xy=15,$
$x-2y=1.$

3. $x^2+y^2=52,$
$-2x+y=8.$

4. $2x-3y=4,$
$y^2+7y=2xy-2.$

5. $6x^2-17xy+5y^2=171,$
$3x-y=9.$

6. $2x-5y=3,$
$(x+5y)^2=5xy+1.$

7. $3x+2y=10,$
$2x^2+3y^2=35.$

8. $x^2-y^2+4x=-4y,$
$4y-3x=8.$

9. $9y-2x=7,$
$x^2+3xy=2.$

10. $3xy+4=4y^2,$
$9x=4y+10.$

11. $9x^2 - 2y^2 = 1,$
$15x - 8y + 1 = 0.$

12. $2x^2 - y^2 = 1,$
$3x + 2y = 1.$

13. $3x + 2y = 7,$
$x^2 - xy + y^2 = 3.$

14. $4x - 3y = 1,$
$16x^2 + 9y^2 = 25.$

15. $(2y - 9)(x - 1) = 0,$
$5x = 2y + 1.$

16. $4x^2 + 3xy - 3y^2 = 4,$
$10 - 3y - 5x = 0.$

17. $3x^2 + 4xy + 5y^2 = 12,$
$5y + 3x = 8.$

18. $3y^2 + x^2 = 76,$
$3x - 2y + 7 = 0.$

19. $3x + 1 = 4y,$
$7x^2 + 4xy = 20y^2.$

20. $4x^2 + 27y^2 - 36x = 171,$
$2x + 9y = 15.$

21. $4x + 3y = 1,$
$6x^2 + 6xy + y^2 = 1.$

22. $3y^2 + 40x = 18xy,$
$6x = 2y + 1.$

23. $3x + 2y + 6 = 0,$
$9x^2 - 9xy - 2y^2 + 6y + 4 = 0.$

24. $x^2 + y^2 = 16,$
$2x - 3y = 5.$

25. $(x + y)^2 - x^2 + y^2 - 6(x - y)^2 = 0,$
$3x - 2y = 4.$

26. $2x - 3y = 7,$
$2x^2 - 3xy - 14y^2 = 0.$

27. $4x^2 - 10xy + 25y^2 = 21,$
$4x - 5y = 3.$

28. $-x + y + 1 = 0,$
$2x^2 - xy + 9y^2 = 2.$

29. $y = 6x - 3,$
$9x^2 + 6xy + y^2 = 10.$

30. $3y = 2x - 3,$
$16x^2 - 3y^2 = 4x.$

134. (B) **Simultaneous equations with three unknowns.**

Three equations containing three unknowns may be reduced to two equations containing two unknowns by eliminating one unknown. The elimination may be performed by substitution or as in Ex. 3 below. The two equations so obtained may then be solved by the methods previously described.

Example 3. *Solve the equations* $7x - 5y - 7z = -8,$ ....(1)

$4x - 2y - 3z = 0,$ ....(2)

$5x + 4y + 4z = 35.$ ....(3)

Multiply (1) by 2, then $14x - 10y - 14z = -16.$ ....(4)

Multiply (2) by 5, then $20x - 10y - 15z = 0.$ ....(5)

$\therefore$ from (4) and (5) by subtraction $6x - z = 16.$ ....(6)

Multiply (2) by 2, then $8x - 4y - 6z = 0.$ ....(7)

$\therefore$ from (3) and (7) by addition $13x - 2z = 35.$ ....(8)

Solving (6) and (8) as usual, we obtain $x = 3$, $z = 2$.

Substituting these values in (2), we obtain $y = 3$.

The solution is therefore $x = 3$, $y = 3$, $z = 2$.

The check is left to the pupil.

EXERCISE 68. a

Solve the equations :

1. $x-2y+z=7,$
$2x-6y-z=0,$
$3x-8y+2z=17.$

2. $4x+2y-z=16,$
$5x-3y-z=3,$
$14x+2y-3z=40.$

3. $x+y-z=3,$
$2x-3y+9z=60,$
$7x+3y+3z=69.$

4. $3x+2y-z=6,$
$2x-y=2,$
$5x+z=2.$

5. $3x-4y+5z=4,$
$2x+3y+8=4z,$
$x+14=2y+3z.$

6. $2a+b+c+19=0,$
$a+2b+c+20=0,$
$a+b+9=0.$

7. $y-5z-x=1,$
$x-3y-5z=1,$
$x+y+10z=2.$

8. $x-y-3z=9,$
$11z-x-3y=7,$
$x-13y+21z=53.$

9. $\frac{x}{2}-\frac{2y}{3}+z=10,$
$2x+y=3,$
$\frac{x}{2}-\frac{5y}{3}-2z=16.$

10. $x-y-3z=8,$
$11z-x-3y=7,$
$x-13y+21z=53.$

11. $2x+y-3z=9,$
$x+2y+3z=3,$
$x^2+y^2+9z^2=26.$

12. $x-y=-1,$
$5x-y^2+z^2=1,$
$x-z^2=1.$

EXERCISE 68. b

Solve the equations :

1. $2x+y+z=8,$
$8x-y-3z=26,$
$4x+y+4z=8.$

2. $5x-2y+z=3,$
$6x+y-4z=62,$
$x+2y+z=15.$

3. $2x+y+z=0,$
$3x+y=1,$
$x+z=1.$

4. $4x+5y+2z=3,$
$8x+7y-z=9,$
$7x+8y+3z=6.$

5. $4x+y=6,$
$y-3z=10,$
$-3z+4x=20.$

6. $3x-2y+z=72,$
$-3y+2z-x=48,$
$3x+2z=60.$

7. $4x+3y+8z=2,$
$3x+y+32z=-9,$
$x+y+8z=-3.$

8. $x+\frac{y}{2}-\frac{z}{3}=-3,$
$x+y-\frac{z}{3}=0,$
$3x+\frac{y}{2}-2z=0.$

9. $5x-3y+z=6,$
$13x-7y-3z=14,$
$7x-4y=5.$

10. $2x+y-z=4\frac{1}{2},$
$x+2y+z=1\frac{1}{2},$
$2(x^2+y^2+z^2)=13.$

11. $2x-3y=5,$
$2x-y+2z=5,$
$2x+yz-2z^2=5.$

12. $5x-3y+4z=6,$
$13x-7y-12z=14,$
$4y+8=7x.$

135. (C) Two equations with two unknowns, both quadratic. There is no simple general method, but solutions may be obtained, as follows, if the terms of the first degree are absent from both equations.

Example 4. *Solve* $x^2+5xy+y^2=-5,$ ......(1)
$2x^2+9xy+2y^2=-7.$ ......(2)

Multiply each side of (1) by 7 and each side of (2) by 5,

$\therefore\ 7x^2+35xy+7y^2=-35,$ ......(3)

and $10x^2+45xy+10y^2=-35.$ ......(4)

$\therefore$ by subtraction, $3x^2+10xy+3y^2=0,$

$\therefore\ (3x+y)(x+3y)=0,\quad \therefore\ 3x+y=0$ or $x+3y=0.$

If $3x+y=0$, substituting in (1), we have

$x^2-15x^2+9x^2=-5,\quad \therefore\ -5x^2=-5,\quad \therefore\ x^2=1,$

$\therefore\ x=1$ or $-1.$ ......(5)

If $x=1$, $y=-3$; if $x=-1$, $y=3$, since $3x+y=0$.

If $x+3y=0$, substituting in (1), we have

$9y^2-15y^2+y^2=-5,\quad \therefore\ -5y^2=-5,\quad \therefore\ y^2=1,$

$\therefore\ y=1$ or $-1.$ ......(6)

If $y=1$, $x=-3$, if $y=-1$, $x=3$, since $x+3y=0$.

The solutions are therefore $(1,-3)$, $(-1,3)$, $(-3,1)$, $(3,-1)$.

The check is left to the pupil.

Note 1. After obtaining (5), the corresponding value of y must be obtained from $3x+y=0$, i.e. from the equation which has been used to get (5).

Similarly, after obtaining (6), the corresponding values of x must be obtained from $x+3y=0$, i.e. from the equation which has been used to get (6).

Note 2. As in Art. 133, Ex. 2, the work may be much simplified if one of the quadratics has factors.

Other cases may be solved, if it is possible to deduce an equation of the first degree from the given equations.

Example 5. *Solve* $2x+y=3xy.$ ....(1)

$4x+y=9xy.$ ....(2)

Multiply (1) by 3 and subtract from (2), we then have

$-2x-2y=0$, i.e. $x=-y.$ ....(3)

Substituting (3) in (2), we have $-3y=-9y^2$, $\therefore\ 9y^2-3y=0$,

$\therefore\ 3y(3y-1)=0.$ $\therefore\ y=0$ or $\frac{1}{3}$. ....(4)

If $y=0$, $x=0$; if $y=\frac{1}{3}$, $x=-\frac{1}{3}$, substituting in (3), i.e. in the equation which has been used to get (4).

The solutions are $(0, 0)$, $(-\frac{1}{3}, \frac{1}{3})$. The check is left to the pupil.

EXERCISE 69. a

Solve the equations :

1. $4x^2+6xy+y^2=1$,
$4x^2-2xy+y^2=13.$

2. $y^2+2xy=16$,
$2x^2+y^2-xy=16.$

3. $3x^2=5y^2+7$,
$3xy+4y^2+2=0.$

4. $4xy=3+13y^2$,
$13x^2=42xy+40.$

5. $8x^2-6xy+2y^2=11$,
$4x^2+y^2=8xy-2.$

6. $5y^2=17+7x^2$,
$6x^2+5xy+6=0.$

7. $(x-2)(x+1)=0$,
$9y^2=4x^2.$

8. $xy+5x=2y+10$,
$x^2+y^2=29.$

9. $12x^2-7xy+y^2=0$,
$20x^2-10xy+y^2+1=0.$

10. $x^2-xy=24$,
$xy-y^2=8.$

11. $-xy-x-9y+21=0$,
$20x^2+31xy+12y^2=0.$

12. $x^2-xy+2y^2=2y$,
$x^2+4xy=5y.$

13. $4x^2+y^2=5$,
$4x^2-2xy=6.$

14. $3x^2-3xy-y^2=15x$,
$9x^2-2y^2=3x.$

15. $2x^2-3xy=26$,
$3y^2-2xy=39.$

16. $x^2-3xy=7$,
$y^2+xy=2.$

17. $3x^2-xy=120$,
$3xy-y^2+216=0.$

18. $4x^2+15xy+29=0$,
$xy+5y^2=22.$

EXERCISE 69. b

Solve the equations :

1. $x^2-2xy+4y^2=7$,
$x^2-2xy-4y^2=-1.$

2. $x^2-4xy=12$,
$x^2+12=16y^2.$

3. $4x^2+3xy+5=0$,
$4x^2+2xy-y^2+11=0.$

4. $2x^2+xy=14$,
$3xy+2y^2=36.$

5. $x^2-2xy+10y^2=145$,
$y^2=24+xy.$

6. $3x^2+xy+y^2=15$,
$3x^2-31xy+5y^2+45=0.$

7. $6x^2-19xy+15y^2=0$,
$5x^2+20xy-52y^2=-172.$

8. $(2x+3)(x-5)=0$,
$25y^2=(x+1)^2.$

9. $6x^2-19xy+3y^2=0,$
$x^2+16y^2=25.$

10. $4xy+21=14x+6y,$
$2(x^2+y^2)=29.$

11. $4x^2-9y^2=64,$
$2xy+3y^2=32.$

12. $x^2-3xy-3y^2=5x,$
$3x^2-6y^2=x.$

13. $18x^2-3xy+y^2=2y,$
$18x^2+12xy=5y.$

14. $2x^2-3xy=8,$
$4y^2=2xy+105.$

15. $-4xy+y^2=16,$
$8x^2+2xy=12.$

16. $9x^2=14-3xy,$
$y^2+3xy+10=0.$

17. $x^2-xy=5,$
$y^2+3xy=76.$

18. $x^2+8y=21,$
$x+y^2=5\frac{1}{4}.$

EXERCISE 69. c

Miscellaneous Simultaneous Equations

Solve:

1. $3x^2-4xy+2y^2=1,$
$3x+2y=5.$

2. $xy-9y=1,$
$2xy+3x=14.$

3. $(2y-7)(y-3)=3(x+2)(3x+1),$
$3x+2y=0.$

4. $x^2-xy+y^2=14,$
$x^2+2y^2=22.$

5. $2x+3y=1,$
$x^2-3y^2+2=0.$

6. $4x-3y=1,$
$6x^2+xy-3y^2=4.$

7. $3x+5y=6,$
$2x^2-5y^2=8x+y.$

8. $3x^2+4y^2=19,$
$x^2-xy-5y^2=-17.$

9. $x^2+2xy=6y^2-4x-6,$
$7=2x+3y.$

10. $xy-2x+3y=6,$
$x^2+xy+y^2=12.$

11. $4x^2+20xy+19=100y^2,$
$5y-1=3x.$

12. $2x+3y=6,$
$4x^2+3y^2=4x+6y+24.$

13 $x+y-5z=2,$
$3x+y-5z=8,$
$x-y+10z=-6.$

14. $3x+4y-z=1,$
$4x+6y+4z=-3,$
$2x-2y-5z=-2.$

15. $3x-2y=10,$
$3y+2z=-26,$
$y-2z=18.$

16. $9x^2+2xy+30x=108,$
$9x-2y=48.$

17. $(x+3)(y-5)=8,$
$(x+2)(y-6)=3.$

18. $9x^2-9xy+y^2+1=0,$
$9x-2y+4=0.$

19. $x+y=a-b,$
$(x+a)^2+(y-b)^2=4(a^2+b^2).$

20. $8x^2-6xy-7y^2=122,$
$3y+2x=14.$

21. $\frac{3x}{4}+\frac{xy}{3}=\frac{2y}{3},$
$\frac{x}{2}-y=7.$

22. $\frac{1}{x}-\frac{2}{y}+\frac{2}{z}=5,$
$\frac{2}{x}-\frac{10}{y}-\frac{2}{z}=7,$
$4x+3y+4=0.$

23. $x+y+z=48,$
$x^2-y^2+z^2=12x,$
$x-y=4.$

24. $\dfrac{1}{x+1}+\dfrac{1}{y+2}=\dfrac{1}{5},$
$\dfrac{11}{2}(x+1)+3(y+2)=1.$

TEST PAPERS VI

A

1. A dealer takes off y per cent. of the marked price for cash. At what price must he mark an article so as to obtain x shillings cash for it?

2. Simplify (i) $\left\{a-b-\dfrac{a^3-b^3}{a^2+b^2}\right\}\div\left\{(a+b)^2-(a-b)^2\right\},$

(ii) $\dfrac{3b}{a^2-9b^2}-\dfrac{1}{3b-a}.$

3. For a solid which has two parallel square faces with edges a ft. and b ft. respectively and the perpendicular distance between them h in., V, the volume in cubic ft., may be calculated from the formula

$$V=\frac{h}{72}\left[a^2+b^2+4\left(\frac{a+b}{2}\right)^2\right].$$

Calculate V, if $a=2$, $b=5$, $h=4$.

4. Solve (i) $\dfrac{2}{3(x-2)}-\dfrac{3}{2(x+2)}=\dfrac{3}{x^2-4},$

(ii) $4x+y=3\,;\quad 3x^2-xy=2y^2.$

5. A vacuum-cleaner, bought for £x, is sold for £24 at a profit of x per cent. Find x.

6. If $\dfrac{ax+b}{cx+d}=\dfrac{m}{n}$, find x in terms of the other letters.

B

1. Factorise (i) l^3+l^2-156l, (ii) $m^2-4mn+4n^2-9l^2n^2$.

2. In what proportion should tea at a shillings per lb. be mixed with tea at b shillings per lb., so that the mixture may be worth c shillings per lb.?

3. Simplify (i) $\dfrac{(2-3x)^2}{(3x-2)^3},$

(ii) $\dfrac{4}{(2x+3)(2x-5)}-\dfrac{3}{4x^2-9}-\dfrac{1}{(2x-3)^2}.$

4. Solve (i) $\dfrac{x-1}{x-2}=\dfrac{x-2}{x-1}+\dfrac{5}{6}$,

(ii) $4x^2-6xy-9y^2=1,\ x+2y=1.$

5. A man rode the first half of the distance from X to Y at a speed of 16 m.p.h. and the second half at a speed of $13\frac{1}{3}$ m.p.h. The second half took 27 min. more than the first. Find the distance from X to Y.

6. The following formula is in actual use by engineers:

$$P=\frac{9900\times T}{3\times D}\left[5-\frac{L+12}{60\times T}\right].$$

(1) Prove that T can be expressed in the form $aPD+bL+c$.

(2) Find L, if $P=250$, $D=5$, $T=\frac{1}{8}$.

C

1. Solve (i) $\dfrac{1}{x-a}+\dfrac{1}{x-2a}=\dfrac{2}{x-3a}$,

(ii) $2\left(\dfrac{x}{b}+\dfrac{y}{a}\right)=3\left(\dfrac{x}{a}-\dfrac{y}{b}\right)=6.$

2. An article is marked for sale at a price which gives a profit of x per cent. on the cost price. A discount for cash of y per cent. is given on the marked price. In the case of an article paid for by cash, what is the percentage profit on the cost price?

3. Solve (i) $\dfrac{3x+2}{x+3}=\dfrac{x-1}{3x+4}+\dfrac{3}{2}$,

(ii) $x^2+y^2-6x-y=1;\quad 2x-3y=1.$

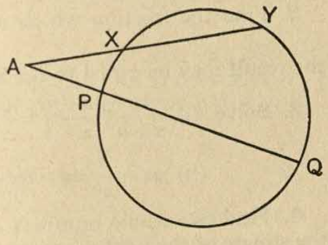
4. Simplify $\left\{\dfrac{4a^2}{(a+b)^2}-1\right\}\left\{\dfrac{4a^2}{(a-b)^2}-1\right\}-1.$

5. A restaurant keeper finds that his daily expenses amount to £12, together with 1s. 4d. per customer. His daily receipts are given by the equation $y=\frac{1}{8}x-7$, where x is the number of customers and y the number of £'s taken. Plot graphs of receipts and expenditure, taking a range from 300 to 400 daily customers, and find for what number of customers his daily receipts and daily expenditure are equal. Find also from your graph in £'s his daily profit when the number of customers is 360.

6. The area of a rectangular field is 5 acres, and a longer side exceeds a shorter side by 36 yards. Find the lengths of the sides, correct to the nearest yard.

D

1. If $AX=a$ in., $XY=b$ in., $AP=c$ in., $PQ=d$ in., find a relation between a, b, c, d.



2. Factorise
   (i) $x^2-y^2-8y-16$,
   (ii) $1-3(a+b)-4(a+b)^2$.

3. Solve (i) $4x-2y-z=10$,
$3x+4y+7z=22$,
$5x-6z=-14$.
(ii) $\dfrac{x^2+1}{x^2+2x-1}=a$.

4. A is four times as old as B; in 8 years B will be twice as old as C; in 13 more years A will be three times as old as C. What are their present ages? Check your answer.

5. If $x=1-\dfrac{1}{y}$, express $\dfrac{x^2-1}{x^2+1}$ in terms of y in its simplest form.

6. Two kinds of coffee are mixed in the ratio of $p:q$. The price of the first kind of coffee is z shillings per lb. What must be the price of the second kind so that the mixture may be worth a shillings per lb.?

E

1. A tradesman marks his goods at a price which will give him a profit of x per cent. of the cost price. What percentage of the marked price can he deduct from it as discount, so as to have a profit of y per cent. on the cost price?

2. Solve (i) $\dfrac{x-2}{x-1}=\dfrac{2(x-1)}{x-2}-1$, (ii) $7x+8y=3$, $\dfrac{49x^2}{9}+4y^2=1$.

3. Simplify $\left(\dfrac{x^3}{2y}-\dfrac{8y^3}{x}\right)\left(\dfrac{3x+2y}{x+2y}-\dfrac{3x-2y}{x-2y}\right)$.

4. From the formula $2x=\pm\sqrt{(y+1)(y-3)}$, obtain (i) the values of x when $y=6{\cdot}2$, (ii) the values of y when $x=0{\cdot}75$. Obtain also the corresponding formula which gives y in terms of x.

5. Draw the graph of x^3-6x^2+9x+1 for values of x from 0 to 4. Use the graph to solve the equation $x^3-6x^2+9x=2$.

6. The side of one square exceeds the side of a second square by 4 inches; the area of the first square together with three times the area of the second square is 44 square feet. Find the lengths of the sides in feet, correct to one place of decimals.

F

1. Factorise (i) $25-a^2+2ab-b^2$, (ii) $(l^2+4l)^2-9$.

2. Find the fraction which must be added to $\frac{2x-1}{x^2-x-6}$, so that the result may be equal to the sum of $\frac{3}{x+2}$ and $\frac{2}{x-3}$.

3. Solve (i) $\frac{3c}{x+b}+\frac{b-3c}{x-b}=\frac{b}{x}$,

(ii) $ax-\frac{b}{2y}=c=bx+\frac{a}{2y}$.

4. Find two whole numbers such that, in each case, seven times the square of the next whole number above exceeds eight times the square of the next whole number below by 220.

5. Solve (i) $\frac{14x-3}{20}-\frac{2x+3}{4}=\frac{2}{2x-1}$,

(ii) $x^2+xy+y^2=39$, $x^2-y^2=21$.

6. The volume, V cubic inches, of a hollow metal pipe is given by the formula $V=14\pi lt(d+t)$, where d in. is the internal diameter, t in. the thickness of the metal and l ft. the length of the pipe. Find V, if $l=8$, $d=2$, $t=0{\cdot}25$. Take $\pi=3\frac{1}{7}$.

G

1. In a form consisting of x boys and y girls, the average mark for the boys was p and for the girls was q. What was the average mark for the whole form? What should have been the average mark for the boys in order to make the average mark for the form equal to z, the average mark of the girls remaining the same?

2. In each of the formulae $Pt=m(v-u)$, $Ps=\frac{1}{2}m(v^2-u^2)$ express v in terms of the other letters. Assuming that both formulae are true at the same time, show that $2s=(u+v)t$.

3. Solve (i) $(x-2a)(3x+b)+(x+a)(3x-b)=6ax-3ab-3a^2$,

(ii) $12x^2-11xy+2y^2=0$, $2x-y=2$.

4. Simplify $\frac{1}{a}+\frac{1}{2a^2-a-3}-\frac{2(5a-6)}{5a(2a-3)}$.

5. Find the values of a, b, c so that an expression of the form ax^2+bx+c shall have the values 5, 5, 4 when the values of x are 1, 2, 3 respectively. Denoting this expression by y, draw its graph between $x=-3$ and $x=6$. At what points on the graph is $x=y$?

6. Find three consecutive positive integers such that twice the square of their sum exceeds three times the sum of their squares by 723.

H

1. Factorise (i) $(a^2-b^2)(a+2b)+(a^2+2ab)(a+b)$,
(ii) a^4-a^2-2a-1.

2. In a town whose population is a, p per cent. are married. Of the males alone q per cent. are married; of the females alone r per cent. are married. How many males are there in the town?

3. Simplify $$\frac{\frac{1}{x-y}-\frac{1}{2x}}{1-\frac{x}{2x+y}}-\frac{\frac{2y}{x-y}+1}{2x-\frac{2xy}{x+2y}}.$$

4. Solve (i) $x=\dfrac{a+2y}{a-2y}$; $y=\dfrac{ax}{2(x+a)}$.
(ii) $3x+2y=5$; $6xy-9x=4y-6$.

5. A cyclist, who rides at a steady 8 m.p.h., leaves his house at 8 a.m. A man in a car, which he drives at 24 m.p.h., leaves the same house by the same road at 8.30 a.m.; he goes to a town 30 miles off, stops there 10 minutes and starts back. At what time and how far from the town will he pass the cyclist the second time?

6. Find the value of a which will make $2x=3-\dfrac{a}{x}$ when $x=1{\cdot}7$. If a has this value, what other value of x will also make the statement true?

I

1. A retailer often estimates his profits as a percentage of his "turnover" (sum of receipts and expenditure). If his profit found in this manner is a per cent., what is it as a percentage of his expenditure?

2. Simplify (i) $\dfrac{x^2-3xy+y^2}{x-y}-\dfrac{x^2+3xy+y^2}{x+y}+\dfrac{2y^3}{x^2-y^2}$,
(ii) $\left(\dfrac{p^3}{q}-\dfrac{q^3}{p}\right)\div\left(\dfrac{p}{q}+\dfrac{q}{p}+2\right)$.

3. If $2x-y+1$, $x-3y-2$ and $3x+4y+10$ all have the same value, what is that value?

4. (i) Find the positive integral value of x for which the value of $2x^2-9x$ is nearest to 400.
(ii) Solve $x^2-4y^2=13$; $2x+3y=5$.

5. When the depth of liquid in a vessel is d inches, the quantity Q cu. in. which it contains is given by $Q = 1{\cdot}05\,[(d+2)^3 - 8]$. Calculate values of Q for values of d up to $d = 5$, and plot a graph taking 1″ on the d-axis to represent 1″ of depth, and 1″ on the Q-axis to represent 100 cu. in. Taking 1 quart = 70 cu. in., find from the graph the heights above the bottom of the vessel at which graduation marks must be made to show 1, 2, 3, 4, 5 quarts.

6. In a right-angled $\triangle ABC$, in which $\angle ABC = 90°$, the side AB is 4 mm. longer than the side BC, which is 3·6 cm. shorter than the side AC. Find the length of AB.

J

1. Solve (i) $\dfrac{x+b}{4a+b} - \dfrac{x+2a}{2(a+b)} = \dfrac{b-2a}{b+2a}$.

Is the result affected if $b = 2a$?

(ii) $\left.\begin{aligned} (a-b)(x-3y)+(a+b)(x+3y)&=0,\\ ax-3by-2ab&=0.\end{aligned}\right\}$

2. Factorise (i) $25b^2 - z^2 + c^2 - 10bc$,

(ii) $(a^2-2a-5)^2-(a^2+4a+1)^2$.

3. Simplify $\left\{\dfrac{x^2-y^2}{x^2+y^2} \times \dfrac{\dfrac{x}{y}+\dfrac{y}{x}+2}{\left(\dfrac{x}{y}-1\right)\left(\dfrac{y}{x}+1\right)}\right\} - \dfrac{2}{\dfrac{y}{x}+\dfrac{x}{y}}$.

4. Water flows at the rate of v ft. per sec. in a pipe of circular section and diameter x in., and a gallons per minute are delivered.

Taking 1 cubic ft. $= 6\frac{1}{4}$ gallons, prove that $v = \dfrac{192a}{125\pi x^2}$.

Evaluate v, if $a = 91\frac{2}{3}$, $x = 3\frac{1}{2}$, $\pi = 3\frac{1}{7}$.

5. Solve (i) $\dfrac{2x+3}{x-1} - \dfrac{2-3x}{1+x} = \dfrac{25}{4}$,

(ii) $\dfrac{x}{2}+\dfrac{y}{3}=1$, $4xy=3$, correct to one decimal place.

6. The area of a rectangular field is 2 acres, and a longer side exceeds twice a shorter side by 12 yards. Find the lengths of the sides, correct to the nearest yard.

K

1. Find x and y, each in terms of K, from the equations

$$\tfrac{2}{3}(x-2K)+\tfrac{4}{5}(K+2)=0=\tfrac{1}{4}(y+K-1)+\tfrac{1}{3}(2K-1).$$

If, in addition to the above equations, $2x+3y+5K=11$, find the values of K, x, y.

2. Solve (i) $abx^2=(a-b)^2(x+1)$,

(ii) $\frac{2}{x}+\frac{3}{y}+1=0,\ 12xy=-1.$

3. (i) Of the population of a town a per cent. are adults ; of the adults k per cent. are men. What is the percentage of women in the town?

(ii) If $y=a(x-1)(x-2)$ and if $y=4$ when $x=3$, find y when $x=4$.

4. Reduce the fraction $\frac{a-b}{1-ab}$ to its simplest form when

$$a=\frac{x}{z-x},\quad b=\frac{y}{z-y}.$$

5. A rectangular sheet of paper is 10″ long and 8″ wide. A rectangle is described on the paper with its sides parallel to and at the same distance from the corresponding edges of the paper. The area of the rectangle is seven-sixteenths of the area of the paper. Find the distance of each side from the nearest parallel edge of the paper.

6. Write down the values of a, b, c which make the expression

$$a(x-1)(x+3)+b(x+3)(x-3)+c(x-3)(x-1)$$

equal to 3, -1, 1 when x is equal to 3, 1, -3 respectively. Draw the graph of the expression between $x=-4$ and 4.

L

1. A man has n articles, each bought at the same price. He sells p of them at a profit of a per cent. and the rest at a loss of b per cent. Express his profit on the whole stock as a percentage.

2. A circular washer, of thickness t in., is of external diameter x in. and internal diameter y in. The weight W oz. of the washer is given by the formula $W=3{\cdot}5(x^2-y^2)t$. Change the subject of the formula to y, and find the internal diameter of a washer whose weight is $2\frac{3}{16}$ oz., external diameter 3 in. and thickness $\frac{1}{8}$ in.

3. Factorise (i) $x^6+2x^3+1-25x^4$,

(ii) $(a^2-a)^2-5(a^2-a)+6$.

4. Solve (i) $x-2y+z=7,\ 3x-y-z=10,\ x+y=2.$

(ii) $x^2+xy=84\ ;\ x^2-y^2=24.$

5. Simplify $\frac{2x^2-x-15}{2x-5}+\frac{11x-2x^2-15}{2x+5}.$

6. The hypotenuse of a right-angled triangle is 8 in. long ; of the remaining sides, one is 2 in. longer than the other. Calculate the lengths of the sides correct to three significant figures.

CHAPTER XXIII

HARDER MULTIPLICATION AND DIVISION. FUNCTIONAL NOTATION. REMAINDER THEOREM. SQUARE ROOT

136. Long Multiplication and Division were dealt with in Chapter XV. In this chapter we consider harder cases.

The method of detached coefficients

When two expressions contain powers of one letter only, the work may be shortened by using detached coefficients, i.e. by writing down the coefficients only, multiplying them together in the ordinary way and then inserting the successive powers of the letter at the end of the operation. The expressions must be arranged in ascending or descending powers of the common letter and care must be taken to insert zero coefficients to represent terms corresponding to missing powers of that letter. In the following example the work is set out in full and in abbreviated form.

Example 1. *Multiply* $\frac{2}{3}x^3 - \frac{1}{4} + \frac{3}{4}x$ *by* $\frac{1}{5} - \frac{2}{3}x$.

Arrange in descending powers. We have

$$\begin{array}{lllll} \frac{2}{3}x^3 & & +\frac{3}{4}x & -\frac{1}{4} & \\ -\frac{2}{3}x & +\frac{1}{5} & & & \\ \hline -\frac{4}{9}x^4 & & -\frac{1}{2}x^2 & +\frac{1}{6}x & \\ & +\frac{2}{15}x^3 & & +\frac{3}{20}x & -\frac{1}{20} \\ \hline -\frac{4}{9}x^4 & +\frac{2}{15}x^3 & -\frac{1}{2}x^2 & +\frac{19}{60}x & -\frac{1}{20}. \end{array} \qquad \Bigg| \qquad \begin{array}{lllll} \frac{2}{3} & +0 & +\frac{3}{4} & -\frac{1}{4} & \\ -\frac{2}{3} & +\frac{1}{5} & & & \\ \hline -\frac{4}{9} & +0 & -\frac{1}{2} & +\frac{1}{6} & \\ & +\frac{2}{15} & +0 & +\frac{3}{20} & -\frac{1}{20} \\ \hline -\frac{4}{9} & +\frac{2}{15} & -\frac{1}{2} & +\frac{19}{60} & -\frac{1}{20}. \end{array}$$

For the final step with detached coefficients we notice that the first term must contain x^4, and the others follow in descending order. The product is

$$-\frac{4}{9}x^4 + \frac{2}{15}x^3 - \frac{1}{2}x^2 + \frac{19}{60}x - \frac{1}{20}.$$

137. An algebraical expression containing two or more letters is called a **homogeneous expression,** if each term is of the same degree. Thus, $x^3+y^3+z^3-3xyz$, regarded as an expression in x, y, z, is a homogeneous expression of the third degree, since each term is of the third degree in x, y, z. Also x^3+y^3-3axy is a homo-

geneous expression of the third degree, regarded as an expression in x, y, a, but it is not a homogeneous expression, if it is regarded as an expression in x, y only.

The method of detached coefficients may be used to multiply two homogeneous expressions. Thus, the product of

$$\tfrac{2}{3}x^3 - \tfrac{1}{4}y^3 + \tfrac{3}{4}xy^2 \quad \text{and} \quad \tfrac{1}{5}y - \tfrac{2}{3}x$$

is

$$-\tfrac{4}{9}x^4 + \tfrac{2}{15}x^3y - \tfrac{1}{2}x^2y^2 + \tfrac{19}{60}xy^3 - \tfrac{1}{20}y^4,$$

the working being identical with that given in Ex. 1 above. The product of two homogeneous expressions is a homogeneous expression, the degree of which is the sum of the degrees of the original expressions.

138. In Chapter XV, it was shown that $(a+b)^2 = a^2 + 2ab + b^2$. This result may be extended to any number of letters. Thus,

$$\begin{aligned}(a+b+c)^2 &= a^2 + 2a(b+c) + (b+c)^2\\ &= a^2 + 2ab + 2ac + b^2 + 2bc + c^2,\end{aligned}$$

which may be more conveniently written

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

More generally, the square of any multinomial is the sum of the squares of the several terms together with twice the product of each pair of terms. In writing down the product terms, it is best to take each term in succession and multiply it by each of the terms that follow it.

Example 2. *Expand* $(2x-5y-3z)^2$.

$$\begin{aligned}(2x-5y-3z)^2 &= (2x)^2 + (-5y)^2 + (-3z)^2 + 2(2x)(-5y)\\ &\quad + 2(2x)(-3z) + 2(-5y)(-3z)\\ &= 4x^2 + 25y^2 + 9z^2 - 20xy - 12xz + 30yz.\end{aligned}$$

139. The expansions of $(a+b)^3$, $(a-b)^3$, $(a+b)^4$, $(a-b)^4$ etc. The expansions of $(a+b)^3$, $(a-b)^3$, $(a+b)^4$, $(a-b)^4$ etc. may be obtained by continued multiplication. Thus,

$$\begin{aligned}(a+b)^3 &= (a+b)^2 \times (a+b) = (a^2+2ab+b^2) \times (a+b)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

Similarly, $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$,

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4, \text{ etc.}$$

These products occur frequently and it is convenient to be able to write them down without doing the working.

140. Expressions containing several letters.

Example 3. *Multiply* $x^2+y^2+z^2-2xy-2yz-2zx$ *by* $x+y-z$.

Choose one letter, say x, and arrange the expressions in descending powers of that letter. In writing down the second and third partial products, place like terms in the same vertical column.

$$\begin{array}{l}
x^2-2xy-2xz+y^2-2yz+z^2 \\
\underline{x+y-z} \\
x^3-2x^2y-2x^2z+xy^2-2xyz+xz^2 \\
\qquad x^2y \qquad -2xy^2-2xyz \qquad +y^3-2y^2z+yz^2 \\
\qquad -x^2z \qquad +2xyz+2xz^2 \qquad -y^2z+2yz^2-z^3 \\
\hline
x^3-x^2y-3x^2z-xy^2-2xyz+3xz^2+y^3-3y^2z+3yz^2-z^3.
\end{array}$$

Or, using brackets to enclose the coefficients of terms in x,

$$\begin{array}{l}
x^2-2x(y+z)+(y^2-2yz+z^2) \\
\underline{x+(y-z)} \\
x^3-2x^2(y+z)+x(y^2-2yz+z^2) \\
\qquad x^2(y-z)-2x(y^2-z^2) \qquad +(y^3-3y^2z+3yz^2-z^3) \\
\hline
x^3-x^2(y+3z)-x(y^2+2yz-3z^2)+(y^3-3y^2z+3yz^2-z^3).
\end{array}$$

EXERCISE 70. a

Expand the following:

1. $\left(3x^2-\frac{1}{4}\right)\left(\frac{x^2}{2}+\frac{2x}{3}+2\right)$.
2. $\left(\frac{2x^2}{3}-\frac{1}{2}+\frac{3x}{2}\right)\left(\frac{3x^2}{2}+\frac{1}{2}-\frac{x}{2}\right)$.
3. $\left(\frac{x^2}{3}-\frac{x}{4}+\frac{1}{5}\right)\left(\frac{x^2}{3}+\frac{x}{4}+\frac{1}{5}\right)$.
4. $\left(1+\frac{2t}{5}-\frac{5t^2}{2}\right)^2$.

Expand the following, using detached coefficients:

5. $(4x^3-3x^2-9x+5)(3x^2-4x+7)$.
6. $(1+x-2x^2)(2+3x-x^2)$.
7. $(5x^3-4x+2)(x^2+x+5)$.
8. $(7a^3-a^2-3)(2a^2+a-3)$.
9. $(7l^3-4l+5)(5l^2-2)$.
10. $(3c^2d^2-5cd^3+2c^3d)(4c^2-3cd-2d^2)$.

Expand the following in ascending powers:

11. $(1-3x+2x^2-x^3)^2$ as far as x^3.
12. $(2x^3-5x^2+x-2)^2$ as far as x^4.
13. $(1-c+c^3)(1-4c^2-c^3)$ as far as c^4.
14. $(2+3l^2-l^4)(5-2l^3+3l^4)$ as far as l^4.

Find the coefficient of:

15. a^3 in $(1-a^2-a^3)(2-a+5a^2)$.
16. a^4 in $(1-2a-4a^3)(3-a^3-5a^4)$.

17. x^3 in $(1 - 2x - 3x^2 + 5x^3)^2$.

18. x^3y^2 in $(5x^3 - 7xy^2 - 2y^3)(4y^2 - 5xy - 2x^2)$.

Expand the following :

19. $(a - 2b + 3c)^2$. **20.** $(5a - 2b - c)^2$. **21.** $(2a - 5b + 3c - d)^2$.

22. $(a + 4b - 5c - 3d)^2$. **23.** $(2x - 1)^4$. **24.** $(3a^2 + 5b^2)^4$.

25. $(a^2 + b^2 + c^2 - ab - bc - ca)(a + b + c)$.

26. $(3xz + 5x^2 - 2z^2 - 3y^2 - 5yz + 4xy)(2x - 5z + 4y)$.

EXERCISE 70. b

Expand the following :

1. $\left(2x^2 - \frac{1}{3}\right)\left(\frac{2x^2}{3} - \frac{5x}{2} + 3\right)$. **2.** $\left(\frac{2x^2}{3} + \frac{2}{5} - x\right)\left(\frac{2x^2}{3} - \frac{2}{5} + x\right)$.

3. $\left(\frac{a^2}{4} + \frac{b^2}{9} - \frac{ab}{8}\right)\left(\frac{3a^2}{2} + \frac{2b^2}{3} + ab\right)$. **4.** $\left(4 - \frac{2t}{3} + \frac{t^2}{4}\right)^2$.

Expand the following, using detached coefficients :

5. $(4c^3 - 3c^2 - 11c + 2)(2c^2 - 5c + 9)$.

6. $(2 - x + 3x^2)(5 - 2x + 2x^2)$. **7.** $(3x^2 - 9x - 2)(2x^2 - 3)$.

8. $(5a^3 - 9a - 2)(7a^2 + 4)$. **9.** $(8d^3 - 2d^2 - 5)(3d^2 - d - 4)$.

10. $(5x^3 - 2xy^2 - 6y^3)(2x^2 + 3y^2 - 5xy)$.

Expand the following in ascending powers :

11. $(3a^3 - a^2 - 5a + 4)^2$ as far as a^3.

12. $(5 - 2x - 4x^3)^2$ as far as x^4.

13. $(2t - 3t^3 + t^4)(3t - 2t^2 - 5t^4)$ as far as t^5.

14. $(2 - x - 2x^4)(1 + 3x - x^3)$ as far as x^3.

Find the coefficient of :

15. a^3 in $(5 - 2a - 2a^3)(1 - a + 3a^3)$.

16. x^4 in $(3x - 5 + 4x^3)(4x - 3 + 2x^3)$.

17. y^2 in $(6y^2 + 1 - 4y)(1 + y^2 - 2y)$.

18. pq^3 in $(7p^2 - 5pq + 4q^2)(3p^2 + 4pq - 5q^2)$.

Expand the following :

19. $(2a - 3b + c)^2$. **20.** $(a - 4b + 2c)^2$. **21.** $(a - 3b - 2c + 5d)^2$.

22. $(5a - b + c - 7d)^2$. **23.** $(2x - 5)^4$. **24.** $(2a + 3b)^4$.

25. $(3a^2 - b^2 + 5c^2 - 3ab + bc - 2ca)(3a - 2b + 4c)$.

26. $(4ln + l^2 - 5m^2 - 4mn - 2lm)(2l - n + 3m)$.

141. Detached coefficients may be used in division, as in multiplication. In the following example the work is set out in abbreviated form.

Example 4. *Divide* $\frac{3}{4}a^4 - \frac{1}{2}a^3b + \frac{1}{4}ab^3 - \frac{3}{16}b^4$ by $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$.

$$
\begin{array}{l}
\frac{1}{2} - \frac{1}{3} + \frac{1}{4}\ )\ \frac{3}{4} - \frac{1}{2} + 0 + \frac{1}{4} - \frac{3}{16}\ (\ \frac{3}{2} + 0 - \frac{3}{4} \\
\qquad\qquad\quad \underline{\frac{3}{4} - \frac{1}{2} + \frac{3}{8}\qquad\quad} \\
\qquad\qquad\qquad\quad 0 - \frac{3}{8} + \frac{1}{4} \\
\qquad\qquad\qquad\quad \underline{0 + 0 + 0\qquad\quad} \\
\qquad\qquad\qquad\qquad - \frac{3}{8} + \frac{1}{4} - \frac{3}{16} \\
\qquad\qquad\qquad\qquad \underline{- \frac{3}{8} + \frac{1}{4} - \frac{3}{16}}
\end{array}
$$

The quotient is $\frac{3}{2}a^2 - \frac{3}{4}b^2$.

142. Expressions containing several letters.

Example 5. *Divide* $a^3 + b^3 + c^3 - 3abc$ *by* $a + b + c$.

Choose one letter, say a, and arrange the expressions in descending powers of that letter.

$$
\begin{array}{l}
a + (b + c)\ )\ a^3 \qquad\qquad - 3abc + (b^3 + c^3)\ (\ a^2 - a(b + c) + (b^2 - bc + c^2) \\
\qquad\qquad\quad \underline{a^3 + a^2(b + c)\qquad\qquad\quad} \\
\qquad\qquad\qquad - a^2(b + c) - 3abc \\
\qquad\qquad\qquad \underline{- a^2(b + c) - a(b + c)^2\qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad + a(b^2 - bc + c^2) + (b^3 + c^3) \\
\qquad\qquad\qquad\qquad\qquad \underline{+ a(b^2 - bc + c^2) + (b^3 + c^3)}
\end{array}
$$

143. The following results may easily be verified; they are of considerable importance and should be carefully noted.

(A) $(a^2 - b^2) \div (a - b) = a + b$,

$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2$,

$(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3$, and so on.

Any expression of the form $a^n - b^n$, where n is an integer, is divisible by $a - b$. In the quotient all the terms are positive and all the coefficients are 1.

(B) $(a^3 + b^3) \div (a + b) = a^2 - ab + b^2$,

$(a^5 + b^5) \div (a + b) = a^4 - a^3b + a^2b^2 - ab^3 + b^4$,

$(a^7 + b^7) \div (a + b) = a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6$,

and so on.

Any expression of the form $a^n + b^n$, where n is an **odd** integer, is divisible by $a + b$. In the quotient the coefficients are alternately $+1$ and -1.

(C) $(a^2 - b^2) \div (a + b) = a - b$,

$(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3$,

$(a^6 - b^6) \div (a + b) = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$, and so on.

Any expression of the form $a^n - b^n$, where n is an **even** integer, is divisible by $a+b$. In the quotient the coefficients are alternately $+1$ and -1.

(D) The expressions a^2+b^2, a^4+b^4, a^6+b^6, or in general a^n+b^n, where n is an **even** integer, are **not** divisible by $a+b$ or $a-b$.

EXERCISE 71. a

Divide :

1. $c^3 - \frac{2}{3}c^2d + \frac{8}{3}cd^2 + d^3$ by $c + \frac{1}{3}d$. **2.** $\frac{1}{8}x^3 - y^3$ by $\frac{1}{2}x - y$.

3. $\frac{1}{27}l^3 + \frac{1}{8}m^3$ by $\frac{1}{3}l + \frac{1}{2}m$.

4. $t^4 - \frac{5}{6}t^3 - \frac{55}{36}t^2 - \frac{5}{6}t - \frac{1}{6}$ by $\frac{2}{3}t^2 - t - \frac{1}{2}$.

Divide, using detached coefficients :

5. $4 + 6d - 27d^3$ by $2 - 3d$.

6. $2z^4 - 9z^3 + 4z^2 - 25$ by $2z^2 - 3z + 5$.

7. $16 - 22x + 33x^2 - 22x^3 + 3x^4$ by $2 - x + 3x^2$.

8. $y^4 - 2y^3 - 15y^2 + 15y + 32$ by $y^2 - 3y - 5$.

9. $6x^4y^4 + 7x^3y^3 - 21x^2y^2 + 9xy - 1$ by $2x^2y^2 + 5xy - 1$.

10. $6c^5 - c^4 + 10c^3 - 14c^2 - 25$ by $3c^2 + 4c + 5$.

Divide :

11. $x^4 + 4y^4$ by $x^2 - 2xy + 2y^2$.

12. $a^3 - b^3 + c^3 + 3abc$ by $a - b + c$.

13. $6a^2 - 12b^2 - c^2 - ab + ac + 7bc$ by $2a - 3b + c$.

14. $a^3 - b^3 + 8c^3 + 6abc$ by $b - a - 2c$.

15. $16c^4 - d^8$ by $2c + d^2$. **16.** $a^8 - b^8$ by $a^2 - b^2$.

17. $64 - y^6$ by $2 - y$. **18.** $c^6 + 125$ by $c^2 + 5$.

19. $3125s^5 + 32t^5$ by $2t + 5s$. **20.** $x^{10} - y^{10}$ by $x^2 + y^2$.

EXERCISE 71. b

Divide :

1. $\frac{1}{2}a^3 - \frac{2}{3}a^2b - \frac{8}{3}ab^2 - \frac{4}{3}b^3$ by $\frac{1}{2}a + \frac{1}{3}b$.

2. $\frac{1}{27}x^3 + 8y^3$ by $\frac{1}{3}x + 2y$. **3.** $27l^3 - \frac{1}{64}m^3$ by $3l - \frac{1}{4}m$.

4. $\frac{1}{2}x^4 - 2 - \frac{1}{2}x^3 + x$ by $\frac{1}{3}x^2 + \frac{2}{3} - \frac{1}{3}x$.

Divide, using detached coefficients :

5. $2c^3 + 3c^2 - 19c + 15$ by $2c - 3$.

6. $5a^4 - 4a^3b + 3a^2b^2 + 22ab^3 + 55b^4$ by $5b^2 - 3ab + a^2$.

7. $12c^4 + c^3 - 8c^2 - 2 + 7c$ by $1 - 2c + 3c^2$.

8. $-3 + 14p^3 - 25p^2 + 13p + 10p^4$ by $4p - 3 + 2p^2$.

9. $4y^4+9y^3-35y^2+44y-10$ by y^2+4y-3.

10. $6t^2-2t-t^4-4t^3+t^5$ by t^3-4t+2.

Divide:

11. $a^2+b^2+c^2+2ab-2bc-2ac$ by $a+b-c$.

12. $27d^3+18cd+8-c^3$ by $c-3d-2$.

13. $8x^3-8y^3+36xy+27$ by $2x-2y+3$.

14. $a^2b-a^2c+b^2c-ab^2+ac^2-bc^2$ by $a-b$.

15. x^7+1 by $x+1$.

16. $243-a^5$ by $3-a$.

17. $64a^6-b^6$ by $2a+b$.

18. $243x^5-3125a^5$ by $5a-3x$.

19. a^8-b^8 by a^2+b^2.

20. $a^4+a^2b^2+b^4$ by a^2+ab+b^2.

Functional notation

144. A function of x is often represented by putting x in a bracket and prefixing some letter, usually f or F.

Thus $f(x)$, $F(x)$ denote functions of x. It should be particularly noted that f, F are symbols denoting functionality and **not multipliers.**

The value of $f(x)$ when $x=a$ is represented by $f(a)$. Thus, if

$$f(x)=(2x-1)(3x+2),\ f(a)=(2a-1)(3a+2),$$
$$f(5)=(2\cdot 5-1)(3\cdot 5+2)=9\cdot 17=153,$$
$$f(a+3)=[2(a+3)-1][3(a+3)+2]=(2a+5)(3a+11).$$

Similarly, $F(x, y)$ denotes a function of x and y. Thus, if $F(x, y)=x^2-y^2$, $F(a, b)=a^2-b^2$, $F(3, 1)=3^2-1^2=8$.

The Remainder Theorem

145. The following example in division is very important.

$$\begin{array}{l} x-h\,]\; ax^3+bx^2+cx+d\;[\; ax^2+(b+ah)x+(c+bh+ah^2) \\ \qquad\quad \underline{ax^3-ahx^2} \\ \qquad\qquad (b+ah)x^2+cx \\ \qquad\qquad \underline{(b+ah)x^2-(bh+ah^2)x} \\ \qquad\qquad\qquad\qquad (c+bh+ah^2)x+d \\ \qquad\qquad\qquad\qquad \underline{(c+bh+ah^2)x-(ch+bh^2+ah^3)} \\ \qquad\qquad\qquad\qquad\qquad\qquad d+ch+bh^2+ah^3 \end{array}$$

Here the division has been carried on until the remainder does not contain x, and its value is the result obtained by replacing x by h in the dividend. This is a particular case of an important theorem known as the **Remainder Theorem**: if any rational integral function $F(x)$ is divided by $x-h$ until the remainder does not contain x, the remainder is $F(h)$.

This may easily be proved for all rational integral functions. For, if $F(x)$ is divided by $x-h$, it is possible to continue the division until a remainder is obtained which does not contain x. Call this remainder R, and let $Q(x)$ represent the quotient. Then

$$F(x)=(x-h)\,Q(x)+R.$$

[Just as in Arithmetic when we divide 17 by 5, $17=5\times 3+2$.]

This is an identity which is true for all values of x.

[In any given instance it may be verified by multiplication. Although we naturally obtain $Q(x)$, if necessary, by division, the final result is not dependent upon division by $x-h$. It is therefore legitimate to put $x=h$ in the final result, although division by $h-h$, or zero, is impossible.]

Let $x=h$. Then we get

$F(h)=R$, for R is independent of x, and $(x-h)Q(x)$ becomes $0\times Q(h)=0$, since $Q(h)$ is finite.

In other words, the remainder is obtained by substituting h for x in the dividend.

More generally, if we divide by $hx+l$ until the remainder does not contain x, that remainder is equal to $F\left(-\frac{l}{h}\right)$. For

$$F(x)=(hx+l)\,Q(x)+R \text{ for all values of } x.$$

Put $x=-\frac{l}{h}$; then

$$F\left(-\frac{l}{h}\right)=0\times Q\left(-\frac{l}{h}\right)+R,$$

$$\text{i.e. } R=F\left(-\frac{l}{h}\right).$$

Example 6. *Find the remainder when* $2x^3-5x^2+6x-3$ *is divided by* $2x+1$.

The remainder is obtained by substituting $-\frac{1}{2}$ for x in the dividend and is

$$2(-\tfrac{1}{2})^3-5(-\tfrac{1}{2})^2+6(-\tfrac{1}{2})-3=-\tfrac{1}{4}-\tfrac{5}{4}-3-3=-7\tfrac{1}{2}.$$

It is possible to extend this method to divisors of higher degree than the first. Whatever the degree of the dividend and divisor, each successive remainder in the process of division is of lower dimensions than the preceding one. Thus the division can be carried on until the remainder is of lower dimensions than the divisor.

Example 7. *Find the remainder when* $4x^4 - 2x^3 + 2x^2 - x + 5$ *is divided by* $2x^2 - 3x + 1$.

Let $Q(x)$ be the quotient. The remainder is of lower degree than the divisor; it is therefore of the first degree and of the form $ax + b$, where a and b do not contain x;

$$\therefore\ 4x^4 - 2x^3 + 2x^2 - x + 5 = (2x^2 - 3x + 1)\,Q(x) + ax + b. \quad \text{......(i)}$$

The factors of $2x^2 - 3x + 1$ are $(2x - 1)(x - 1)$ and we substitute in (i) the values of x which make the divisor zero, i.e. $x = 1$, $x = \frac{1}{2}$. The object of this is to make the term containing $Q(x)$ zero, thus making it unnecessary to calculate the quotient.

Put $x = 1$, then $8 = 0 + a + b$. Put $x = \frac{1}{2}$, then

$$\frac{1}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2} + 5 = 0 + \frac{a}{2} + b,$$

$$\text{i.e. } 5 = \frac{a}{2} + b.$$

Solving, we obtain $a = 6$, $b = 2$;

$$\therefore \text{ the remainder is } 6x + 2.$$

This method is not of much practical value unless the divisor breaks up into factors of the first degree with rational coefficients.

EXERCISE 72. a

1. If $f(x) = x^2 - x + 1$, find the values of $f(5)$, $f(-2)$, $f(\frac{1}{2})$, $f(0)$.

2. If $f(x) = 3x^2 - 2x - 5$, find the values of $f(4)$, $f(\frac{1}{3})$, $f(-3)$, $f(a)$.

3. If $f(x) = (4x + 1)(3x - 5)$, find the values of $f(-1)$, $f(-\frac{1}{4})$, $f(a + 1)$, $f(x^2)$.

4. If $F(n) = \frac{1}{2}n(n + 1)$, find the values of $F(30)$, $F(n - 1)$, $F(-7)$, $F(n + 1) - F(n)$.

5. If $F(n) = 2^n - 1$, find the values of $F(6)$, $F(-1)$, $F(n + 1)$, $F(2n - 1)$.

6. If $F(x, y) = x^3 - y^3$, find the values of $F(3, 2)$, $F(1, 1)$, $F(a, b)$, $F(a + 1, b + 1)$.

Find the remainder (not containing x) obtained on dividing:

7. $x^3 - 4x^2 + 8x - 5$ by $x - 2$.

8. $3x^3 - 2x^2 - 4x + 6$ by $x + 4$.

9. $8x^3 - 6x - 33$ by $x + 1$.

10. $9x^3 - 5x^2 - 7$ by $x + 3$.

11. $2x^4 - 5x^3 + 2x^2 - x - 2$ by $2x - 5$.

12. $3x^4 + 2x^3 - 3x^2 + 2x - 7$ by $3x + 2$.

13. $2x^3 - 9x^2 - 7x + 14$ by $2x + 1$.

14. $4x^3 + x^2 - 9x - 11$ by $4x - 3$.

15. $2x^4 - 6x^3 - x^2 + 12x - 5$ by $x + 2$.

16. $21x^4 - 213x^3 + 197x - 4$ by $x - 1$.

Find the remainder (not containing x) obtained on dividing: (What conclusion do you draw in each case?)

17. $4x^3 - 8x^2 - 9x - 9$ by $x - 3$. **18.** $x^3 + 8x^2 + 19x + 12$ by $x + 3$.

19. $4x^3 - 4x^2 - 23x + 30$ by $2x - 3$.

20. $2x^4 + 3x^3 - 51x^2 + 22x + 24$ by $2x + 1$.

What is the value of a, if

21. $2c^4 - 3c^3 + ac - 6$ is exactly divisible by $c + 2$?

22. $ax^3 - 7x^2 - 7x + 3$ is exactly divisible by $x - 3$?

23. $14x^4 - 27x^3 - 9x^2 + 12x + a$ is exactly divisible by $x - 2$?

24. $3x^4 + ax^2 + 58x + 40$ is exactly divisible by $x + 5$?

Find the remainder (of the first degree in x) obtained on dividing:

25. $4x^4 - 16x^3 + 9x^2 + 3x - 33$ by $2x^2 - 5x - 7$.

26. $2x^4 - 7x^3 + 12x^2 - 11x - 7$ by $2x^2 - 3x - 2$.

Determine a and b in order that:

27. $3x^4 + x^3 + ax^2 + 5x + b$ may be exactly divisible by $x - 1$ and $x + 2$.

28. $ax^4 - 2x^3 + bx^2 - 6x - 9$ may be exactly divisible by $x^2 - 2x - 3$.

EXERCISE 72. b

1. If $f(x) = 2x^2 - x - 1$, find the values of $f(7), f(-1), f(\frac{3}{2}), f(b)$.

2. If $f(x) = 2x^2 - 5x + 1$, find the values of $f(3), f(-3), f(\frac{1}{5}), f(0)$.

3. If $f(x) = (3x - 1)(5x + 4)$, find the values of $f(2), f(-2), f(-\frac{4}{5})$, $f(a - 2)$.

4. If $F(n) = 3^n - 1$, find the values of $F(1), F(3), F(2a), F(3n - 1)$.

5. If $F(n) = (3n + 4)(2n - 5)$, find the values of $F(5)$, $F(-3)$, $F(n - 1)$, $F(n + 4)$.

6. If $F(x, y) = (x - y)^2$, find the values of $F(5, 2)$, $F(2, -2)$, $F(x + 1, y + 1)$, $F(2a, 3b)$.

Find the remainder (not containing x) obtained on dividing:

7. $x^3 - 3x^2 + 7x - 4$ by $x - 3$. **8.** $2x^3 - x^2 - 3x + 5$ by $x - 2$.

9. $4x^3 - 3x - 9$ by $x + 2$. **10.** $5x^3 - x^2 - 3$ by $x + 1$.

11. $2x^4 - 3x^3 + 8x^2 - 4x - 3$ by $2x - 3$.

12. $3x^4 + x^3 - 9x^2 + 6x - 5$ by $3x + 1$.

13. $3x^3 - 5x^2 - 3x - 1$ by $3x + 2$. **14.** $5x^3 + 4x^2 - 11x - 3$ by $5x - 3$.

15. $x^4 - 7x^3 - 2x^2 + 11x - 5$ by $x + 3$.

16. $17x^4 - 93x^3 + 117x^2 - 51x - 10$ by $x - 1$.

Find the remainder (not containing x) obtained on dividing: (What conclusion do you draw in each case?)

17. $x^3+3x^2-10x-24$ by $x+4$. **18.** $x^3-8x^2+9x+18$ by $x-3$.

19. $4x^3+8x^2+9x+9$ by $2x+3$.

20. $4x^4-2x^3-2x^2-x+1$ by $2x-1$.

What is the value of k if

21. $2x^4-kx^3-x-6$ is exactly divisible by $x-2$?

22. $3x^3+7x^2+kx-3$ is exactly divisible by $x+3$?

23. $kx^4-2x^3-x^2+2x-4$ is exactly divisible by $x+1$?

24. $5x^4-2x^3-16x^2+8x+k$ is exactly divisible by $x+2$?

Find the remainder (of the first degree in x) obtained on dividing:

25. $3x^4+7x^3+15x^2+2x-15$ by $3x^2+x-2$.

26. $12x^4-13x^3-57x^2+32x+8$ by $4x^2+5x-6$.

Determine a and b in order that

27. $ax^4+2x^3+bx^2-14x+24$ may be exactly divisible by $x-3$ and $x+4$.

28. $3x^4+ax^3-27x^2+18x+b$ may be exactly divisible by x^2+2x-8.

Square root

146. It is usually possible, when given an expression which is a perfect square, to write down its square root by inspection and check by the rule stated in Art. 138. The general method is shown in the following example. The pupil who is familiar with the arithmetical method will have no difficulty in following the work, but the general method should only be used in the last resort. It is better to use methods of inspection, if possible.

Example 8. *Find the square root of* $4x^4+4x^3-7x^2-4x+4$.

$$
\begin{array}{lrl}
 & 2x^2+x-2 & \\
\cline{2-2}
 &)\;4x^4+4x^3-7x^2-4x+4 & \ldots\ldots(\text{i}) \\
 & 4x^4 \qquad\qquad\qquad\qquad & \ldots\ldots(\text{ii}) \\
\cline{2-2}
\dagger\, 4x^2+x\;) & 4x^3-7x^2-4x+4 & \ldots\ldots(\text{iii}) \\
 & 4x^3+\;\;x^2 \qquad\qquad & \ldots\ldots(\text{iv}) \\
\cline{2-2}
\dagger\, 4x^2+2x-2\;) & -8x^2-4x+4 & \ldots\ldots(\text{v}) \\
 & -8x^2-4x+4 & \ldots\ldots(\text{vi}) \\
\cline{2-2}
 & 0 & \ldots\ldots(\text{vii})
\end{array}
$$

The square root is $\pm(2x^2+x-2)$.

† The divisor at each stage is obtained by doubling the terms of the square root already found and adding on the new term that is then being obtained.

In the above working

(i) = the given expression, say y,

(ii) $=(2x^2)^2$, (iii) $=y-(2x^2)^2$, (iv) $=(2\,.\,2x^2+x)x$,

(v) $=$ (iii) $-$ (iv) $=y-(2x^2)^2-(2\,.\,2x^2+x)x=y-(2x^2+x)^2$,

(vi) $=[2(2x^2+x)-2](-2)$,

(vii) $=$ (v) $-$ (vi) $=y-(2x^2+x)^2+4(2x^2+x)-4=y-(2x^2+x-2)^2$.

But (vii) $=0$, $\therefore\ y=(2x^2+x-2)^2$.

EXERCISE 73. a

Find, if possible by inspection, the square roots of:

1. $a^2+4b^2+c^2-4bc+2ac-4ab$.

2. $9x^2+16y^2+4z^2+24xy-12xz-16yz$.

3. $4l^2-12lm+9m^2+16ln-24mn+16n^2$.

4. $25x^4-30x^2y^2+9y^4-10x^2z^2+6y^2z^2+z^4$.

5. $4a^4-4a^3+5a^2-2a+1$.

6. $4a^4-12a^3b+25a^2b^2-24ab^3+16b^4$.

7. $25x^6-10x^5-9x^4+32x^3-5x^2-6x+9$.

8. $16x^6-40x^5+49x^4-46x^3+29x^2-12x+4$.

9. $49+x^6+4x^2+14x^3-4x^4-28x$.

10. $16x^6+y^6-24x^5y-4xy^5+25x^4y^2+10x^2y^4-20x^3y^3$.

11. $9l^6+22l^4m^2+l^2m^4+12lm^5+4m^6-12l^5m$.

12. $x^4+\frac{1}{x^4}-2x^3+\frac{4}{x^3}+4x+\frac{4}{x^2}-\frac{2}{x}-2+x^2$.

EXERCISE 73. b

Find, if possible by inspection, the square roots of:

1. $a^2+9b^2+c^2+6ab-2ac-6bc$.

2. $9x^2+y^2+4z^2-6xy-12xz+4yz$.

3. $4l^2-20lm+25m^2-4ln+10mn+n^2$.

4. $4x^4-16x^2y^2+16y^4+20x^2z^2-40y^2z^2+25z^4$.

5. $a^4-6a^3+13a^2-12a+4$.

6. $l^4-2l^3m+5l^2m^2-4lm^3+4m^4$.

7. $4x^6-4x^5+13x^4-26x^3+19x^2-30x+25$.

8. $9x^6-12x^5-2x^4+28x^3-15x^2-8x+16$.

9. $9y^6+x^6-8x^5y-12xy^5-22x^3y^3+28x^2y^4+20x^4y^2$.

10. $25+16x^6+4x^2+20x-40x^3-16x^4$.

11. $a^6+49b^6-6a^4b^2-14a^3b^3+9a^2b^4+42ab^5$.

12. $4x^4+9x^2+\frac{16}{x^2}-12x^3+16x-24$.

CHAPTER XXIV

HARDER FACTORS

THE FACTOR THEOREM

147. In Chapter XXIII we proved the Remainder Theorem, i.e. that if any rational integral function $F(x)$ is divided by $hx+l$ until the remainder does not contain x, the remainder is $F\left(-\frac{l}{h}\right)$. If $F\left(-\frac{l}{h}\right)=0$, then $hx+l$ is a factor of $F(x)$. **This result is known as the Factor Theorem.**

Example 1. *Find the factors of* $2x^4-3x^3-3x-2=F(x)$.

The only possible factors of the first degree are $(x-1)$, $(x+1)$, $(x-2)$, $(x+2)$, $(2x-1)$, $(2x+1)$. [There is no need to consider $(2x-2)$, $(2x+2)$ for these contain the factor 2, which is not a factor of the given expression, and when this factor has been taken out, the remaining factors $(x-1)$, $(x+1)$ have already been considered.]

We therefore work out in succession $F(1)$, $F(-1)$, $F(2)$, $F(-2)$, $F(\frac{1}{2})$, $F(-\frac{1}{2})$, until a zero result is obtained.

$$F(1)=2-3-3-2=-6, \quad \therefore (x-1) \text{ is not a factor.}$$
$$F(-1)=2+3+3-2=6, \quad \therefore (x+1) \text{ is not a factor.}$$
$$F(2)=32-24-6-2=0, \quad \therefore (x-2) \text{ is a factor.}$$

By division the other factor is $2x^3+x^2+2x+1$, $=f(x)$. We now test for factors of this. There is no need to test for $(x-1)$ and $(x+1)$, so we now work out $f(2)$, $f(-2)$, $f(\frac{1}{2})$, $f(-\frac{1}{2})$.

$$f(2)=16+4+4+1=25, \quad \therefore x-2 \text{ is not a factor.}$$
$$f(-2)=-16+4-4+1=-15, \quad \therefore x+2 \text{ is not a factor.}$$
$$f(\tfrac{1}{2})=\tfrac{1}{4}+\tfrac{1}{4}+1+1=2\tfrac{1}{2}, \quad \therefore 2x-1 \text{ is not a factor.}$$
$$f(-\tfrac{1}{2})=-\tfrac{1}{4}+\tfrac{1}{4}-1+1=0, \quad \therefore 2x+1 \text{ is a factor.}$$

By division the remaining factor is x^2+1, which cannot be factorised any further,

$$\therefore 2x^4-3x^3-3x-2=(x-2)(2x+1)(x^2+1).$$

148. If the coefficient of the highest power of x and the constant term have a large number of prime factors, the work may be long, but there is no need to work out in detail the test for every factor. Thus, in searching for the factors of $2x^3+x^2+2x+1$, it is obvious

that there cannot be any factor of the type $(hx+l)$, if h and l have opposite signs. For in that case $-\frac{l}{h}$ is positive; $f\left(-\frac{l}{h}\right)$ is then the sum of four positive terms and cannot be equal to zero. In the above working, therefore, we need not have worked out $f(2)$ and $f(\frac{1}{2})$.

Similarly, in searching for the factors of $F(x)=3x^3-3x^2+x-5$, there is no need to work out $F(\frac{1}{3})$, $F(-\frac{1}{3})$, $F(\frac{5}{3})$, $F(-\frac{5}{3})$, for in each case, after cancelling, the first term has a denominator 9 and all the other terms have denominators less than 9. It is therefore impossible for the expression to equal zero.

If the pupil constantly bears in mind such considerations, the work is not often unduly laborious.

EXERCISE 74. a

Use the Factor Theorem to prove that:

1. $x-1$ is a factor of $4x^3-9x^2+3x+2$.

2. $x+1$ is a factor of $4x^3+9x^2+7x+2$.

3. $2x+3$ is a factor of $2x^3+x^2-9x-9$.

4. $(5x-2)$ is a factor of $5x^3-7x^2-8x+4$.

Use the Factor Theorem to find the factors of:

5. x^3-5x+4. **6.** x^3-4x^2+x+6. **7.** $3x^3+2x+5$.

8. $x^3+7x^2+16x+12$. **9.** $a^3-19a-30$. **10.** $2l^3+13l^2-36$.

11. $c^3-3c^2-10c+24$. **12.** $l^4+l^3-2l^2+2l+4$.

13. $2x^4+5x^3-9x^2+11x-4$. **14.** $5x^4+11x^3-18x^2-19x-3$.

15. $6x^4+13x^3y+3x^2y^2+4xy^3+4y^4$. **16.** $24t^3-14t^2-t+1$.

17. $30x^3-x^2-6x+1$. **18.** $30a^3+13a^2-40a+12$.

Show that the following expressions have no factors of the first degree with rational coefficients:

19. x^4-3x^2+4x-3. **20.** $x^4+2x^3-3x^2-4x+3$.

EXERCISE 74. b

Use the Factor Theorem to prove that:

1. $x-2$ is a factor of $x^3-6x^2+11x-6$.

2. $x+1$ is a factor of $5x^3+11x^2+4x-2$.

3. $4x-1$ is a factor of $4x^3-5x^2-11x+3$.

4. $3x+5$ is a factor of $20+17x-2x^2-3x^3$.

Use the Factor Theorem to find the factors of :

5. $2x^3+3x^2-1$.

6. $x^3+6x^2+11x+6$.

7. $x^3+3x^2-4x-12$.

8. x^3-6x+5.

9. $2l^3+3l^2-12l-20$.

10. $a^3-a^2-10a-8$.

11. $l^4+4l^3+l^2+14l+40$.

12. $c^3+8c^2+9c-18$.

13. $2x^4-13x^3+19x^2+5x-4$.

14. $4x^4+11x^3-47x^2-9x+5$.

15. $6t^3-11t^2+6t-1$.

16. $9x^4+15x^3y-35x^2y^2+5xy^3+6y^4$.

17. $30a^3+67a^2-28a-20$.

18. $30x^3+19x^2-1$.

Show that the following expressions have no factors of the first degree with rational coefficients :

19. x^4-3x^2-2.

20. $x^4-2x^3+5x^2-4x+3$.

The sum or difference of two cubes

149. If $f(x)=x^3-a^3$, $f(a)=0$; $\therefore\ x-a$ is a factor of x^3-a^3.

The other factor, obtained by division, is x^2+ax+a^2.

Similarly, if $F(x)=x^3+a^3$, $F(-a)=0$; $\therefore\ x+a$ is a factor of x^3+a^3.

The other factor, obtained by division, is x^2-ax+a^2.

We have therefore the following identities :

$$(\mathbf{x^3-a^3})=(\mathbf{x-a})(\mathbf{x^2+ax+a^2}).$$
$$(\mathbf{x^3+a^3})=(\mathbf{x+a})(\mathbf{x^2-ax+a^2}).$$

It should be noted carefully that the sign of the middle term in the second bracket is opposite to the sign of the second term in the first bracket.

By means of these identities, any expression which is the sum or difference of two cubes may be factorised.

Example 2. *Find the factors of* (i) x^3+27y^3, (ii) $8x^3-125y^3$.

(i) $x^3+27y^3=(x+3y)(x^2-x(3y)+(3y)^2)$ $\quad x$

$=(x+3y)(x^2-3xy+9y^2)$, $\quad 3y$

(ii) $8x^3-125y^3=(2x-5y)\{(2x)^2+(2x)(5y)+(5y)^2\}$ $\quad 2x$

$=(2x-5y)\{4x^2+10xy+25y^2\}$. $\quad 5y$

It is a good plan to write in the margin the numbers which are cubed.

Example 3. *Factorise* $(x-2y)^3+27(3x-2y)^3=E$.

$E=[(x-2y)+3(3x-2y)]$ $\quad x-2y$

$\times[(x-2y)^2-3(3x-2y)(x-2y)+9(3x-2y)^2]$ $\quad 3(3x-2y)$

$$=[10x-8y][x^2-4xy+4y^2-3(3x^2-8xy+4y^2)$$
$$+9(9x^2-12xy+4y^2)]$$
$$=2(5x-4y)[x^2-4xy+4y^2-9x^2+24xy-12y^2+81x^2$$
$$-108xy+36y^2]$$
$$=2(5x-4y)[73x^2-88xy+28y^2].$$

EXERCISE 75. a

Find the factors of :

1. a^3+1.
2. $8a^3-1$.
3. $1-l^3m^3$.
4. $1+8l^3m^3$.
5. y^3-27.
6. $8y^3+27$.
7. $343-n^3$.
8. b^3-125c^3.
9. $8x^6+125y^3z^3$.
10. b^4+64b.
11. $27l^3-m^3n^3$.
12. $343x^6-1000a^3$.
13. $2l^3+250m^3$.
14. $343o+10t^3$.
15. $32c^3-4z^3$.
16. $512a^3b^3+125$.
17. c^6+1.
18. l^6-m^6.
19. $27c^6-d^6$.
20. $8t^3-1000x^6$.
21. $64x^3-(3x-5y)^3$.
22. $8l^3+(5l+3m)^3$.
23. $(7x+5y)^3-(3x+2y)^3$.
24. $27(8x-3y)^3-8(2x-4y)^3$.

EXERCISE 75. b

Find the factors of :

1. x^3-1.
2. $8x^3+1$.
3. $1+c^3$.
4. $8-c^3$.
5. $8a^3-27b^3$.
6. z^3+27.
7. a^3-64.
8. $27a^3+b^3c^3$.
9. a^3+125.
10. $343+8x^3$.
11. $27x^3+1000y^6$.
12. $125l^6-8m^3n^3$.
13. $24a^3-3b^3$.
14. $2x^3+54y^3$.
15. $512-27x^3y^3$.
16. $10c^3+640d^3$.
17. a^6-1.
18. x^6+y^6.
19. $4p^6+500r^3$.
20. $8x^6-125y^6$.
21. $125a^3-(2a+3b)^3$.
22. $27y^3+(5x-2y)^3$.
23. $(8x-3y)^3-(2x-4y)^3$.
24. $8(7x+5y)^3-125(3x+2y)^3$.

Factors by grouping terms. Harder cases

150. Many expressions may be resolved into factors by choosing one letter, and arranging the expression in ascending or descending powers of that letter.

As a rule it is better to choose first the letter which occurs in the lowest degree, but each letter should be tried in turn before giving up the attempt to factorise.

Example 4. *Factorise* $2ax^2 - 3x - 2ax - 6 - 12a$.

The highest power of a is the first, and of x is the second. So we arrange in descending (or ascending) powers of a.

The expression equals $2ax^2 - 2ax - 12a - 3x - 6$

$$= 2a(x^2 - x - 6) - 3(x+2) = 2a(x-3)(x+2) - 3(x+2).$$

[It is now clear that $(x+2)$ is a factor of the whole expression]

$$= (x+2)[2a(x-3) - 3] \text{ or } (x+2)(2ax - 6a - 3).$$

Example 5. *Factorise* $a^3(b-c) + b^3(c-a) + c^3(a-b) = E$.

The highest power of a is the third. So also is the highest power of b and of c. We may therefore choose any letter, say a, and rearrange in descending powers of a.

$$E = a^3b - a^3c - ab^3 + ac^3 + b^3c - bc^3$$
$$= a^3(b-c) - a(b^3 - c^3) + bc(b^2 - c^2).$$

[It is now clear that $(b-c)$ is a factor of the whole expression, for $(b^3 - c^3)$ and $(b^2 - c^2)$ are each divisible by $(b-c)$.]

$$= (b-c)[a^3 - a(b^2 + bc + c^2) + bc(b+c)].$$

In the big bracket the highest power of a is the third, of b is the second and of c is the second. We therefore choose either b or c, say b, and rearrange in descending powers of b.

$$E = (b-c)[-ab^2 + b^2c - abc + bc^2 + a^3 - ac^2]$$
$$= (b-c)[b^2(c-a) + bc(c-a) - a(c^2 - a^2)].$$

[It is now clear that $(c-a)$ is a factor.]

$$= (b-c)(c-a)[b^2 + bc - a(c+a)].$$

In the big bracket the highest power of c is the first, so we rearrange in descending powers of c.

$$E = (b-c)(c-a)[bc - ac + b^2 - a^2]$$
$$= (b-c)(c-a)[c(b-a) + b^2 - a^2].$$

[It is now clear that $(b-a)$ is a factor.]

$$= (b-c)(c-a)(b-a)[c+b+a].$$

An expression like this, which is unaltered by changing a into b, b into c, and c into a, is called a **cyclic expression**, with respect to the letters a, b, c.

The interchange of letters is called a **cyclic interchange.**

In dealing with cyclic expressions it is usual to keep to the cyclic order: $a-b$, $b-c$, $c-a$. Thus, in the above expression the third bracket $(b-a)$ is usually replaced by $-(a-b)$, and the expression written $$-(b-c)(c-a)(a-b)(a+b+c).$$

Expressions such as $a^3(b-c)+b^3(c-a)+c^3(a-b)$ may also be resolved into factors by using the Factor Theorem. We shall return to this later in this chapter.

151. It is sometimes a good plan to group together terms of the same degree.

Example 6. *Factorise* $x^2+2x+4y+2y^2+3xy=E$.

Group together the terms of the second degree, and also the terms of the first degree.

$E=(x^2+3xy+2y^2)+(2x+4y)=(x+y)(x+2y)+2(x+2y)$.

[It is now clear that $(x+2y)$ is a factor of the whole expression]

$=(x+2y)(x+y+2)$.

EXERCISE 76. a

Factorise :

1. $2ax^2-5x+5+6a-8ax$.
2. $3bx^2-2-2x-6bx-9b$.
3. $6ax^2-2x+2a+1-7ax$.
4. $6-6cx+8c-3x+cx^2$.
5. $5b-6x+4bx^2-3+12bx$.
6. $9x^2y-8y-6x+6xy+4$.
7. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
8. $ab(a+b+c)+bc(a+b+c)+ca(a+b+c)-abc$.
9. $x^2(y-z)-y^2(z-x)+z^2(x+y)-2xyz$.
10. $(a-b)(a+b)^2+(b-c)(b+c)^2+(c-a)(c+a)^2$.
11. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
12. $l^3(m^2-n^2)+m^3(n^2-l^2)+n^3(l^2-m^2)$.
13. $x^2+5x+5xy+6y^2+15y$.
14. $x^2-2x-2y+6xy+5y^2$.
15. $3y^2-y-2x+7xy+2x^2$.
16. $5x^2-y^2+10x-2y+4xy$.
17. $6x^2-12y-6y^2+8x-5xy$.
18. $6y^2+3x-2y-17xy+12x^2$.
19. $5xy-6y^2-12x+8y+6x^2$.
20. $8x^2y+4x^2-30xy-10x+25y$.
21. $7xy-15x^2+3x-2y+15x^2y$.
22. $10y^2+15x^2+4y-10x-31xy$.
23. $15x^2+4y+2x-8-6xy$.
24. $14x^2-6x+17xy-6y^2-9y$.
25. $14x^2-25xy-35x-25y-25y^2$.
26. $2+3y+4x+12xy-16x^2$.
27. $4y+14x^2+17x-14xy-6$.
28. $10lx^2-25x-9l-15-9lx$.

EXERCISE 76. b

Factorise :

1. $5ax^2-3x-2a-6+9ax$.
2. $4ax^2-x+5a+1-9ax$.
3. $2bx^2+21b-9-13bx+3x$.
4. $12-11cx-8x-6c+10cx^2$.
5. $6x^2y-10-15y-6x+xy$.
6. $11xy-7-2x-35y+6x^2y$.

7. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

8. $ab(a-b)+bc(b-c)+ca(c-a)$.

9. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.

10. $x^2(y+z)-y^2(z+x)+z^2(x-y)$.

11. $l^2(m^3-n^3)+m^2(n^3-l^3)+n^2(l^3-m^3)$.

12. $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$.

13. $x^2-3y^2-3x+9y-2xy$. **14.** $x^2+3x+xy-12y-20y^2$.

15. $3x^2-y^2+5y-15x+2xy$. **16.** $y+4x^2-5y^2-4x+19xy$.

17. $6y^2-3y-4x+17xy+12x^2$. **18.** $8x^2+14x-15y^2-35y-14xy$.

19. $15x^2y+4y+20x^2-17xy-16x$.

20. $10y^2+6x+31xy+10y+15x^2$.

21. $14x^2+21x-6y^2+6y-17xy$.

22. $25x^2+13xy+30x+30y-10x^2y$.

23. $30x-2xy-24y^2+36y+15x^2$. **24.** $6x^2+42y-5x-21-18xy$.

25. $5x^2-25xy-63-38x-35y$.

26. $14x^2-25y^2+35x-25y+25xy$.

27. $4+6y-28x-27xy+45x^2$. **28.** $21y^2+3y-8x-50xy-16x^2$.

Harder trinomial factors

152.* The method given in Chapter XVI (Type III, Method 1) is very effective for reasonably small numbers, but may be very tedious if the numbers are large. The numbers may be large either because of the presence of a very large prime factor, in which case the work remains reasonably short, or because a number of prime factors occur several times. The following examples show how the working may be shortened.

Example 7. *Factorise* $72x^2+17xy-72y^2$.

We must find two numbers whose sum is $+17$ and whose product is $-72\times 72=-2^6\cdot 3^4$. We might proceed as before and make a table in which the first column contains unity and multiples of 3, the highest prime factor. But the work may be shortened by noticing that 17 is divisible neither by 2 nor by 3. It therefore cannot be the sum of two even numbers or of two numbers divisible by 3.

It follows that one of the required numbers must contain 2^6 as a factor, and one must contain 3^4 as a factor. In other words, for the purpose of forming the table, 2^6 and 3^4 may be used as if they were prime factors. We may therefore use either of the tables :

-1	72×72 (obviously too large)	-1	72×72
-64	81	81	-64

The numbers required are 81 and -64, and

$$72x^2 + 17xy - 72y^2 = 72x^2 - 64xy + 81xy - 72y^2$$
$$= 8x(9x - 8y) + 9y(9x - 8y) = (9x - 8y)(8x + 9y).$$

Example 8. *Factorise* $64x^2 + 144xy - 243y^2$.

We have to find two numbers whose sum is $+144$ and whose product is $-64 \times 243 = -2^6 \cdot 3^5$. The work may be simplified by noticing that some of the factors of 144 are contained twice in 64 (and in this case also in 243), i.e. 8 is a factor of 144 and 8^2 is a factor of 64. Also 9 is a factor of 144 and 9^2 is a factor of 243.

We therefore write $8x = X$, $9y = Y$. The expression then becomes $X^2 + 2XY - 3Y^2$, and the factorisation is made to depend upon that of the simpler expression $X^2 + 2XY - 3Y^2$, which equals

$$(X + 3Y)(X - Y) \text{ or } (8x + 27y)(8x - 9y).$$

153. If the original rule does not quickly lead to a solution, it will nearly always be found that the method either of Ex. 7 or of Ex. 8 is applicable. In the last resort, however, the pupil may fall back upon the method of rewriting the expression as the difference of two squares.

Example 9. *Factorise* $96x^2 - 20x - 875$, $= E$.

$$E = 96\left\{x^2 - \frac{5x}{24} - \frac{875}{96}\right\}$$
$$= 96\left\{x^2 - \frac{5x}{24} + \left(\frac{5}{48}\right)^2 - \left(\frac{5}{48}\right)^2 - \frac{875}{96}\right\}$$
$$= 96\left\{\left(x - \frac{5}{48}\right)^2 - \frac{(25 + 21000)}{48^2}\right\}$$
$$= 96\left\{\left(x - \frac{5}{48}\right)^2 - \left(\frac{145}{48}\right)^2\right\}$$
$$= 96\left(x - \frac{5}{48} + \frac{145}{48}\right)\left(x - \frac{5}{48} - \frac{145}{48}\right)$$
$$= 96\left(x + \frac{35}{12}\right)\left(x - \frac{25}{8}\right) = 96\,\frac{(12x + 35)}{12}\,\frac{(8x - 25)}{8}$$
$$= (12x + 35)(8x - 25).$$

$$\begin{array}{r} \underline{1}\ \underline{4}\ \underline{5} \\ 2\ 10\ 25 \\ \overline{1} \\ 24\,)\ 1\ 10 \\ \underline{96} \\ 285\,)\ 14\ 25 \\ \underline{14\ 25} \end{array}$$

This method is always applicable, but it should be used sparingly, as it is nearly always possible to find a shorter method.

154. The following device is sometimes useful.

Example 10. *Factorise* (i) $x^4+x^2y^2+y^4=E$; (ii) $x^4+y^4=F$.

(i) $E=x^4+2x^2y^2+y^4-x^2y^2=(x^2+y^2)^2-x^2y^2$
$=(x^2+y^2+xy)(x^2+y^2-xy)$.

(ii) $F=x^4+2x^2y^2+y^4-2x^2y^2=(x^2+y^2)^2-(\sqrt{2}\,xy)^2$
$=(x^2+y^2+\sqrt{2}\,xy)(x^2+y^2-\sqrt{2}\,xy)$.

155. The method of splitting up the middle term may be used to factorise the general expression of the second degree in x and y.

Example 11. *Factorise* $8x^2-6xy-9y^2+10x+21y-12=E$.

This may be regarded as a quadratic expression in x (or y).

We have $E=8x^2-2x(3y-5)-(9y^2-21y+12)$.

We have therefore to find two numbers whose sum is $-2(3y-5)$ and whose product is $-8(9y^2-21y+12)$

$$=-2^3\cdot 3(3y^2-7y+4)=-2^3\cdot 3\cdot(3y-4)(y-1).$$

Since the sum is of the first degree in y it is clear that $(3y-4)$ must be a factor of one number and $(y-1)$ must be a factor of the other. We therefore make a table containing **numerical** factors of $(3y-4)$ in one column. One factor is positive and one negative. We place the $-$ sign where it leads to a negative coefficient of y in the sum. Thus :

$(3y-4)$ $\quad -24(y-1)$
$2(3y-4)$ $\quad -12(y-1)$
$-3(3y-4)$ $\quad 8(y-1)$
$-4(3y-4)$ $\quad 6(y-1)$, and this is the pair required.

Then $E=8x^2-4x(3y-4)+6x(y-1)-3(3y-4)(y-1)$
$=4x[2x-(3y-4)]+3(y-1)[2x-(3y-4)]$.

[It is now clear that $2x-(3y-4)$, or $2x-3y+4$ is a factor.]

$=(2x-3y+4)[4x+3(y-1)]=(2x-3y+4)(4x+3y-3)$.

The factors of the above expression may also be found as follows :

The terms of the second degree, $8x^2-6xy-9y^2$, must be obtained as the product of the first degree terms in the required factors. But $8x^2-6xy-9y^2=(2x-3y)(4x+3y)$;

$\therefore$ the factors must be $2x-3y+a$ and $4x+3y+b$, where a and b stand for numbers and do not contain x and y.

The product of $2x-3y+a$ and $4x+3y+b$ is

$$8x^2-6xy-9y^2+x(4a+2b)+y(3a-3b)+ab.$$

Comparing this with the given expression, we see that we require values of a and b, such that

$$4a+2b=10, \quad \text{.................................(i)}$$
$$3a-3b=21, \quad \text{.................................(ii)}$$
$$ab=-12. \quad \text{.................................(iii)}$$

Solving (i) and (ii), we obtain $a=4$, $b=-3$.

These values satisfy (iii) and are the values required;

$$\therefore\ E=(2x-3y+4)(4x+3y-3).$$

Note. It should be most carefully noted that we must find values of a and b to satisfy the three equations (i), (ii), (iii). It is essential to verify that the values of a and b found from (i) and (ii) satisfy (iii).

Example 12. *Factorise* $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4=E$.

Arrange E as a quadratic in a^2, in ascending powers.

$$\begin{aligned}
E&=-(b^2-c^2)^2+2a^2(b^2+c^2)-a^4\\
&=\text{(splitting up the middle term in the usual way)}\\
&\quad -(b^2-c^2)^2+a^2(b-c)^2+a^2(b+c)^2-a^4\\
&=-(b-c)^2[(b+c)^2-a^2]+a^2[(b+c)^2-a^2]\\
&=[(b+c)^2-a^2][-(b-c)^2+a^2]\\
&=(b+c+a)(b+c-a)(a+b-c)(a-b+c),\\
\text{or}\quad &(a+b+c)(b+c-a)(c+a-b)(a+b-c).
\end{aligned}$$

*EXERCISE 77. a

Factorise:

1. $45x^2-106x+45$.
2. $72x^2+65x-112$.
3. $32x^2+60x+27$.
4. $50x^2+225xy-243y^2$.
5. $27x^2-165xy-100y^2$.
6. $40x^2-438x+189$.
7. $100x^2-641x+100$.
8. $18x^2-165xy+375y^2$.
9. $30x^2+161xy-396y^2$.
10. $147a^2-42ab-144b^2$.
11. $54x^2+105x-125$.
12. $24x^2+310xy+125y^2$.
13. $250a^2-435a+189$.
14. $108l^2-271lm+63m^2$.
15. $80x^4-51x^2y-275y^2$.
16. $96a^2+812ab-245b^2$.
17. $160x^2-1148x+1029$.
18. $175x^2+190xy-24y^2$.
19. $300x^2-140xy-1029y^2$.
20. $144x^2-337xy+144y^2$.
21. $225x^2-706x+225$.
22. $315x^2-66xy-24y^2$.
23. $96x^2-79x-135$.
24. $64x^2-276xy+135y^2$.
25. $a^4-3a^2b^2+b^4$.
26. $x^4-3x^2y^2+9y^4$.

27. $a^4+9a^2b^2+25b^4$.

28. $4x^4-16x^2y^2+9y^4$.

29. $c^4-c^2d^2+16d^4$.

30. a^4+64.

31. $2x^2+xy-6y^2-4x-y+2$.

32. $6x^2+13xy-5y^2-x+23y-12$.

33. $12x^2+xy-6y^2-31x-2y+20$.

34. $15x^2+16xy-15y^2-9x+19y-6$.

35. $4x^2-4xy-35y^2-14x+13y+12$.

36. $6x^2-31xy+5y^2+16x+7y-6$.

37. $6x^2-13xy-5y^2-16x+23y+10$.

38. $2x^2-xy-6y^2+7y-2$.

39. $15x^2+16xy-15y^2+x-21y-6$.

40. $12x^2-xy-6y^2-30x-3y+18$.

*EXERCISE 77. b

Factorise :

1. $27x^2-96x+64$.

2. $72x^2-145x+72$.

3. $150x^2+175xy-294y^2$.

4. $36x^2+97xy+36y^2$.

5. $125x^2+220xy+96y^2$.

6. $98x^2-189xy-405y^2$.

7. $294l^2+455l+125$.

8. $108c^2-215cd-63d^2$.

9. $54x^2+165xy-250y^2$.

10. $36x^2-19x-80$.

11. $243x^2+630xy-125y^2$.

12. $54a^2+231ab+245b^2$.

13. $512x^2+144x-45$.

14. $80a^4-237a^2cd+175c^2d^2$.

15. $160x^2-444xy+189y^2$.

16. $108x^2+713xy-108y^2$.

17. $294x^2+595xy+125y^2$.

18. $175x^4-190x^2-24$.

19. $392c^2+223cd-105d^2$.

20. $640x^2-1624xy-1029y^2$.

21. $64x^2+204xy-135y^2$.

22. $96x^2-241x+135$.

23. $512x^2-176x-1815$.

24. $640x^2-616xy+147y^2$.

25. $a^4-6a^2b^2+b^4$.

26. $x^4+7x^2y^2+16y^4$.

27. $x^4+2x^2y^2+9y^4$.

28. x^4+4.

29. $a^4+4a^2b^2+16b^4$.

30. $x^4-15x^2y^2+9y^4$.

31. $2x^2+xy-6y^2-5x+4y+2$.

32. $6x^2+13xy-5y^2+6x-19y-12$.

33. $12x^2+xy-6y^2-32x-7y+20$.

34. $15x^2-16xy-15y^2-37x+5y+20$.

35. $4x^2-4xy-35y^2-14x+y+12$.

36. $6x^2-31xy+5y^2-9x-13y-6$.

37. $6x^2-13xy-5y^2-19x+5y+10$.

38. $2x^2-xy-6y^2-3x-8y-2$.

39. $15x^2 - 16xy - 15y^2 - 35x + 13y + 20$.

40. $12x^2 - xy - 6y^2 - 33x + 12y + 18$.

Further use of the factor theorem. Symmetrical and alternating functions

156.* It has already been shown in Chapter XXIII, that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab).$$

This is an important identity which enables us to factorise any expression which consists of the sum of the cubes of three quantities diminished by three times their product.

Example 13. *Factorise* $8x^3 - 1 + y^3 + 6xy$.

This expression is the same as $a^3 + b^3 + c^3 - 3abc$, if $a = 2x$, $b = -1$, $c = y$, so that it equals

$$(2x - 1 + y)(4x^2 + 1 + y^2 + y - 2xy + 2x)$$

or $$(2x + y - 1)(4x^2 - 2xy + y^2 + 2x + y + 1).$$

An important corollary is that $\mathbf{a^3 + b^3 + c^3 = 3abc}$, if $\mathbf{a + b + c = 0}$.

In particular, $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

Symmetrical and alternating functions

157.* A function is said to be **symmetrical** with respect to any set of letters it contains, if its value remains unaltered when any two of the letters are interchanged.

Thus, $a^2 + b^2$, $a^2 - 5ab + b^2$, $2a^3 + 5a^2b + 5ab^2 + 2b^3$ are symmetrical with respect to a and b;

$k(x + y + z)$, $k(x^2 + y^2 + z^2) + l(yz + zx + xy)$, $4(x^3 + y^3 + z^3) - 7xyz$ are symmetrical with respect to x, y, z.

The most general homogeneous symmetrical functions of the first, second and third degrees respectively in a, b, c are

$$k(a + b + c)\ ; \quad k(a^2 + b^2 + c^2) + l(bc + ca + ab)\ ;$$

$$k(a^3 + b^3 + c^3) + l(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) + mabc,$$

where k, l, m are constants independent of a, b, c.

The sum, difference and product of any two symmetrical functions are also symmetrical functions.

158.* A function E is said to be **alternating** with respect to any set of letters it contains, if it is transformed into $-E$ when any two of those letters are interchanged.

Thus $x^2 - y^2$, $(x - y)(y - z)(z - x)$, $5a - 5b$ are alternating functions.

The sum and difference of any two alternating functions with respect to the same letters, if not zero, are alternating functions.

The product of any two alternating functions with respect to the same letters is a symmetrical function.

The product of a symmetrical function and an alternating function with respect to the same letters is an alternating function. It follows that if one alternating function is divided by another (with respect to the same letters), the quotient must be symmetrical.

The pupil should verify these statements by considering a number of simple examples.

159.* Σ and Π notation. The sum of a number of quantities forming a symmetrical or alternating function may be conveniently denoted by writing down one of the terms preceded by the symbol Σ. Thus Σa stands for the sum of all the terms of which a is the type, i.e. $a+b$, if the function contains the two letters a, b.

Similarly $\Sigma a = a+b+c$, $\Sigma bc = bc+ca+ab$,

$$\Sigma a(b^2-c^2) = a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2),$$

if the function contains the three letters a, b, c. If there is any doubt about the number of letters, these letters may be written below the Σ, e.g. $\sum_{abc}(b-c)^3$.

Likewise, the product of a number of quantities forming a symmetrical or alternating function may be denoted by writing down one of the terms preceded by the symbol Π. Thus $\Pi(b^2-c^2)$ stands for the product of all the factors of which (b^2-c^2) is the type, i.e. $(b^2-c^2)(c^2-a^2)(a^2-b^2)$, if the function contains the three letters a, b, c.

Example 14. *Factorise* $\Sigma a^3(b^2-c^2) = E$.

$$E = a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$$

This may be regarded as a function of a. By trial $E=0$ when $a=b$; $\therefore$ $(a-b)$ is a factor of E. Similarly, it may be shown that $(b-c)$ and $(c-a)$ are factors of E. E is therefore divisible by $(b-c)(c-a)(a-b)$, which is a homogeneous alternating function of the third degree.

But E is a homogeneous alternating function of the fifth degree;

$\therefore\ E \div (b-c)(c-a)(a-b)$ is a homogeneous symmetrical function of the second degree.

It must therefore be of the form

$$k(a^2+b^2+c^2)+l(bc+ca+ab),$$

where k, l are constants independent of a, b, c;

$$\therefore\ a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$
$$=[k(a^2+b^2+c^2)+l(bc+ca+ab)](b-c)(c-a)(a-b).$$

Since k, l are constants independent of a, b, c, their values may be found by giving particular values of a, b, c, avoiding of course values which make the expressions zero.

Putting $a=0$, $b=1$, $c=2$, we get

$$4-8=[5k+2l](-1)(2)(-1), \text{ or } 5k+2l=-2. \quad \ldots\ldots\text{(i)}$$

Putting $a=0$, $b=1$, $c=-1$, we get

$$1+(-1)(-1)=[2k-l](2)(-1)(-1), \text{ or } 2k-l=1. \quad \text{(ii)}$$

From (i) and (ii), $k=0$, $l=-1$;

$$\therefore\ \Sigma a^3(b^2-c^2)=-(bc+ca+ab)(b-c)(c-a)(a-b).$$

*EXERCISE 78. a

Resolve into factors:

1. $a^3+b^3-c^3+3abc$. **2.** $8l^3-1-y^3-6ly$.

3. $8a^3-27b^3-c^3-18abc$.

4. $(a-3b)^3+27(b-2c)^3+(6c-a)^3$.

5. $a^3+(2b+3c)^3-(a+2b+3c)^3$.

6. $(3a-b)^3+(3b-c)^3+(3c-a)^3$, if $a+b+c=0$.

7. $\Sigma a^2(b-c)$. **8.** $\Sigma a\,.\,\Sigma bc-abc$.

9. $\underset{abc}{\Sigma}(a-b)(a+b)^2$. **10.** $\Sigma a^3(b-c)$.

11. $\Sigma x^4(y-z)$. **12.** $\underset{xyz}{\Sigma}x^2y^2(x-y)$.

13. $\underset{abc}{\Sigma}(a-b)(a+b)^3$. **14.** $(\underset{abc}{\Sigma}a)^5+\underset{abc}{\Sigma}(a-b-c)^5$.

15. $\Sigma(b-c)(b+c-2a)^2$. **16.** $\Sigma(a+b)^2(a+c)^2(b-c)$.

*EXERCISE 78. b

Resolve into factors:

1. $a^3+b^3+8c^3-6abc$. **2.** $27a^3-b^3+c^3+9abc$.

3. $a^3-8b^3+27+18ab$.

4. $(5a-4b)^3+(4b-3c)^3-(5a-3c)^3$.

5. $8(a-b)^3+(2b-c)^3+(c-2a)^3$.

6. $(5a-2b-c)^3+(5b-2c-a)^3+(5c-2a-b)^3$, if $a+b+c=0$.

7. $\Sigma a^2(b+c)+2abc$.

8. $\sum_{abc} ab(a-b)$.

9. $\Sigma a(b-c)^3$.

10. $\Sigma l^2(m^3-n^3)$.

11. $\sum_{xyz}(x-y)^5$.

12. $(\sum_{abc} a)^3+\sum_{abc}(a-b-c)^3$.

13. $(\sum_{abc} a)^5-\sum_{abc} a^5$.

14. $(\sum_{abc} a)^4-\sum_{abc}(b+c)^4+\sum_{abc} a^4$.

15. $\Sigma(b-c)(b+c-2a)^3$.

16. $\sum_{abc}(b-c)^3(a-x)^3$.

EXERCISE 79

Miscellaneous Factors

Factorise completely:

1. $(5x-1)yz+xz^2-5y^2$.

2. $4a^5-36ab^2$.

3. $42a^2-ab-30b^2$.

4. $5a^4-40a$.

5. $a^3-13a^2+5a-65$.

6. $30(x^2-1)-32x$.

7. $9a^4-l^2-4m^2+4lm$.

8. $1-(a-b^2)c^2-ab^2c^4$.

9. $6a^2-19ab-36b^2$.

10. $(x-y)^3+(y-z)^3+(z-x)^3$.

11. $3x^3y-12xy^3$.

12. $4(x^2-1)-5x-5$.

13. $x^3-7(x+3)+27$.

14. $x^4+6x^3+13x^2+12x+4$.

15. $(ab-2c)^2+(ac+2b)^2$.

16. $(x+y)^3-8(x-y)^3$.

17. $4(c^2-d^2)-20c+25$.

18. $17x^2+201x-36$.

19. $a^2+b^2-(x^2+1)-2(ab+x)$.

20. $6x^3-19x^2+11x+6$.

21. $18x^2+9xy-20y^2$.

22. $32-2(2a-1)^4$.

23. $8a^3-26a+12$.

24. $48a^2-88a-45$.

25. $(2x-3y)^3+(3x-2y)^3$.

26. $x^4+x^3+27x+27$.

27. $a^2(2b-1)-4b^2(a-1)-2b+a$.

28. $32x^2-52x-45$.

29. $x^3+2ax^2-9b^2x-18ab^2$.

30. $108x^2-24x-175$.

31. $4x^3+10x^2-14x-9$.

32. $9x^2-9x-(y-2)(y+1)$.

33. x^4-41x^2+400.

34. $y^2+6y-9(x^2-1)$.

35. $3x^2-10xy-8y^2+2x-8y$.

36. $(9x^2+9x-2)^2-(9x^2-9x+2)^2$.

37. $4x^2-y^2-2x+y$.

38. $2a^2-3ab-ac+b^2+bc$.

39. $(2a-3b)^2-(a-b)^2$.

40. $a(a-4)-b(b-4)$.

41. $a^2+ab+b-1$.

42. $m(l+m)-n(n-l)$.

43. $x(y^2-4)-y(x^2-4)$.

44. $(1+8x+8x^2)^2-(2x+1)^2$.

45. $4x^3-8x^2y-9xy^2+18y^3$.

46. $9x^2-a+3ax-1$.

47. $a^2+ab-2b^2+a-b$.

48. $72a^6-14a^4-8a^2$.

49. $9x^2(3x+5a)^2 - 36a^4$. **50.** $a(x^2+a-1) - x(a^2+x-1)$.

51. $a^2 - ab - ac - 2b^2 + 2bc$. **52.** $(x^2+y^2-l^2-m^2)^2 - 4(xy-lm)^2$.

53. $6x^2 + x - 15 - 4xy + 6y$. **54.** $(x^2+ax+b)^3 + (x^2-ax+b)^3$.

55. $(7x+8)^2 - 2(7x+8) - 15$. **56.** $\Sigma a(b+c)^2 - 4abc$.

57. $(4x^2 - 3xy + y^2)^2 - 4(x^2 + 2xy - y^2)^2$.

58. $4 - 5x - 5y + x^2 + 2xy + y^2$.

59. $(x^2-y^2)a^2 - 2(x^2-yz)a + x^2 - z^2$. **60.** $x^4 + 2x^2 + 9$.

61. $2a^2 - 3ab + b^2 + 3a - 2b + 1$. **62.** $x^2 + 4xy - x - 12y - 6$.

63. $1 - 4c - 16c^2 + 64c^3$. **64.** $x^2 - 5y^2 + 3ax - 4xy - 15ay$.

65. $6x^3 - 11x^2 - 37x + 70$. **66.** $(4x-5y)y^3 + (5x+4y)x^3$.

67. $(y+z)^3 + (z+x)^3 - (x+y+2z)^3$.

68. $x^2 - 3ax - 4xy + 6ay + 4y^2$. **69.** $4y^3 - 39yz^2 + 35z^3$.

70. $15a^2 + 11ab - 8b - 12b^2 - 6a$. **71.** $24x^3 - 2x^2y - 31xy^2 - 12y^3$.

72. $6x^2 - 3x + 10xy - 2y + 4y^2$.

73. $18x^2 + 9xy - 9y^2 - 27x + 45y - 56$.

74. $24x^3 - 50x^2y - 3xy^2 + 36y^3$.

75. $45x^2 + 56x - 45$. **76.** $36x^2 - 65xy - 36y^2$.

77. $15x^2 + 2xy - 18x - 24y^2 - 24y$.

78. $6x^2 + 31xy + 5y^2 - 32x - 15y + 10$.

79. $108a^2 + 745a + 108$. **80.** $250c^2 - 705cd^2 + 189d^4$.

81. $4x^2 + 4xy - 35y^2 - 16x - 20y + 15$.

82. $7y + 16x^2 - 50xy - 2x - 21y^2$.

83. $x^2y - 3x^2 + xy^2 - xy - 8x - 2y - 4$.

84. $6x^2y - 12x^2 - 27x - 2xy^2 + 10xy + 5y - 15$.

85. $6x^2 + 31xy + 5y^2 - 17x - 27y + 10$.

86. $12mx^2 + 23mx + 72 - 9m + 32x$.

87. $x^2y + 2xy^2 + 5x^2 + 11xy - 6y + 2x - 3$.

88. $15x^2y - 6xy^2 - 5xy - 10x^2 + 8y - 14x + 12$.

89. Find p, so that $3x^3 + px^2 + 9x - 9$ may be divisible by $x+3$.

90. Factorise $3x^2 + 14x + 15$ and deduce the prime factors of 31415.

91. Find k, so that $x^3 - 6x + k$ may be divisible by $x - 2$.

92. Find the two values of p for which the quadratic expressions $10x^2 - 21x - 10$, $100x^2 + 10x + p$ have a common factor.

93. $F(x) \div (x^2 - 3x + 2)$ leaves remainder $ax + b$. If $F(1) = 4$, $F(2) = 2$, find a and b.

94. Find p and q, so that $x^4 - x^3 + px^2 + qx + 6$ may be divisible by $x+2$ and $x-3$.

CHAPTER XXV

HARDER FRACTIONS. H.C.F. BY LONG METHOD. HARDER FRACTIONAL EQUATIONS

160. In the first part of this chapter the work of Chapter XIX is continued. No new principles are introduced, but the work involves a knowledge of the factors dealt with in Chapter XXIV. The worked examples illustrate a number of devices for shortening the working.

Example 1. *Simplify* $\dfrac{1-a+a^2}{1+a^3}-\dfrac{a+a^2}{(1+a)^3}=E.$

$$E=\frac{(1-a+a^2)}{(1+a)(1-a+a^2)}-\frac{a(1+a)}{(1+a)^3}=\frac{1}{1+a}-\frac{a}{(1+a)^2}$$

$$=\frac{1+a-a}{(1+a)^2}=\frac{1}{(1+a)^2}.$$

Note. It is most important that the pupil should reduce each fraction to its lowest terms before adding the fractions.

Example 2. *Simplify* $\dfrac{1}{2x}-\dfrac{3}{2(x+1)}+\dfrac{3}{2(x+2)}-\dfrac{1}{2(x+3)}=E.$

It is easier to combine the fractions in pairs, instead of finding the L.C.M. of all the denominators at once.

$$E=\frac{1}{2}\left[\frac{1}{x}-\frac{1}{x+3}\right]-\frac{3}{2}\left[\frac{1}{x+1}-\frac{1}{x+2}\right]$$

$$=\frac{3}{2x(x+3)}-\frac{3}{2(x+1)(x+2)}$$

$$=\frac{3[(x+1)(x+2)-x(x+3)]}{2x(x+3)(x+1)(x+2)}$$

$$=\frac{3\cdot 2}{2x(x+1)(x+2)(x+3)}=\frac{3}{x(x+1)(x+2)(x+3)}.$$

Example 3. *Simplify* $\dfrac{1}{1-a}+\dfrac{1}{1+a}+\dfrac{2}{1+a^2}+\dfrac{4}{1+a^4}=E.$

$E=\dfrac{2}{1-a^2}+\dfrac{2}{1+a^2}+\dfrac{4}{1+a^4}$, taking the first two terms together,

$$=\frac{4}{1-a^4}+\frac{4}{1+a^4}=\frac{8}{1-a^8}.$$

Example 4. *Simplify*

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}=E.$$

Keeping the usual cyclic order of the expressions, we write

$$(x-z)=-(z-x),\ (y-x)=-(x-y),\ (z-y)=-(y-z).$$

We then have

$$E=-\frac{1}{(x-y)(z-x)}-\frac{1}{(x-y)(y-z)}-\frac{1}{(z-x)(y-z)}$$

$$=-\frac{y-z+z-x+x-y}{(x-y)(z-x)(y-z)}=0.$$

Example 5. *Simplify* $\dfrac{6x-1}{3x-1}-\dfrac{12x-6}{3x-2}+\dfrac{6x-5}{3x-3}=E.$

$$\frac{6x-1}{3x-1}=\frac{6x-2+1}{3x-1}=2+\frac{1}{3x-1}.$$

$$\frac{12x-6}{3x-2}=\frac{12x-8+2}{3x-2}=4+\frac{2}{3x-2}.$$

$$\frac{6x-5}{3x-3}=\frac{6x-6+1}{3x-3}=2+\frac{1}{3x-3}.$$

$$\therefore\ E=2-4+2+\frac{1}{3x-1}-\frac{2}{3x-2}+\frac{1}{3x-3}$$

$$=0+\frac{6x-4}{(3x-1)(3x-3)}-\frac{2}{(3x-2)}=2\left\{\frac{(3x-2)^2-(3x-1)(3x-3)}{(3x-1)(3x-2)(3x-3)}\right\}$$

$$=\frac{2}{(3x-1)(3x-2)(3x-3)},$$ after simplifying the numerator.

EXERCISE 80. a

Simplify:

1. $\dfrac{(a+b)^2}{a^2+ab+b^2}\times\dfrac{a^3-b^3}{a^2+ab}\div\left(1-\dfrac{b^2}{a^2}\right).$ 2. $\dfrac{a^3-b^3}{a^2b+b^3}\div\dfrac{a-b}{a^2+b^2}.$

3. $\dfrac{x^3-8y^3}{x+3y}\div\left(\dfrac{x-2y}{4x}\times\dfrac{8x^4}{x^2+3xy}\right).$

4. $\left(\dfrac{64x^3-y^3}{2x^2}\div\dfrac{6x^3}{4x+y}\right)\times\dfrac{8z^2}{16x^2+4xy+y^2}.$

5. $\left(2a+\dfrac{b^2}{2a-b}\right)\left(1-\dfrac{2b^3}{8a^3+b^3}\right)\div\left(\dfrac{4a^2}{2a+b}+b\right).$

6. $\left\{\dfrac{x^4+x^2y^2+y^4}{x^2-2xy-15y^2}\div\dfrac{x^3-y^3}{x^2+2xy-3y^2}\right\}\div\dfrac{1}{x-5y}.$

7. $\left\{\dfrac{x^3+8}{x^2-2x-24} \div \dfrac{x^2+8x+12}{x^2+8x+16}\right\} \div \dfrac{x^4+4x^2+16}{x^2-36}.$

8. $\dfrac{12x^2-4x-1}{8x^3-4x^2-2x+1}.$

9. $\dfrac{8a^3+b^3}{4a^2-2ab+b^2}+\dfrac{8a^3-b^3}{4a^2+2ab+b^2}.$

10. $\dfrac{a^3+4a^2b-5b^3}{a^3-3ab^2+2b^3}.$

11. $\dfrac{x^3-y^3}{x^4-y^4}-\dfrac{x-y}{x^2-y^2}-\dfrac{x+y}{2(x^2+y^2)}.$

12. $\dfrac{1-a}{1-a^2}+\dfrac{a^2}{a-a^3}-\dfrac{1+a^2}{1+a^3}.$

13. $\dfrac{5l}{l^2-1}+\dfrac{3l}{l-l^2}-\dfrac{2}{1+l}.$

14. $\left[\dfrac{a}{a+b}+\dfrac{a}{a-b}\right] \div \left[\dfrac{(a+b)^2}{2(a-b)}+\dfrac{(a-b)^2}{2(a+b)}\right].$

15. $\dfrac{x-y}{x-2y}-\dfrac{x+y}{x+2y}-\dfrac{4y^2}{4y^2-x^2}+\dfrac{2y}{2y-x}.$

16. $\dfrac{4}{a+b}-\dfrac{1}{4b-a}+\dfrac{4}{b-a}-\dfrac{1}{4b+a}.$

17. $\dfrac{x+5y}{x^2+5xy}+\dfrac{1}{x-3y}-\dfrac{3y}{3xy-x^2}.$

18. $\dfrac{1}{c+2d}+\dfrac{1}{2d-c}+\dfrac{8d}{c^2-4d^2}.$

19. $\dfrac{1}{x-1}-\dfrac{3}{x}+\dfrac{3}{x+1}-\dfrac{1}{x+2}.$

20. $\dfrac{x}{x-1}-\dfrac{2x-2}{x-2}+\dfrac{x-2}{x-3}.$

21. $\dfrac{6xy}{(3x-2y)(2y-z)}+\dfrac{3xz}{(3x-z)(z-2y)}.$

22. $\dfrac{5}{(a-c)(b-c)}+\dfrac{1}{(b-a)(c-a)}+\dfrac{1}{(a-b)(c-b)}.$

23. $\dfrac{x}{(x-y)(x-z)}+\dfrac{y}{(y-z)(y-x)}+\dfrac{z}{(z-x)(z-y)}.$

24. $\dfrac{b+c}{(l-m)(l-n)}+\dfrac{c+a}{(m-l)(m-n)}+\dfrac{a+b}{(n-l)(n-m)}.$

EXERCISE 80. b

Simplify:

1. $\dfrac{\dfrac{4x^2+y^2}{y}-2x}{\dfrac{1}{y}-\dfrac{1}{2x}} \times \dfrac{4x^2-y^2}{8x^3+y^3}.$

2. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}} \times \dfrac{a^2-b^2}{a^3+b^3}.$

3. $\left\{\dfrac{1-x+x^2}{1+x+x^2}+\dfrac{1-x}{1+x}\right\} \div \left\{\dfrac{1-x+x^2}{1+x+x^2}+\dfrac{1+x}{1-x}\right\}.$

4. $\dfrac{(a^2-ab)^2}{(a^2+ab)^2} \times \dfrac{a^3+b^3}{a^3-b^3} \div \dfrac{a-b}{a+b}.$

5. $\dfrac{8c^4 - 27cd^3}{4c^2 - 9d^2} \div \dfrac{4c^2 + 6cd + 9d^2}{2(2c + 3d)}$.

6. $\dfrac{4x^2y^2 - 2xy^3 + y^4}{8x^3 + y^3}$.

7. $\left(\dfrac{4ab}{4a^2 + b^2} - 1\right) \times \left(\dfrac{b}{2a} + \dfrac{2a}{b}\right) \div \left(\dfrac{8a^3 - b^3}{2a - b} - 6ab\right)$.

8. $\left\{\dfrac{a^4 + a^2b^2 + b^4}{a^2 - b^2} \div \dfrac{a^3 + b^3}{(a + b)^3}\right\} \div \dfrac{a^3 - 8a^2b - 9ab^2}{a - b}$.

9. $\dfrac{x^3 + 2x^2 + 4x - 24}{x^3 + 6x^2 + 20x + 24}$.

10. $\dfrac{(b - 2a)^2(4a^2 + 2ab + b^2)}{(4a^2 - b^2)(8a^3 - b^3)}$.

11. $\left\{\dfrac{x}{y^2} - \dfrac{y}{x^2}\right\} \div \left\{\dfrac{x}{y} + 1 + \dfrac{y}{x}\right\}$.

12. $\dfrac{1}{t + 4} - \dfrac{2t + 6}{(t + 3)^2} + \dfrac{1}{t + 2}$.

13. $\dfrac{8x^2 - 3y^2}{x^3 - y^3} - \dfrac{7x + 5y}{x^2 + xy + y^2} - \dfrac{2}{x - y}$.

14. $\dfrac{b(a^2 + b^2)}{a^3 - b^3} - \dfrac{ab}{b^2 - a^2} - \dfrac{b}{a - b}$.

15. $\dfrac{9 - a}{3(1 - a)} - \dfrac{9 + a}{3(1 + a)} - \dfrac{16a - 3}{3(1 - a^2)}$.

16. $\dfrac{5x}{25x^2 - y^2} - \dfrac{1}{3(5x + y)} - \dfrac{1}{3(5x - y)}$.

17. $\dfrac{1}{x + 4y} - \dfrac{3}{x + 3y} + \dfrac{3}{x + 2y} - \dfrac{1}{x + y}$.

18. $\dfrac{3a}{a^2 - 1} - \dfrac{a}{a + a^2} + \dfrac{3}{2(1 - a)}$.

19. $\dfrac{1}{a + 3b} - \dfrac{2a + 6b}{a^2 + 3ab} + \dfrac{1}{a - 3b}$.

20. $\dfrac{4x - 3}{x - 1} - \dfrac{8x - 14}{x - 2} + \dfrac{4x - 11}{x - 3}$.

21. $\dfrac{1}{(x + 5y)(x - 2y)} + \dfrac{1}{(y - x)(2y - x)}$.

22. $\dfrac{q - r}{(p - q)(p - r)} + \dfrac{r - p}{(q - r)(q - p)} + \dfrac{p - q}{(r - p)(r - q)}$.

23. $\dfrac{x + y}{x^2 + 2xy + 2y^2} - \dfrac{2x^3}{x^4 + 4y^4}$.

24. $\sum_{abc} \dfrac{(x - b)(x - c)}{(a - b)(a - c)}$.

EXERCISE 80. c

Simplify:

1. $\dfrac{1}{a - 3} + \dfrac{1}{3(a - 1)} + \dfrac{1}{a + 3} + \dfrac{1}{3(a + 1)}$.

2. $\dfrac{5b - 3c}{15bc} - \dfrac{10b + 3c}{25b^2 - 15bc} - \dfrac{5b - 6c}{15bc + 9c^2}$.

3. $\dfrac{1 + 2x + 2x^2 + x^3}{1 - x - x^3 - x^4}$.

4. $\dfrac{7}{3x - 5y} - \dfrac{3}{7x + 5y} + \dfrac{7}{3x + 5y} - \dfrac{3}{7x - 5y}$.

5. $\dfrac{1}{2a(2a-3b)}+\dfrac{1}{3b(2a+3b)}-\dfrac{4a^2-9b^2}{6ab(4a^2+9b^2)}$.

6. $\dfrac{4x^3-8x^2+5x-1}{2x^3-3x^2-3x+2}$.

7. $\left(a^2-1-\dfrac{6}{a^2}\right)\div\left(a^2+2a+3+\dfrac{4}{a}+\dfrac{2}{a^2}\right)$.

8. $\dfrac{2+a}{1+a+a^2}+\dfrac{2-a}{1-a+a^2}+\dfrac{1}{1+a}+\dfrac{1}{1-a}$.

9. $\dfrac{bc(x-a)}{(a-b)(a-c)}+\dfrac{ca(x-b)}{(b-c)(b-a)}+\dfrac{ab(x-c)}{(c-a)(c-b)}$.

10. $\Sigma\dfrac{a^2}{bc(a-b)(a-c)}$.

11. $\dfrac{2b}{a-b}-\left\{\dfrac{1-\dfrac{a}{b}+\dfrac{a^2}{b^2}}{1+\dfrac{a}{b}+\dfrac{a^2}{b^2}}\times\dfrac{b^3-a^3}{b^3+a^3}\right\}+\dfrac{2b}{a+b}$.

12. $\dfrac{1}{x^2-5}-\dfrac{2}{x^2-3}+\dfrac{2}{x^2+3}-\dfrac{1}{x^2+5}$.

13. $\dfrac{a^4-7a^2+1}{(a^2+1)^2-4a(a^2+1)+3a^2}$.

14. $\Sigma\dfrac{(b+c)(b^2+c^2-a^2)}{bc}$.

15. $\underset{abc}{\Sigma}\dfrac{(a-b)\sqrt{x-c}}{\sqrt{(x-a)(x-b)}}$ (Take + sign for each root).

16. $\dfrac{\Sigma(y^2-z^2)(y+z-2x)}{\Sigma x^2(y-z)}$.

17. If $(x+1)y=1$, express $\dfrac{x^2}{x+1}-\dfrac{y^2}{y+1}$ in terms of x only, reducing the answer to its lowest terms.

18. If $x=a+b$, find in its simplest form the value of

$$\frac{a}{x^2+b^2-a^2}+\frac{b}{x^2+a^2-b^2}-\frac{a+b}{x^2-a^2-b^2}.$$

19. If $y=\dfrac{1}{1-x}$, express $\dfrac{x^3-3x+1}{x^2-x}$ in terms of y in its simplest form.

20. If $x=a-b$, find in its simplest form the value of

$$\frac{1}{x^2+a^2-b^2}-\frac{1}{x^2+b^2-a^2}+\frac{1}{x^2-a^2-b^2}.$$

GENERAL METHOD OF FINDING THE H.C.F.

161. We have previously found the H.C.F. of two or more expressions by resolving the expressions into their factors, and then selecting the H.C.F. by inspection. When the expressions are not easily resolved into factors, it is better to use a more general method. This method, which is similar to the " Division Method " used in Arithmetic, depends upon the following theorem :

If $\mathbf{X = YQ + R}$, where X, Y, Q, R are rational integral functions, then the common factors of X and Y are the same as those of Y and R.

(1) Let A be a common factor of Y and R; then $Y=AC$, $R=AD$, and $X=ACQ+AD=A(CQ+D)$, i.e. A is a factor of X.

(2) Let K be a common factor of X and Y; then, $X=KL$, $Y=KM$, and $R=X-YQ=KL-KMQ=K(L-MQ)$, i.e. K is a factor of R.

Hence (1) every common factor of Y and R is a factor of X, and is therefore also a common factor of X and Y; (2) every common factor of X and Y is a factor of R, and is therefore also a common factor of Y and R. Hence the common factors of X and Y are the same as those of Y and R.

In particular, if X and Y are rational integral expressions in x, and R is any one of the remainders in the process of dividing X by Y, the common factors of X and Y are the same as those of Y and R; for if Q is the quotient corresponding to the remainder R, we have $X=YQ+R$.

The application of this theorem is best illustrated by numerical examples. The method is based on the fact that R is of lower degree than either X or Y and it is therefore easier to find the H.C.F. of R and Y (or X) than of X and Y. The process may be repeated until the result is obtained.

Example 6. *Find the H.C.F. of* x^3-x^2-3x+6 (X) *and* x^3-3x+2 (Y).

We first divide Y into X and obtain the remainder R_1,

$$\begin{array}{r|l} x^3-3x+2 & x^3-x^2-3x+6 \;(\,1 \\ & \underline{x^3 \qquad\; -3x+2} \\ & \quad -x^2 \qquad\; +4 \quad (R_1). \end{array}$$

The H.C.F. of X and Y is the same as the H.C.F. of Y and R_1. We apply the same method to find the H.C.F. of Y and R_1.

$$\begin{array}{r|l} -x^2+4 \,) & x^3 - 3x + 2 \,(-x \\ & \underline{x^3 - 4x} \\ & \phantom{x^3-{}} x + 2 \quad (R_2). \end{array}$$

The H.C.F. of Y and R_1 is the same as the H.C.F. of R_1 and R_2. We apply the same method to find the H.C.F. of R_1 and R_2.

$$\begin{array}{r|l} x+2 \,) & -x^2 \qquad +4 \,(-x+2 \\ & \underline{-x^2 - 2x} \\ & \phantom{-x^2-{}} 2x + 4 \\ & \phantom{-x^2-{}} \underline{2x + 4} \end{array}$$

$x+2$ (R_2) is a factor of R_1, and therefore the common factor of R_2 and R_1, and therefore of R_1 and Y, and therefore of Y and X. The required H.C.F. is therefore $x+2$.

Note. The process has been written out in full in order to illustrate the principle. But in practice it is often possible to shorten the work. Thus, in the example above, it is easily seen that $R_1 = -(x-2)(x+2)$.

Since we only require the *common* factor or factors of R_1 and Y, we need only find whether $x-2$ and $x+2$ are factors of Y.

We have $\quad Y = f(x) = x^3 - 3x + 2, \quad \therefore f(2) = 8 - 6 + 2 = 4,$

$$\therefore x - 2 \text{ is not a factor of } Y.$$

Again, $\quad f(-2) = -8 + 6 + 2 = 0, \quad \therefore x + 2$ is a factor of Y,

$\therefore x+2$ is the common factor of Y and R_1, and therefore of X and Y.

162. In the process outlined above R_1, R_2 ... need not be the *final* remainders in the divisions. *Any* remainder is sufficient for the purpose. Again, we are only concerned with the *common* factors of X and Y, Y and R_1, R_1 and R_2, etc.

Consider any pair chosen out of the quantities X, Y, R_1, $R_2 \cdots$, say Y and R_1. The final result of the process will not be altered by changing Y or R_1, or both Y and R_1, in any way which does not introduce or remove a common factor.

Hence, without altering the final result, we may multiply or divide Y or R_1 (or any other pair) by a constant or by a function of x, provided that these are not factors of R_1 or Y respectively.

Example 7. *Find the H.C.F. of* $3+4x-16x^2+9x^3 (X)$ *and* $4+7x-19x^2+8x^3 (Y)$.

If we divide X by Y or Y by X, we introduce fractional coefficients. To avoid this, we introduce a suitable multiplier, e.g. we may multiply X by 4 and divide $4X$ by Y. It is clear that 4 is not a factor of Y, and therefore the common factor of X and Y is the same as the common factor of $4X$ and Y. Dividing $4X$ by Y, we get

$$\begin{array}{r} 4+7x-19x^2+8x^3\,)\;12+16x-64x^2+36x^3\;(\,3 \\ \underline{12+21x-57x^2+24x^3} \\ -\;5x-\;7x^2+12x^3\;(R_1) \\ =-x(5+7x-12x^2). \end{array}$$

The factor $-x$ is now removed, since it is clearly not a factor of Y, and to avoid introducing fractional coefficients, Y is multiplied by 5 (which is not a factor of $5+7x-12x^2$) before we divide by $5+7x-12x^2$. We have

$$\begin{array}{r} 5+7x-12x^2\,)\;20+35x-95x^2+40x^3\;(\,4 \\ \underline{20+28x-48x^2} \\ 7x-47x^2+40x^3\;(R_2) \\ =x(7-47x+40x^2). \end{array}$$

As before, we reject the factor x, and multiply by 5 (which is not a factor of $5+7x-12x^2$) before completing the division by $5+7x-12x^2$. Thus,

$$\begin{array}{r} 5+7x-12x^2\,)\;35-235x+200x^2\;(\,7 \\ \underline{35+\;49x-\;84x^2} \\ -284x+284x^2 \\ =-284x(1-x). \end{array}$$

As above, 284 and $-x$ may be rejected, and it is easily seen that $1-x$ divides exactly into $5+7x-12x^2$, $\therefore$ the H.C.F. is $1-x$.

Note 1. The above example has been set out in full to show how (1) factors containing x may be rejected without affecting the result, (2) factors may be introduced without affecting the result.

But it is much shorter to consider the factors of R_1

$$=-x(1-x)(5+12x).$$

By the Factor Theorem it is easily seen that $1-x$ is the only common factor of Y and R_1 and therefore of X and Y.

Note 2. With a different order of working, we might obtain $x-1$ as the H.C.F. It does not matter which result we take, since -1 may be considered a factor of any algebraical expression.

Note 3. If X and Y contain obvious common factors, they should be at once removed. Thus, to find the H.C.F. of

$$6x+8x^2-32x^3+18x^4 (X) \quad \text{and} \quad 8x+14x^2-38x^3+16x^4,$$

we notice that $X=2x(3+4x-16x^2+9x^3)$

and $Y=2x(4+7x-19x^2+8x^3)$.

$2x$ is therefore a common factor; the other common factor is the H.C.F. of $3+4x-16x^2+9x^3$ and $4+7x-19x^2+8x^3$, which has been found above to be $1-x$, $\therefore$ the H.C.F. is $2x(1-x)$.

Note 4. The work may be shortened by the use of Detached Coefficients.

Note 5. One of the principal uses of H.C.F. is to reduce fractions to their lowest terms.

163. The H.C.F. of more than two expressions must be a factor of the H.C.F. of any two of them. It may therefore be obtained as follows:

(1) Find the H.C.F. of any two of the given expressions.

(2) Take this result and a third expression, and find their H.C.F., and so on.

The H.C.F. last found must be the H.C.F. required, for it is the highest factor contained in all the expressions.

EXERCISE 81. a

Find the H.C.F. of:

1. $x^3+7x^2+14x+8$ and $x^3+6x^2+11x+6$.

2. $8x^3-10x^2+5x-3$ and $16x^3-28x^2+12x-9$.

3. $4c^3+11c^2+25c-7$ and $2c^3+5c^2+11c-7$.

4. $x^3-18x-35$ and $x^3-21x-20$.

5. $16x^3-16x+30$ and $72x^3-48x^2+75$.

6. $6a^4+9a^3-39a^2-36a$ and $2a^4+13a^3-28a^2-32a$.

Reduce to their lowest terms:

7. $\dfrac{18x^3+32x^2+8x-6}{8x^3+19x^2+7x-4}$. **8.** $\dfrac{2a^3-a^2-25}{4a^4+5a^2-38a+5}$.

Find the H.C.F. of:

9. $8x^3-16x^2+8x-3$, $8x^3-4x^2-14x+3$, $8x^3-20x^2+18x-9$.

10. $1-c-c^3+c^5$, $1-c^4-c^6-c^7$.

EXERCISE 81. b

Find the H.C.F. of :

1. $x^3+7x^2+11x+5$ and $x^3+8x^2+13x+6$.

2. $8l^3-24l^2-172l+35$ and $8l^3-20l^2-198l+40$.

3. $4x^3-9x^2-2x+1$ and $4x^3-10x^2+2$.

4. $12x^3-2x^2y+12xy^2-70y^3$ and $3x^3+7x^2y-47xy^2+45y^3$.

5. $3a^4-6a^3-12a^2+24a$ and $4a^4-14a^3+8a^2+8a$.

6. $3x^4-11x^3+15x^2-6x$ and $2x^3-12x^2+24x-18$.

Reduce to their lowest terms :

7. $\dfrac{2x^3-7x^2-10x+24}{2x^3-3x^2+2x-3}$. **8.** $\dfrac{x^4-5x-6}{3x^4+5x^2-8}$.

Find the H.C.F. of :

9. $27a^3+9a^2-21a+2$, $54a^3-9a^2-21a+2$, $9a^3-12a^2+7a-2$.

10. $6x^5-10x^3+4$, $6x^5-15x^2+9$.

HARDER FRACTIONAL EQUATIONS

164. The following example illustrates a method of shortening the work involved in solving fractional equations.

Example 8. *Solve* $\dfrac{6x-47}{x-8}+\dfrac{2x-7}{x-4}=\dfrac{3x-14}{x-5}+\dfrac{5x-34}{x-7}$.

We have

$$\frac{(6x-48)+1}{x-8}+\frac{(2x-8)+1}{x-4}=\frac{(3x-15)+1}{x-5}+\frac{(5x-35)+1}{x-7},$$

$$\therefore\ 6+\frac{1}{x-8}+2+\frac{1}{x-4}=3+\frac{1}{x-5}+5+\frac{1}{x-7},$$

$$\therefore\ \frac{1}{x-8}+\frac{1}{x-4}=\frac{1}{x-5}+\frac{1}{x-7},$$ etc., as on p. 266.

EXERCISE 82. a

Solve the equations :

1. $\dfrac{1}{2x-5}-\dfrac{1}{2x-9}=\dfrac{1}{2x-3}-\dfrac{1}{2x-7}$.

2. $\dfrac{1}{x-12}-\dfrac{1}{x-10}-\dfrac{1}{x-6}+\dfrac{1}{x-4}=0$.

3. $\dfrac{3}{x-3}-\dfrac{6}{x-6}=\dfrac{9}{x-9}-\dfrac{12}{x-12}$.

4. $\dfrac{3}{2x+1}-\dfrac{7}{6(x+1)}=\dfrac{1}{2x+3}-\dfrac{1}{6(x+1)}$.

5. $\dfrac{3x+40}{3x+1}+\dfrac{4x-24}{2x+1}=\dfrac{2x+3}{x+2}+\dfrac{6x+5}{6x-1}.$

6. $\dfrac{1}{x+2}-\dfrac{2}{x+6}=\dfrac{1}{2x+3}-\dfrac{3}{2(x+3)}.$

7. $\dfrac{4x^2-2x+1}{2x-1}-\dfrac{4x^2+2x+1}{2x+1}=\dfrac{28x}{4x^2-1}.$

8. $\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

9. $\dfrac{2}{2x+1}-\dfrac{5}{2x+3}=\dfrac{2}{2x-1}-\dfrac{5}{2x-2}.$

10. $\dfrac{4x-27}{2x-14}-\dfrac{2x-12}{2x-13}=\dfrac{4x-17}{2x-9}-\dfrac{2x-7}{2x-8}.$

11. $\dfrac{4x^2-2x+1}{2x-1}+\dfrac{4x^2+2x+1}{2x+1}=4x.$

EXERCISE 82. b

Solve the equations :

1. $\dfrac{1}{3x-10}-\dfrac{1}{3x-5}=\dfrac{1}{3x-7}-\dfrac{1}{3x-2}.$

2. $\dfrac{1}{3x+2}+\dfrac{1}{3x+7}=\dfrac{1}{3(x+1)}+\dfrac{1}{3(x+2)}.$

3. $\dfrac{2}{x+1}-\dfrac{3}{x+2}=\dfrac{4}{x+3}-\dfrac{5}{x+4}.$

4. $\dfrac{1}{x-9}+\dfrac{1}{x-5}-\dfrac{1}{x-6}-\dfrac{1}{x-8}=0.$

5. $\dfrac{1}{2x+1}-\dfrac{1}{4x+1}=\dfrac{2}{2x+5}-\dfrac{3}{4(x+1)}.$

6. $\dfrac{2x}{4x-1}+\dfrac{8x-5}{12x-3}=\dfrac{4x}{6x+4}-\dfrac{5-6x}{12x+8}.$

7. $\dfrac{2x+3}{4x+3}+\dfrac{3x+2}{4x+2}=\dfrac{5(x+2)}{4x+5}.$

8. $\dfrac{5x-63}{x-12}+\dfrac{8x-30}{2x-9}=\dfrac{6x-69}{2x-21}+\dfrac{6x-15}{x-3}.$

9. $\dfrac{2x+3}{2x-3}+\dfrac{2x-3}{2x+3}=c+\dfrac{1}{c}.$

10. $\dfrac{x+7}{x+8}+\dfrac{x+3}{x+4}=\dfrac{x+8}{x+9}+\dfrac{x+2}{x+3}.$

11. $\dfrac{3x-1}{3x-2}-\dfrac{3x}{3x-1}=\dfrac{3x-8}{3x-9}-\dfrac{3x-7}{3x-8}.$

CHAPTER XXVI

HARDER PROBLEMS

165. We shall now consider some harder problems.

Example 1. *A certain number of men agreed to travel by private omnibus, each paying his own fare. If there had been 8 fewer, each would have paid 10d. more; and if there had been 2 fewer, each would have paid 2d. more. Find the number of men and what each had to pay.*

Let x be the number of men and y shillings the amount each had to pay. Then the total number of shillings paid is xy. But this is also equal to $(x-8)\left(y+\frac{5}{6}\right)$ and to $(x-2)\left(y+\frac{1}{6}\right)$,

$$\therefore\ xy=(x-8)\left(y+\frac{5}{6}\right),\quad \therefore\ 8y-\frac{5x}{6}=-\frac{20}{3},\ \text{.....................(i)}$$

$$\text{and}\quad xy=(x-2)\left(y+\frac{1}{6}\right),\quad \therefore\ 2y-\frac{x}{6}=-\frac{1}{3}.\ \text{.....................(ii)}$$

Multiply (ii) by 4, $\quad \therefore\ 8y-\dfrac{4x}{6}=-\dfrac{4}{3}.$ ....(iii)

Subtract (i) from (iii), $\quad \therefore\ \dfrac{x}{6}=\dfrac{16}{3},\quad \therefore\ x=32.$

Substitute $x=32$ in (ii), $\quad \therefore\ 2y-\dfrac{16}{3}=-\dfrac{1}{3},$

$$\therefore\ 2y=5,\quad \therefore\ y=2\tfrac{1}{2}.$$

Thus, there were 32 persons and each paid 2s. 6d.

Check. The total cost for 32 persons at 2s. 6d. each is £4.
" " " 24 " " 3s. 4d. " £4.
" " " 30 " " 2s. 8d. " £4.

Thus the conditions are satisfied and the solution is correct.

Example 2. *The perimeter of a rectangular playground was* 324 *metres. The length was increased by* 6 *metres and the breadth by* 3 *metres. The area was thereby increased by one-ninth of its former value. Find the original dimensions of the playground.*

Suppose that x metres was the length of the original *shorter* side; then $(162-x)$ metres was the length of the original longer side. The new sides are $(x+3)$ metres and $(168-x)$ metres;

$$\therefore\ (x+3)(168-x)=\frac{10x(162-x)}{9},$$

$$\therefore\ 9(504+165x-x^2)=1620x-10x^2,$$

$$\therefore\ x^2-135x+4536=0,$$

$$\therefore\ (x-72)(x-63)=0, \quad \therefore\ x=72 \text{ or } 63.$$

Thus the dimensions may be

either 90 metres by 72 metres *or* 99 metres by 63 metres.

The check is left to the pupil.

Note. Both roots of the quadratic lead to valid solutions of the problem. This is not always the case, and the pupil must decide in each instance whether each root of the equation corresponds to a valid solution of the problem.

Example 3. *A train runs* 120 *miles at a uniform rate; if the rate had been* 5 *miles an hour more, it would have taken* 20 *minutes less for the journey. Find the rate of the train.*

Let the rate of the train be x m.p.h.; then the time taken is $\frac{120}{x}$ hours. If the rate is increased by 5 m.p.h., the time taken is $\frac{120}{x+5}$ hours;

$$\therefore\ \frac{120}{x}-\frac{120}{x+5}=\frac{1}{3}, \text{ ....(i)}$$

whence

$$360\times 5=x(x+5),$$

$$\therefore\ x^2+5x-1800=0, \quad \therefore\ (x-40)(x+45)=0,$$

$\therefore\ x=40$ or -45. Hence the train travels at 40 m.p.h. The negative root does not correspond to a valid solution of the problem. The check is left to the pupil.

Note 1. In most problems, though not in all, the quadratic equation may be solved by factors. The pupil should always try to factorise, unless there is reason to suppose that the roots may be

irrational (see Note 3, below). In all other cases, if the equation cannot be solved by factors, he should look through his work to make sure that there is no error. Only after this should he proceed to solve by completing the square.

Note 2. The most common source of error is a mistake in sign in (i). The pupil should always stop to consider whether what he has written at this stage is sensible. Thus, in this instance, since x is necessarily positive, $\frac{120}{x}$ must be greater than $\frac{120}{x+5}$. It would therefore be a gross blunder to write $\frac{120}{x+5}-\frac{120}{x}=\frac{1}{3}$. Neglect of this precaution often leads to serious loss of time, if not to failure to solve the problem.

Note 3. In deciding whether to solve by factors or by completing the square, the nature of the answer expected is often a guide. Thus, if we are asked to find how many eggs are sold for a shilling, we expect the answer to be a rational number and therefore that the equation may be solved by factors. On the other hand, if we are asked to find the length of the side of a square, correct to one decimal place, the presumption is that the answer will be the approximate value of an irrational number, and that the equation must be solved by completing the square.

EXERCISE 83. a

(*Unless otherwise stated all speeds are to be taken as uniform*)

1. A person buys 60 yd. of cloth and 75 yd. of canvas for £5 12s. 6d. By selling the cloth at a gain of 15 per cent. and the canvas at a gain of 20 per cent., he gains 18s. 9d. Find the price of each per yd.

2. The perimeter of a rectangle is 68 ft. If the length were increased by 4 ft. and the breadth diminished by 4 ft., the area would be diminished by 40 sq. ft. Find the length and breadth.

3. In an action between two battleships A and B, A fired 3 times as many shells as B. The total number of misses was 7 times the total number of hits. The number of B's misses was 357, but B's hits exceeded A's hits by 66. What was the number of A's hits?

4. If the breadth of a certain rectangle were increased by 10 yd., and its length diminished by 20 yd., its area would be increased by 800 sq. yd.; whilst if its breadth were diminished by 10 yd. and its length increased by 30 yd., its area would be decreased by 300 sq. yd. What are its dimensions?

5. A sum of money is to be divided equally among a certain number of boys. If there were 5 fewer, each would get 4s.; if there were 4 more, each would get 2s. 6d. What is the sum of money?

6. At 1.15 p.m. a train leaves X and arrives at Y at 5.15 p.m. Another train leaves Y at 2.35 p.m. and arrives at X at 5.15 p.m. Find when they meet.

[HINT. Let n miles be the distance between X and Y, and let the trains meet x hours after 1.15 p.m.]

7. At 10.10 a.m. A sets out from P and reaches Q at 3.30 p.m. At 9.48 a.m. B leaves Q for P and arrives there at 2.36 p.m. Find when they meet.

8. In walking from A to B at the rate of 4 m.p.h., X meets Y when he has gone three-quarters of the way. He rides back with Y in his car for 2 miles at the rate of 24 m.p.h. Resuming his walk, he increases his speed and reaches B 4 hr. 17 min. after he left A. If he had walked straight through at the faster rate, his time would have been 3 hr. 12 min. Find his faster rate of walking and the distance from A to B.

9. The perimeter of a rectangular playground was 216 yd. The length was increased by 4 yd. and the breadth by 2 yd.; the area was thereby increased by one-ninth of its former value. Find the original dimensions of the playground.

10. Find the price of eggs, if giving two less for a shilling increases the cost of 100 by 1s. 8d.

11. The difference in the average speeds of two trains is 25 m.p.h. The faster train takes 2 hr. less to travel 150 miles than the slower train takes to travel 125 miles. Find the speed of the two trains.

12. Two turnstiles A and B admit to a football ground. On an average it takes a spectator 0·2 sec. longer to pass through A than through B, and B admits on an average 10 more spectators per minute than A. How many spectators can enter the ground in a quarter of an hour?

13. A man, having to walk 8 miles, increases his speed by $\frac{1}{4}$ mile per hr. immediately after the first 2 miles, thus reducing by 6 min. his time for the journey. How long does he actually take?

14. The circumference of each front wheel of a traction engine is 5 ft. less than that of each back wheel. In travelling 250 yd. a back wheel makes $12\frac{1}{2}$ fewer revolutions than a front wheel. Find the circumference of each back wheel.

15. How long will it take each of two pipes to fill a cistern, if one of them alone takes 9 minutes longer to fill it than the other, and 25 minutes longer than the two together?

16. A man set apart £24 for a certain length of holiday, but, wishing to extend it without extra expense, found that he could manage an extra 6 days by reducing his expenditure by 4s. per day throughout. What length of holiday did he plan at first?

17. A man converted some English money into francs; if he had done so 2 days later, he would have obtained 4 francs less for each £1, because the value of a franc had risen $\frac{1}{8}$d. How many francs, correct to 1 decimal place, did he actually obtain for each £1?

18. A motorist completes a journey of 200 miles in two stages, with a rest of $\frac{1}{2}$ hr. before the second stage. His average speed during the first stage of 150 miles is 36 m.p.h., while during the second stage the average speed is 5 m.p.h. less than the average speed for the whole journey. Find the average speed for the whole journey.

19. Find a number consisting of 2 digits such that the sum of twice this number and 3 times the number formed by interchanging the digits is 183, and such that the number exceeds by 5 twice the product of its digits.

20. *ABCD* is a rectangular field. Three men start from *A* and run at the same speed, the first man running along the diagonal *AC*, the second along *AB* and *BC*, and the third along *AD* and *DC*. When the first man reaches *C* the second is on *BC* and 6 yd. from *B*, while the third is on *DC* and 36 yd. from *C*. Find the lengths of the sides of the rectangle.

21. An aeroplane flying between 2 towns takes 20 min. more than its usual time when its normal speed is reduced by 30 m.p.h., and 10 min. less than its usual time when its normal speed is increased by 20 m.p.h. Find the normal speed and the distance between the towns.

22. A garden path, bounded by two circles, is gravelled at 2s. per sq. yd., the cost being £6 12s. Along both edges of the path is a stone edging at 3s. per yd., the cost of this being £19 16s. Find the width and inner radius of the path. [Take $\pi = 3\frac{1}{7}$.]

23. *A* and *B* run a 100 yd. race, *B* having 5 yd. start. During the earlier part of the race their speeds are as 21 to 20, and during the remainder as 19 to 18. How far has *B* run at the instant the change in speed takes place, if the race ends in a dead heat?

24. A convoy a mile in length is travelling at the rate of 3 m.p.h. A cyclist carries a message from the rear to the front and *at once* returns to the rear, riding at the same speed throughout, and taking 10 min. 40 sec. over the double journey. Find his speed in m.p.h.

25. A man rows to a place 12 miles down a river and back in 5 hours. If the speed of the current had been 3 times as great, he would have taken $7\frac{1}{2}$ hours. Find the rate at which the man rows.

EXERCISE 83. b

(*Unless otherwise stated all speeds are to be taken as uniform*)

1. A grocer buys 22 lb. of figs and 35 lb. of currants for £1 10s. By selling the figs at a loss of 20 per cent. and the currants at a gain of 50 per cent., he gains 2s. 2d. How much per lb. does he pay for each?

2. A train 52 yd. long passed in 8 sec. another train 91 yd. long which was travelling in the same direction. If the slower train had been travelling one-third as fast again, the faster train would have passed it in 12 sec. Find the speeds of the two trains in m.p.h.

3. A certain subscription is raised in a girls' school; had each girl given 3d. less, the money would have been obtained from 100 more, and if each girl had given 4d. more, from 40 fewer subscribers. How many subscribers were there and what did each give?

4. A train travelled a certain distance; had the speed been 2 miles an hour less, the journey would have taken 2 hours more; had the speed been 8 miles an hour more, the journey would have taken 6 hours less. Find the distance and speed.

5. If the floor of a room were 18 ft. longer and 12 ft. narrower, it would take 16 sq. yd. less carpet; but if it were 12 ft. shorter and 12 ft. wider, it would not change its area. Find its dimensions.

6. X starts to cycle from P at 10 a.m. and reaches Q at 2 p.m. Y leaves Q at 10.45 a.m. and reaches P at 1.15 p.m. Find when they meet. [See hint to Ex. 83 *a*, No. 6.]

7. A train leaves A at 1.11 p.m. and reaches B at 4.56 p.m. Another train leaves B at 2.6 p.m. and reaches A at 5.26 p.m. Find when they meet.

8. In walking from X to Y at the rate of $3\frac{1}{2}$ m.p.h., A meets B at half-way and rides back with him in his car for one-seventh of the total distance at 30 m.p.h. Resuming his walk he increases his speed and arrives at Y 4 hr. 19 min. after he left X. If he had walked straight through at the faster speed, he would have taken $3\frac{1}{2}$ hr. over the walk. Find his faster rate of walking and the distance from X to Y.

9. The denominator of a fraction exceeds the numerator by 3, both being positive. A new fraction is formed by adding 10 to the numerator and 9 to the denominator. The product of the two fractions is $\frac{1}{2}$; find the original fraction.

10. A swimming bath is rectangular at the surface and the water is 5 ft. deep at the shallow end, but the bottom slopes uniformly, so that it is x ft. deeper at the other end. After half the water it contains has been drained off, the depth at the deep end is x times the depth at the other. Find the two possible values of x.

11. When the price of a certain kind of tea was reduced by 3d. a lb. it was found that at the lower price 3 lb. more could be bought for £4 7s. 10d. than at the higher price. Find the original price per lb. of the tea.

12. A merchant spends £100 in buying coal of one grade and £51 in buying a poorer grade at 7s. 6d. per ton cheaper. In all, he buys 64 tons. Find the cost per ton of each grade of coal.

13. A man who can swim 48 yd. per min. in still water swims 200 yd. against the current and 200 yd. with the current. If the difference between the two times is 10 min., find the speed of the current in yd. per min.

14. A train is scheduled to run 350 miles in a certain time. At the end of the first 150 miles it is 10 min. behind time. For the remainder of the journey the speed is increased by 2 m.p.h. beyond that at which the train was scheduled to run, and the whole distance is covered in the appointed time. Find the speed at which the train was scheduled to run.

15. Working alone A could do a piece of work in 20 hours less time than B alone would take for the same work; working together A and B could do it in $18\frac{3}{4}$ hours, if each worked at the same rate as when working separately. How long would each take to do it separately?

16. A certain number of persons promised to subscribe equally to a gift of £30. Five of them failed to keep their promise, and each of the others increased his subscription by 10s. The exact amount required was obtained. How many persons subscribed?

17. An expert estimates that the receipts at a show, with a certain entrance fee, will be £500, but that if the entrance fee is reduced by 6d., 3000 more people will attend and the receipts will be £100 more. What is the number of people expected at the lower fee, and what is that fee?

18. A motorist starts to ride 108 miles in a given time. After riding five-sixths of the distance he is detained for $2\frac{1}{2}$ min., but by increasing his speed thereafter by 12 m.p.h. he finishes the journey in 5 min. less than the given time. Find his speeds.

19. The area of a rectangular plot of land is 12,000 sq. yd., and its diagonal is 170 yd.; find the length and breadth of the plot.

20. Two numbers, each consisting of 2 digits, are such that each is obtained from the other by reversing its digits. If the sum and product of the two numbers are 121 and 3154 respectively, find the numbers.

21. The difference between the squares of two numbers is 21. If one number is reduced by 2, its ratio to the other number is then 3 : 2. Find the numbers.

22. In a trapezium one of the parallel sides is three-quarters of the other, and the other two sides are equal. The height is 4 cm., and the perimeter is 52 cm. Find the lengths of the sides.

23. In a 100 yd. race when A gives B $\frac{1}{2}$ yd. start he wins by $\frac{1}{5}$ sec., but when he gives B $4\frac{1}{2}$ yd. start B wins by $\frac{1}{5}$ sec. What are their times respectively for running 100 yd.?

24. A walks along a road from P to Q, starting at 10.30 a.m. B cycles from Q to P and straight back again, starting at 11.42 a.m. and passing A at 11.50 a.m. They arrive at Q at the same time. What is that time?

25. A motorist and a cyclist start together from a place A to travel by the same road to a place B 18 miles away. The motorist starts at a uniform speed 15 m.p.h. greater than that of the cyclist. When the motorist has got half-way he is delayed for 30 min., and thereafter travels at a speed 20 per cent. less than his original speed; he reaches B 15 min. before the cyclist. Find the cyclist's speed.

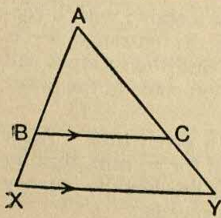
TEST PAPERS VII

A

1. Factorise : (i) $a^4 - x^2 + 9b^2 - 6a^2b$,
(ii) $14x^2 - 65x + 56$,
(iii) $8x^3 + 22x^2 + 7x - 10$.

2. Simplify

$$\frac{x-2a}{2x^2+7ax+6a^2} - \frac{2x+3a}{4(x^2+3ax+2a^2)} + \frac{4x+3a}{2(2x^2+5ax+3a^2)}.$$



3. If $AB = x$ in., $BX = y$ in., $AC = t$ in., $CY = v$ in., $BC = u$ in. and $XY = z$ in., and if BC, XY are parallel, find u and v in terms of x, y, z, t.

4. Solve (i) $\dfrac{(2-x)(3-x)}{(1-x)(5-x)} = 1$,

(ii) $\dfrac{x^2+1}{x} = \dfrac{4t^2+1}{2t}$.

5. Two men started at the same time to meet each other from points which were 22 miles apart. If one took 3 minutes longer than the other to walk a mile, and they met 3 hours after starting, find the speed of each in m.p.h.

6. Find the square root of $(x+1)(x+2)(x+3)(x+4)+1$, and solve the equation $(x+1)(x+2)(x+3)(x+4)+1=0$, correct to two places of decimals.

B

1. Simplify $\dfrac{2a^2-ab-3b^2}{a^2-b^2}$, and use the result to calculate the value of $\dfrac{2\times 875^2-875\times 250-3\times 250^2}{875^2-250^2}$.

2. Factorise : (i) $162x^2-9x-10$,
(ii) $[x^2+3x+3]^2-[2x+3]^2$,
(iii) $3+6cx-4c-3x-2cx^2$.

3. A and B can do a piece of work together in a days. After working together for b days, A falls ill and B is left to finish the work, which he does in c more days. How long would it have taken (i) A, (ii) B separately to have done the work?

4. Solve : (i) $\dfrac{2c-d}{x-2c+d}=\dfrac{2c}{x-2c}-\dfrac{d}{x+d}$,
(ii) $9x^2-16y^2=65, \quad 3x-4y=13$.

5. Find the H.C.F. of $2x^3-x^2-15x+18$ and $4x^3-20x^2+27x-9$. Complete the factorisation of the expressions and solve the equation $2x^3-x^2-15x+18=0$.

6. Two towns X and Y are 30 miles apart. A man takes 8 hr. 45 min. to travel from X to Y and back again to X, his average rate on the return journey being 2 m.p.h. slower than on the outward journey. At what average rate did he travel on the outward journey?

C

1. Factorise : (i) $(3x-7y)^3-8y^3$,
(ii) $x^4-4z^4+9y^4-6x^2y^2$,
(iii) $3y^2-y+10x-35xy+50x^2$.

2. Find the square root of
$$x^6+6x^5+x^4-20x^3+28x^2-16x+4.$$

3. Simplify $\dfrac{9x^2+a^2}{a^2-3ax}+\dfrac{9x^2+2a^2}{3ax+a^2}+\dfrac{9a^2}{9x^2-a^2}$.

4. The area of a rectangular floor was 396 sq. ft. If it had been 2 ft. shorter and also 2 ft. wider, the area would have been 11 sq. ft. more. Find its length and width.

5. Solve : $\dfrac{2}{2x-3}-\dfrac{7}{2x+3}=\dfrac{13}{2x+9}.$

6. Solve : $3x+2y=1, \quad 7x^2-10xy+15y=2.$

D

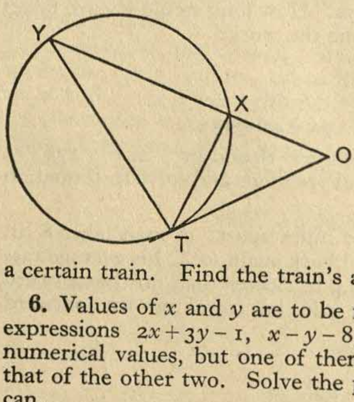
1. Factorise : (i) $6x^2+77xy+245y^2,$
(ii) $(x^2+x+10)^2-(7x+2)^2,$
(iii) $x^4+3x^3+x^2+4x+3.$

2. (*a*) Find the value of c, if the L.C.M. of $x^2+3x-10$, $x^2-2x-35$, and $x^3-5x^2-29x+c$ is $(x-2)(x+5)(x-3)(x-7)$.

(*b*) Simplify $\left(l+\dfrac{1}{l}\right)^2+\left(m-\dfrac{1}{m}\right)^2-\left(\dfrac{l}{m}+\dfrac{m}{l}\right)\left(lm+\dfrac{1}{lm}\right).$

3. (*a*) If $b=x+\dfrac{y}{a}$, $c=x+\dfrac{y}{b}$, find a in terms of x, y, c.

(*b*) If $t^2+(4a-3)t+4a^2-6a+2=0$, find t in terms of a.



4. OT is a tangent to the circle XYT. $OX=q$ in., $XY=p$ in., $OT=r$ in., $XT=v$ in., $YT=t$ in.

Find a relation between p, q, r; and one between q, r, t, v.

5. In a journey of 240 miles to increase the average speed by 15 m.p.h. would shorten by 80 minutes the time taken by a certain train. Find the train's average speed.

6. Values of x and y are to be found which will make the three expressions $2x+3y-1$, $x-y-8$, $x-3y-2$ all have the same numerical values, but one of them is to have its sign opposite to that of the other two. Solve the problem in as many ways as you can.

E

1. What sum of money amounts to £K in T years at C per cent. per annum Compound Interest?

2. Factorise : (i) $24x^2-10x-25,$
(ii) $5b+2x+4bx^2-1-12bx,$
(iii) $1+2a+2bc+a^2-b^2-c^2.$

3. Find the H.C.F. of

$$6x^4-x^3-47x^2+30x \quad \text{and} \quad 36x^4-27x^3-16x^2+12x.$$

4. Solve : (i) $\dfrac{3x-4}{2(x+3)}=1+\dfrac{x}{2(x-1)}$,

(ii) $3x+5y+3=0,\quad 9x^2-y^2=45$.

5. If $\frac{1}{u}+\frac{1}{v}=\frac{2}{r}$, express (1) r in terms of u and v, (2) $r-u$ and $r-v$ in terms of u and v. Show that $\dfrac{1}{r-u}+\dfrac{1}{r-v}=\dfrac{1}{u}+\dfrac{1}{v}$.

6. Two motor-cars make the journey between two towns which are 112 miles apart, both starting from the same town. One car travels 10 m.p.h. faster than the other, but leaves the starting point 3 hr. 20 min. later. If both cars arrive at the second town at the same time, find the pace of each car.

F

1. Factorise : (i) $16x^3+2(2x-3y)^3$,

(ii) $54x^2+8y-15xy-6y^2+36x$,

(iii) $2x^2+3xy-54y^2-21y-2$.

2. Find the square root of

$$\frac{x^6-6x^5+21x^4-44x^3+60x^2-48x+16}{4x^4}.$$

3. Solve : (i) $ab(25-x^2)=5x(a^2-b^2)$,

(ii) $4x^2+7xy+12y^2=14,\quad x+y=1$.

4. Simplify $\dfrac{5}{(5x-3)(5x+2)}-\dfrac{4}{25x^2-4}-\dfrac{1}{(5x-3)(5x-2)^2}$.

5. A cyclist sets out to ride from A to B, a distance of 30 miles. After riding 25 miles, he has a puncture and walks the remainder of the distance, and finds that the journey takes 4 hr. 10 min. Had he ridden 2 m.p.h. faster, and the puncture occurred 1 mile earlier, the journey would have taken 4 hours. What is his rate of riding?

6. Draw the graph of $(x+1)(x+2)$ for values of x from -3 to $1·5$. Between what values of x within this range does the expression $(x+1)(x+2)$ decrease in value as x increases? Use your graph to solve $x^2+3x+0·4=0$. Draw (and label) the line whose points of intersection with the curve would give solutions of the equation $x^2+3x+1·5=0$.

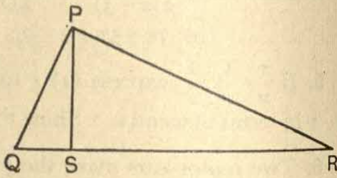
G

1. Factorise : (i) $40(4l-3m)^3+5(l+3m)^3$,

(ii) $4x^4+5x^3-42x^2+11x+12$.

2. Simplify : $\dfrac{2x^3-4x^2+x-2}{x^2-x-2}$.

3. $\angle QPR = 90° = \angle PSQ$. $PQ = a$ in., $PS = x$ in., $QS = b$ in., $SR = c$ in. Find a in terms of (i) b, x, (ii) b, c.



4. A man bought a number of sheep for £240. He lost 5, but was able to make up the deficiency by selling the rest at 4 shillings profit per head. How many did he buy?

5. Solve : $3x + 2y = 2xy$, $9x + 4y = 5xy$.

6. (*a*) If $3x - y = 4$, prove that $27x^3 - 36xy - y^3 = 64$.

(*b*) If $2s = a + b + c$, find in its simplest form, in terms of a, b, c only, the value of $s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2$.

H

1. (*a*) If $x = \dfrac{2a}{1-a}$ and $y = \dfrac{1-a}{1+a}$, express $\dfrac{x+y}{1-xy}$ in terms of a in its simplest form.

(*b*) If $f(a, b, c) = a^2 + b^2 + c^2 - ab - bc - ca$, evaluate

$$f(a+x, b+x, c+x) - f(a, b, c).$$

2. Factorise : (i) $14a^4 - 33a^2 - 56$,

(ii) $8ax^2 - x + 10a + 1 - 18ax$,

(iii) $a^4(b+c) - b^4(c+a) + c^4(a-b)$.

3. Find the H.C.F. of

$$2x^4 - 3x^3 + 9x^2 - 7x + 15 \quad \text{and} \quad 4x^4 - 7x^3 + 13x^2 - 3x + 9.$$

4. Solve $\dfrac{2x-11}{x-2} + \dfrac{x+4}{x-3} = \dfrac{x-5}{x+2} + \dfrac{2x+9}{x+1}$.

5. A farmer bought a number of horses for £720. Four of them died, and he sold the rest at £14 per head more than he gave for them. On the whole transaction he made a profit of £80. Find the price he originally paid for each horse.

6. The floor levels of a building are 15 ft. apart, and the lift moves at an average rate of 3 ft. per sec. upwards and 5 ft. per sec. downwards. A man ascends from the ground floor by the staircase, rising vertically at a uniform rate of 1 ft. per sec. At the same moment the lift leaves the ground floor, stops for 5 sec. at the second floor, and returns to the ground. After 9 sec. it ascends to the first floor, remaining there 5 sec., and finally ascends to the fourth floor. Find graphically when the man and the lift are simultaneously at the same height from the ground.

I

1. (*a*) A man invests £X in a c per cent. stock at A. What is his income?

(*b*) Find the values of a and b, if the expression $2x^3 - 15x^2 + ax + b$ is divisible by both $x - 4$ and $2x - 1$.

2. Factorise : (i) $(7a - 3b)^3 - (5a + 2b)^3$,

(ii) $12y + 14x^2 + 17x - 42xy - 6$,

(iii) $4c^2 + 12xy - 9x^2 + d^2 + 4cd - 4y^2$.

3. Solve (i) $\dfrac{x+6}{x} - \dfrac{6-x}{2(x+2)} = \dfrac{7}{3}$,

(ii) $4x^2 + 3xy - 3y^2 = 4, \quad 5x + 3y = 10$.

4. A man's total income is £A. On £220 of this he pays no income-tax, on the next £135 he pays tax at 1/6 in the £, and on the remainder he pays tax at 4/6 in the £. What is his total tax and what percentage is this of his total income (1) if $A > 355$, (2) if $355 > A > 220$, (3) if $A < 220$.

5. A man walks a certain distance. Had he walked half the distance further at the same rate, he would have taken 40 min. longer, and had he walked at the rate of $\frac{1}{2}$ m.p.h. slower, he would have taken 16 min. longer to cover the original distance. Find the distance and the time of walking.

6. Solve $x + y = a,\ y + z = b,\ x^2 - z^2 = c^2$.

J

1. (*a*) If $x = \dfrac{4}{2-y}$ and $y = \dfrac{4}{2-z}$, prove that $z = \dfrac{4}{2-x}$.

(*b*) Determine a and b so that $12x^4 - 31x^3 + 14x^2 + ax + b$ may be divisible by $(x-1)^2$ without remainder ; and factorise the quotient.

2. If a suitable value be given to λ, the square root of the expression $9x^6 - 6x^5 + 25x^4 - 50x^3 + 30x^2 + \lambda x + 49$ works out exactly. Work the square root and find the value of λ.

3. Factorise : (i) $600x^2 - 130xy - 63y^2$,

(ii) $(12x^2 - 7x + 45)^2 - (53x - 3)^2$,

(iii) $4x + 24x^2y - 22xy - 7 - 35y$.

4. (*a*) If $\dfrac{1+x}{1+zt} = \dfrac{1+y}{1+wt}$, find t in terms of the other letters.

(*b*) Solve $\dfrac{1}{x-3} + \dfrac{1}{x-6} = \dfrac{1}{x-4} + \dfrac{1}{x-5}$.

5. A bookseller bought a certain number of books of equal value for £13 10s. 0d. He sold all but 20 at 9d. each more than he paid for them, and the remaining 20, which were shop soiled, he sold for half as much each as he obtained for the others. His profit was £3. How many books did he buy?

6. Show that the values of x and y which satisfy the two equations $(b+c)x+by=ax+(a+c)y=-c$, also satisfy the equation

$$a(x+1)=b(y+1).$$

K

1. Factorise : (i) $16-4cd-c^2-4d^2$,
(ii) $48x^2-56xy+16y^2$,
(iii) $4-6y+28x-27xy+45x^2$.

2. Simplify $\dfrac{5x-4}{25x^2-4}+\dfrac{2}{5x}-\dfrac{15x-6}{25x^2+5x-6}$.

3. (*a*) Expand $\left(\dfrac{1}{2}-\dfrac{t}{3}+\dfrac{t^2}{4}\right)^3$.

(*b*) Divide

$a^3-b^3+c^3+3abc+3c^2+3ab+3c+1$ by $a-b+c+1$.

4. Solve (i) $(k^2+k)x^2-(k^3+k^2-1)x+(k^2-1)=0$,
(ii) $4x^2-10xy+25y^2=21$, $4x-5y=3$.

5 (*a*) If $F(x, y)=x^3+y^3$, find $F(2a+1, a-3)$.

(*b*) Determine a and b in order that $ax^4+bx^3-28x^2-32x+21$ may be exactly divisible by $2x^2+5x-3$.

6. A man rented a certain number of acres for £60. He worked 4 acres himself, and by letting the rest for 10s. per acre more than he paid for it, he received for this portion an amount equal to the rent of the whole. How many acres did he rent altogether?

L

1. Find the remainder (of the first degree in x) obtained on dividing $6x^5-19x^4-x^3+49x^2-42x+5$ by $2x^2-5x+1$.

2. Factorise : (i) $50(x-3)^2-95(x-3)(2x+1)+42(2x+1)^2$,
(ii) $4x^4+12x^3+3x^2-3x-1$,
(iii) $12x^2-xy-6y^2-31x+2y+20$.

3. Find the H.C.F. of

$a^3-3a^2b-7ab^2+12b^3$ and $2a^4-a^3b-5a^2b^2+13ab^3-12b^4$.

4. The sum of two numbers is nine times their difference, and their product is less by 4 than nine times their sum. Find the numbers.

5. Solve (i) $\dfrac{a^2}{(x-b)(x+c)}+3=\dfrac{x+a-b}{x-b}+\dfrac{x+a+c}{x+c}$,

(ii) $\dfrac{6}{5x+2y}+\dfrac{8}{x-2y}=\dfrac{1}{2}$, $3x+4y=2$

6. Draw in the same diagram and with the same scales the graphs of $\frac{1}{5}x^3$ and $\frac{1}{5}(25-x^2)$ for values of x from 0 to 4. Find a value of x which satisfies $x^3+x^2=25$.

PART III

CHAPTER XXVII

INDICES, SURDS, IRRATIONAL EQUATIONS

166. Indices. In Chapter XI, the following laws of indices were proved :

(1) $a^m \times a^n \times a^p \times a^r \times \ldots = a^{m+n+p+r+\ldots}$,

(2) $a^m \div a^n = a^{m-n}$, provided that $m > n$ and $a \neq 0$,

or, $a^m \div a^n = \frac{1}{a^{n-m}}$, provided that $n > m$ and $a \neq 0$,

(3) $(a^m)^n = a^{mn}$.

These laws have only been proved, if all the letters $m, n, p, r \ldots$ represent positive integers, for the proofs were based on the definition that a^m stands for the product of m factors each equal to a, and this definition is unintelligible, unless m is a positive integer. Up to the present no meaning has been assigned to such expressions as $a^{-\frac{3}{4}}$, a^{-2}, a^0 ; if we wish to use such symbols, we must first define them.

Now it would clearly be very inconvenient, if we had one set of rules for positive integral indices and a different set of rules for fractional or negative indices. We therefore determine meanings for symbols such as

$$a^{\frac{3}{4}}, \quad a^{-2}, \quad a^0, \quad a^{\frac{p}{q}}, \quad a^{-n}$$

in the following way :

We assume that they obey the fundamental law of indices,

$$a^m \times a^n \times a^p \times a^r \times \ldots = a^{m+n+p+r+\ldots},$$

and accept the meaning derived from this assumption. It will be found that the symbols so interpreted will also obey the other laws given above.

167. To find a meaning for $a^{\frac{p}{q}}$, where p, q are positive integers. Consider first a simple case, say $a^{\frac{2}{3}}$.

Since $a^{\frac{2}{3}}$ obeys the fundamental law of indices, we have

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2.$$

Since the result of cubing $a^{\frac{2}{3}}$ is a^2, it follows that $a^{\frac{2}{3}} = \sqrt[3]{a^2}$. More generally,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \text{to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\dots\text{ to } q \text{ terms}}$$

$$= a^p, \text{ i.e. } a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

In other words, $a^{\frac{p}{q}}$ is equal to the qth root of a^p.

Note. The pupil is already familiar with the fact that the square root of 4 may be $+2$ or -2. This is a particular case of a more general result, viz. that, if n is a positive integer, every quantity has n nth roots.

When n is even, two of these are real, one being positive and the other negative (e.g. two values of $\sqrt[4]{81}$ are $+3$ and -3), and the others are imaginary (e.g. two values of $\sqrt[4]{81}$ are $\sqrt{-9}$ and $-\sqrt{-9}$).

When n is odd there is only one real root and the others are imaginary. The real positive root of a real positive quantity is called the **principal root.**

In using the fractional index notation we consider the principal root only. Thus, $81^{\frac{1}{4}} = +3$, while -3, the other real root, is written $-81^{\frac{1}{4}}$.

When a is a negative quantity, e.g. -27, it cannot have an even real root, since an even number of positive or negative quantities multiplied together gives a positive quantity. But a negative quantity has one real negative odd root; this will be the principal root. Thus, the principal cube root of -27 is -3.

168. To find a meaning for a^0. By the fundamental law of indices, we have $$a^m \times a^0 = a^{m+0} = a^m.$$

Provided that $a \neq 0$, we may divide each side by a^m, obtaining

$$a^0 = a^m \div a^m = 1.$$

It should be particularly noted that we have assigned no meaning to 0^0.

169. To find a meaning for a^{-n}, where n is a positive number.
Consider first a simple case, say a^{-3}.

Since a^{-3} obeys the fundamental law of indices, we have

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1, \text{ provided that } a \neq 0.$$

Dividing each side by a^3, we have

$$a^{-3} = \frac{1}{a^3}, \text{ provided that } a \neq 0.$$

More generally,

$$a^{-n} \times a^n = a^{-n+n} = a^0 = 1, \text{ provided that } a \neq 0.$$

Dividing each side by a^n, we have

$$a^{-n} = \frac{1}{a^n}, \text{ provided that } a \neq 0.$$

It follows that any factor may be transferred from the numerator to the denominator of an expression, or vice versa, provided that the sign of the index is changed.

Thus, $$\frac{x^3y^{-2}}{z^{-4}} = \frac{x^3z^4}{y^2}.$$

It should be particularly noted that we have assigned no meaning to a^{-n}, when $a = 0$.

170. We have now obtained meanings for a^m, when m has any rational value, with the proviso that if m is negative or zero, a must not be zero. It should be noted that we have not defined a^m when m is irrational or unreal, e.g. for $m = \sqrt{7}$ or $m = \sqrt{-4}$.

In the work which follows it is assumed that all indices are rational and real, and that the following propositions are true.

(1) **$a^m \div a^n = a^{m-n} (a \neq 0)$ for all values of m and n.**

(2) **$(a^m)^n = a^{mn}$ for all values of m and n.**

[a must not be zero, if either m or n is negative or zero.]

(3) **$(ab)^n = a^nb^n$ for all values of n.**

[ab must not be zero, if n is negative or zero.]

Note 1. In general, a^mb^n can be simplified only if **either** $a = b$ **or** $m = n$.

[If a and b have a common factor, it may be possible to obtain a simpler expression for a^mb^n.]

Note 2. There is no simple form equivalent to $(a+b)^n$.

The methods to be used are illustrated by the following examples.

171. Example 1. *Simplify*

$$\text{(i) } a^{\frac{3}{4}} \times a^{\frac{4}{5}}, \quad \text{(ii) } c^{x-y} \div c^{x-z}, \quad \text{(iii) } (2d^{-2})^{-3}.$$

$$\text{(i) } a^{\frac{3}{4}} \times a^{\frac{4}{5}} = a^{\frac{3}{4}+\frac{4}{5}} = a^{\frac{31}{20}}.$$

$$\text{(ii) } c^{x-y} \div c^{x-z} = c^{(x-y)-(x-z)} = c^{z-y}.$$

$$\text{(iii) } (2d^{-2})^{-3} = 2^{-3}d^6 = \frac{1}{2^3} \cdot d^6 = \frac{1}{8}d^6.$$

Example 2. *Evaluate* $81^{-\frac{3}{4}}$.

$$81^{-\frac{3}{4}} = \frac{1}{81^{\frac{3}{4}}} = \frac{1}{(81^{\frac{1}{4}})^3} = \frac{1}{3^3} = \frac{1}{27}.$$

Note that we may regard $81^{\frac{3}{4}}$ either as $(81^{\frac{1}{4}})^3$ or as $(81^3)^{\frac{1}{4}}$; the result is the same in either case, but the work is considerably shortened by taking $(81^{\frac{1}{4}})^3$, etc., as above, instead of $\sqrt[4]{531441}$, etc.

Example 3. *Simplify* (i) $\dfrac{\sqrt{a^5} \times \sqrt[3]{b^2}}{\sqrt[6]{b^{-2}} \times \sqrt[4]{a^{10}}}$, (ii) $\dfrac{(2x)^{-2}}{(3y^2)^{-1}}$.

$$\text{(i) } \frac{\sqrt{a^5} \times \sqrt[3]{b^2}}{\sqrt[6]{b^{-2}} \times \sqrt[4]{a^{10}}} = \frac{a^{\frac{5}{2}} \times b^{\frac{2}{3}}}{b^{-\frac{1}{3}} \times a^{\frac{5}{2}}} = a^{\frac{5}{2}-\frac{5}{2}}b^{\frac{2}{3}+\frac{1}{3}} = a^0b^1 = b.$$

$$\text{(ii) } \frac{(2x)^{-2}}{(3y^2)^{-1}} = \frac{2^{-2}x^{-2}}{3^{-1}y^{-2}} = \frac{3^1y^2}{2^2x^2} = \frac{3y^2}{4x^2}.$$

EXERCISE 84

(*Nos.* 1–7 *may be taken orally*)

1. Read off the values of:

$a^{\frac{1}{4}} \times a^{\frac{1}{5}}$; $b^{\frac{2}{3}} \times b^{\frac{1}{6}}$; $c^{-\frac{2}{3}} \times c$; $d^{-\frac{3}{5}} \times d^{\frac{3}{5}}$; $h \times h^{-\frac{2}{7}}$;
$l^{-\frac{1}{2}} \times l^{-\frac{3}{2}}$; $m^{-4} \times m^7$; $n^{1\cdot5} \times n^{-0\cdot5}$; $x^{\frac{4}{3}} \times x^{-\frac{1}{2}}$; $y^{\frac{2}{5}} \times y$;
$a^{\frac{1}{2}} \div a^{\frac{1}{3}}$; $b^2 \div b^{-3}$; $c^{\frac{2}{5}} \div c^{-\frac{3}{5}}$; $d^{-7} \div d^{-10}$; $h^a \div h^b$;
$l^{-\frac{2}{3}} \div l^{-\frac{1}{2}}$; $m \div m^{-4}$; $n^{3x} \div n^{6x}$; $x^{-\frac{2}{3}} \div x^{\frac{4}{3}}$; $y^{-3} \div y^4$.

2. Express in words:

$$\sqrt[3]{a^5};\ \sqrt[a]{x^b};\ \sqrt{c^r};\ \sqrt[7]{y^3};\ \sqrt[6]{b^{18}};\ \sqrt[s]{z^{3s}}.$$

3. Express with a single index:

$(a^4)^2$; $(a^2)^4$; $(a^{\frac{1}{3}})^6$; $(b^{\frac{1}{2}})^2$; $(x^{\frac{1}{7}})^{\frac{5}{7}}$;
$(c^{-\frac{1}{3}})^4$; $(y^{-\frac{2}{5}})^{-10}$; $(d^{-\frac{2}{3}})^{\frac{3}{2}}$; $(l^{\frac{2}{3}})^{-6}$; $(x^{-a})^{\frac{3}{a}}$;
$(a^{-4})^2$; $(a^{-4})^{-2}$; $(a^{-\frac{1}{3}})^6$; $(b^{\frac{1}{2}})^{-2}$; $(x^{-\frac{1}{7}})^{-\frac{5}{7}}$;
$(c^{-\frac{1}{3}})^{-4}$; $(y^{\frac{3}{4}})^{-8}$; $(d^{-\frac{2}{3}})^{-\frac{3}{4}}$; $(l^{-\frac{3}{2}})^{\frac{1}{6}}$; $\left(x^{-\frac{1}{a}}\right)^{a^2}$.

4. Express with positive indices and without root signs :

$\sqrt[4]{a}$; $\sqrt[5]{b^2}$; $\sqrt[7]{c^{-3}}$; $\sqrt[3]{x^{-5}}$; $\sqrt[5]{l^{-8}}$; $\sqrt[10]{z^{-7}}$.

5. Express with root signs and with positive indices :

$a^{\frac{2}{5}}$; $b^{\frac{3}{7}}$; $c^{-\frac{2}{3}}$; $d^{\frac{4}{3}}$; $l^{-\frac{1}{3}}$; $x^{-\frac{3}{4}}$; $y^{\frac{7}{8}}$; $z^{-\frac{a}{3}}$.

6. Express with positive indices and without root signs :

$$a^{-4};\ x^{-5};\ \frac{a^3}{c^{-2}};\ \frac{1}{b^{-\frac{1}{3}}};\ \frac{a^{-2}}{a^5};\ \frac{a^{\frac{1}{3}}}{b^{-\frac{1}{2}}}.$$

7. Find the numerical values of :

$16^{\frac{3}{4}}$; $4^{2\cdot5}$; $25^{-\frac{3}{2}}$; $8^{-\frac{2}{3}}$; $(0\cdot04)^{\frac{3}{2}}$; $216^{-\frac{2}{3}}$;
$32^{\frac{3}{5}}$; $64^{-\frac{2}{3}}$; $(6\cdot25)^{1\cdot5}$; $4^{-1\cdot5}$; $(-8)^{-\frac{5}{3}}$; $343^{-\frac{2}{3}}$;
$(0\cdot25)^{\frac{1}{2}}$; 144^0; $(-512)^{-\frac{2}{3}}$; $32^{-\frac{7}{5}}$; $64^{\frac{5}{6}}$; 7^0;
$(-343)^{\frac{1}{3}}$; $(0\cdot01)^{-\frac{3}{2}}$; $27^{\frac{2}{3}}$; $(0\cdot008)^{\frac{1}{3}}$; $(-3\frac{3}{8})^{-\frac{4}{3}}$;
$(\frac{8}{27})^{-\frac{2}{3}}$; $2^{\frac{1}{2}}\times 2^{\frac{5}{2}}$; $9^{\frac{1}{4}}\times 9^{-\frac{1}{4}}$; $(5^3)^{\frac{2}{3}}$; $25^{\frac{3}{4}}\times 25^{-\frac{1}{4}}$;
$32^{-0\cdot4}$; $3^{\frac{2}{3}}\times 3^{\frac{7}{3}}$; $(4^{\frac{3}{4}})^{-\frac{4}{3}}$; $5^{\frac{1}{3}}\times 5^{-\frac{1}{3}}$; $4^{n-2}\times 4^{3-n}$; $1024^{-0\cdot3}$

Express with positive indices and without root signs :

8. $3x^{-\frac{1}{2}}$. **9.** $5c^{-\frac{3}{4}}$. **10.** $7\div z^{-3}$. **11.** $3x^{-4}a^5$.

12. $\frac{1}{3a^{-7}}$. **13.** $\frac{1}{4c^{-\frac{2}{3}}}$. **14.** $x^2\times 4x^{-\frac{1}{2}}$. **15.** $x^{-4}\times 8x^{\frac{3}{2}}$.

16. $x^{-4}\times(4x)^{\frac{3}{2}}$. **17.** $x^2\times(4x)^{-\frac{1}{2}}$. **18.** $\frac{3a^{-2}b^3}{7c^3d^{-5}}$.

19. $\frac{x^{-l}y^m}{z^{-n}}$. **20.** $2a^{\frac{1}{3}}\times 5a^{-2}$. **21.** $1\div 4x^{-\frac{1}{2}}$.

22. $1\div(4x)^{-\frac{1}{2}}$. **23.** $ab^{-2}\times b^2c^{-3}$. **24.** $b^3x^{-4}\div 3x^{-2}$

25. $b^3x^{-4}\div(3x)^{-2}$. **26.** $1\div\sqrt{4x^3}$. **27.** $3\div 4\sqrt{y^{-5}}$.

28. $\sqrt[5]{x^4}\div\sqrt{x^{-2}}$. **29.** $\sqrt[3]{8y^2}\div\sqrt[6]{y^{-5}}$. **30.** $\sqrt[7]{a^{-4}}\div\sqrt[7]{a^{10}}$.

31. $1\div(9y^2)^{-\frac{3}{2}}$. **32.** $a^{-1}b^2\times b^{-3}c^2$. **33.** $3a^{\frac{1}{2}}\times 4a^{-2}$.

34. $3a^{\frac{1}{2}}\times(4a)^{-2}$. **35.** $9x^{\frac{1}{2}}\div(27x^2)^{-\frac{1}{3}}$. **36.** $c^2y^{-6}\div 3y^{-2}$.

37. $4\div\sqrt[3]{8z^{-2}}$. **38.** $c^2y^{-6}\div(3y)^{-2}$. **39.** $1\div\sqrt{25c}$.

40. $\sqrt[5]{x^{-3}}\div\sqrt[5]{x^2}$. **41.** $\sqrt[4]{2a^5}\div\sqrt[4]{32a^{-3}}$. **42.** $(2ab)^{-2}\times b^2c^2$.

43. Express as a power of 8 : 4, 16, 1, $\frac{1}{2}$, $\sqrt{2}$, 128.

44. Find n, if

(i) $x^n\times x^n=x^3$, (ii) $x^n\times x^n=\frac{1}{x}$, (iii) $(x^n)^3=\sqrt{x^7}$, (iv) $\sqrt[5]{x^{-n}}=x\sqrt{x}$.

172. Example 4. *Divide* $24a^{-3}+13a^{-2}+11a^{-1}-6$ *by* $3a^{-1}-1$.

$$
\begin{array}{r|l}
3a^{-1}-1 & 24a^{-3}+13a^{-2}+11a^{-1}-6 \;(\,8a^{-2}+7a^{-1}+6 \\
 & \underline{24a^{-3}-\;\;8a^{-2}} \\
 & \qquad 21a^{-2}+11a^{-1} \\
 & \qquad \underline{21a^{-2}-\;\;7a^{-1}} \\
 & \qquad\qquad 18a^{-1}-6 \\
 & \qquad\qquad \underline{18a^{-1}-6}
\end{array}
$$

Detached coefficients may be used, if desired.

Example 5. *Simplify* $\dfrac{5^n\times(5^{n-1})^n}{5^{n+1}\times 5^{n-1}}\times\dfrac{1}{25^{-n}}$.

The expression equals $\dfrac{5^n\times 5^{n^2-n}}{5^{n+1}\times 5^{n-1}}\times\dfrac{1}{5^{-2n}}$, for $25^{-n}=(5^2)^{-n}=5^{-2n}$

$$=5^{n+n^2-n-n-1-n+1+2n}=5^{n^2}.$$

Example 6. *Simplify* $\left(\dfrac{4x^{-3}}{y^{-\frac{2}{3}}z}\right)^{-\frac{3}{2}}\div\left(\dfrac{\sqrt{9x^{-\frac{1}{2}}}\,.\,\sqrt[6]{y^3}}{2x^2z^{-1}}\right)^{-2}$.

$$\text{The expression}=\frac{4^{-\frac{3}{2}}x^{\frac{9}{2}}}{yz^{-\frac{3}{2}}}\div\left(\frac{3x^{-\frac{1}{4}}\,.\,y^{\frac{1}{2}}}{2x^2z^{-1}}\right)^{-2}$$

$$=\frac{x^{\frac{9}{2}}}{4^{\frac{3}{2}}yz^{-\frac{3}{2}}}\div\frac{3^{-2}x^{\frac{1}{2}}\,.\,y^{-1}}{2^{-2}x^{-4}z^2}$$

$$=\frac{x^{\frac{9}{2}}}{8yz^{-\frac{3}{2}}}\times\frac{2^{-2}x^{-4}z^2}{3^{-2}x^{\frac{1}{2}}y^{-1}}$$

$$=\frac{3^2}{8\,.\,2^2}x^{\frac{9}{2}-4-\frac{1}{2}}y^{1-1}z^{2+\frac{3}{2}}$$

$$=\frac{9}{32}x^0y^0z^{\frac{7}{2}}=\frac{9z^{\frac{7}{2}}}{32}.$$

EXERCISE 85

Multiply:

1. $3x^{\frac{1}{4}}-7+9x^{-\frac{1}{4}}$ by $2x^{\frac{1}{4}}-5x^{-\frac{1}{4}}$.
2. $a^{\frac{1}{3}}b^{\frac{1}{2}}+a^{-\frac{1}{2}}b^{-\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{2}}-2a^{-\frac{1}{2}}b^{-\frac{1}{3}}$.
3. $2x^{\frac{1}{2}}+3x^{\frac{1}{4}}+1$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}+1$.
4. $a^{\frac{2}{5}}-a^{\frac{1}{5}}b^{\frac{1}{5}}+b^{\frac{2}{5}}$ by $a^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{1}{5}}+b^{\frac{2}{5}}$.

Divide:

5. $16a^{-\frac{3}{4}}+6a^{-\frac{1}{2}}+5a^{-\frac{1}{4}}-6$ by $2a^{-\frac{1}{4}}-1$.

6. $21x^{\frac{3}{5}}+x^{\frac{2}{5}}+x^{\frac{1}{5}}+1$ by $3x^{\frac{1}{5}}+1$.

7. $x^{\frac{3}{2}}+x^{-\frac{3}{2}}$ by $x+x^{-1}-1$.

8. $3x^{-\frac{4}{7}}-5x^{-\frac{3}{7}}+x^{-\frac{1}{7}}-1$ by $3x^{-\frac{2}{7}}-2x^{-\frac{1}{7}}+1$.

Find the square root of :

9. $9x^2-12x+10-4x^{-1}+x^{-2}$.

10. $4x^5+9x^{-5}-24x^{-\frac{5}{2}}-16x^{\frac{5}{2}}+28$.

11. $\dfrac{a^{-2}}{16}+9\sqrt[3]{b^2}-6\sqrt[3]{b}+1+\dfrac{a^{-1}}{2}(1-3\sqrt[3]{b})$.

12. $9x^{\frac{8}{5}}-12x^{\frac{4}{5}}-2+4x^{-\frac{4}{5}}+x^{-\frac{8}{5}}$.

Simplify and express with positive indices :

13. $(\sqrt{x^2y^5})^8$.

14. $(\sqrt[3]{x^{-3}y^7})^{-4}$.

15. $(4a^{-4}\div 25b^2)^{-\frac{1}{2}}$.

16. $\left(\dfrac{8x^2}{y^{-3}}\right)^{-\frac{1}{3}}$.

17. $\left(\dfrac{8c^6}{27d^{-6}}\right)^{-\frac{2}{3}}$.

18. $\left(\dfrac{x^{-\frac{3}{2}}}{9y^{\frac{1}{2}}}\right)^{-2}$.

19. $2(x^{\frac{3}{2}}y^{\frac{1}{2}})^2 \,.\, (2x^{-1}y)^{-2}$.

20. $\sqrt[3]{8p^2q^{-7}}\div\sqrt{4p^{-\frac{1}{2}}q^{\frac{4}{3}}}$.

21. $\left[x^{\frac{n}{1-n}}-x^{\frac{1}{1-n}}\right]\div x^{\frac{n}{1-n}}$.

22. $(225)^{\frac{1}{5}}\times(72)^{-\frac{1}{5}}\times(1000)^{-\frac{2}{15}}$.

23. $(8^{\frac{2}{3}}\times 16^{-\frac{1}{4}}\div 4^{\frac{1}{2}})^x$.

24. $(0{\cdot}4)^2\times(0{\cdot}125)^{\frac{1}{3}}\div(2{\cdot}5)^{-3}$.

25. $(3x^{-\frac{1}{2}}b^{\frac{1}{4}})^{-2}\div\left(\dfrac{x^{\frac{1}{6}}}{2b^{\frac{1}{2}}}\right)^3$.

26. $\dfrac{a^{\frac{1}{3}}b}{\sqrt[6]{64a^{\frac{2}{3}}b^{\frac{3}{2}}}}\div a^{\frac{1}{9}}b^{-\frac{1}{4}}$.

27. $(8a^{\frac{1}{2}}b^{-\frac{3}{4}})^{\frac{4}{3}}\div 4a^{-\frac{1}{3}}b^{-1}$.

28. $\sqrt{a^{3+x}}\times\sqrt[3]{a^{2+x}}\div\sqrt[6]{a^{1-x}}$.

29. $(9x^{-1}y^3)^{-\frac{1}{2}}\times\sqrt{xy^3}$.

30. $\sqrt[3]{x^{-2}\sqrt{y^{\frac{3}{2}}}}\div\sqrt{\sqrt{y}\sqrt[3]{x^2}}$.

31. $\left(\dfrac{x^{-2}y^2}{2x^3y^{-8}}\right)^{-3}\div\left(\dfrac{2xy^{-2}}{x^{-3}y^4}\right)^5$.

32. $\sqrt[5]{(a^2-b^2)^8}\times(a^2-b^2)^{-\frac{3}{5}}$.

33. $(b^{-\frac{3}{2}}\sqrt[3]{c^2})^{-3}\times\sqrt{c^{-4}\sqrt{b^{-18}}}$.

34. $(x^{n^2-1})^{\frac{n}{n+1}}+x^{-1}\,.\,\sqrt[n]{x^{2n^2+n}}$.

35. $\dfrac{5^{2n}\times(5^{2n-1})^{2n}}{5^{n+3}\times 5^{3n-3}}\times\dfrac{1}{25^{-2n}}$.

36. $\dfrac{5\cdot 3^k-9\cdot 3^{k-2}}{3^k-3^{k-1}}$.

Solve the following equations :

37. $ax^5=b^6$.

38. $a^3x^{\frac{2}{3}}=b^{\frac{5}{3}}x^2$.

39. $a^{-\frac{1}{4}}x^{-\frac{3}{4}}=b^{-1}$.

40. $a^2b^3x=c^{\frac{11}{2}}x^{\frac{1}{2}}$.

41. $c^{-\frac{1}{2}}x^{-\frac{1}{3}}=d^{-\frac{1}{3}}x^{-\frac{1}{2}}$.

42. $k^{\frac{5}{6}}x^{-\frac{5}{6}}=k^{-\frac{5}{6}}x^{\frac{5}{6}}$.

43. $x^2y^3=a^5$;
$xy^7=b^8$.

44. $x^{\frac{1}{3}}y^{\frac{2}{3}}=k$;
$y^{\frac{1}{2}}=lx^{-\frac{1}{2}}$.

45. $x^{\frac{1}{5}}y^{-1}=a^{-\frac{4}{5}}$;
$yx^{\frac{4}{5}}=b^{\frac{9}{5}}$.

46. $2^x=32$.

47. $\frac{1}{81}=3^y$.

48. $10^{5n}=1000$.

Write down the value of :

49. $(x^{\frac{1}{2}}-4)(x^{\frac{1}{2}}+5)$.

50. $(4\sqrt{a}-3)(3\sqrt{a}+2)$.

51. $(x^{\frac{3}{2}}-64)\div(x^{\frac{1}{2}}-4)$.

52. $(1+125c^{-6})\div(1+5c^{-2})$.

53. $(a^{\frac{2}{3}}-3a^{-\frac{2}{3}})^2$.

54. $(x^{\frac{1}{4}}+2x^{-\frac{1}{4}})^4$.

Express in simplest form, free from radical signs and negative indices :

55. $\dfrac{a+7\sqrt{a}+10}{a-4}$.

56. $\dfrac{x-5x^{\frac{1}{2}}-14}{x-4x^{\frac{1}{2}}-21}$.

57. $\dfrac{x^{-6}+x^{-3}-12}{x^{-6}-x^{-3}-6}$.

58. $\dfrac{x^{\frac{4}{3}}-8x^{\frac{1}{3}}y}{x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+4y^{\frac{2}{3}}}$.

59. $\dfrac{x^{\frac{3}{2}}+xy}{xy-y^3}-\dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}-y}$.

60. $(x^{\frac{1}{2}}+7)(x-7x^{\frac{1}{2}}+49)$.

Simplify and express with positive indices :

61. $(\sqrt{x^3y^6})^4$.

62. $(16c^{-6}\div t^4)^{-\frac{1}{4}}$.

63. $(\sqrt[6]{a^{-2}b^5})^3$.

64. $\left(\dfrac{81x^{-5}}{16y^{-2}}\right)^{-\frac{3}{4}}$.

65. $\left(\dfrac{3l^{-\frac{3}{8}}}{m^{\frac{3}{4}}}\right)^{-4}$.

66. $\left(\dfrac{125a^{-5}}{8b^{-2}}\right)^{\frac{2}{3}}$.

67. $32^{\frac{3}{5}}x^{-\frac{2}{3}}y^2\div(216x^4y^2)^{\frac{1}{3}}$.

68. $24^{\frac{1}{2}}\cdot16^{-\frac{1}{3}}\cdot54^{-\frac{1}{6}}$.

69. $\sqrt{(2x)^{-2}y}\times\sqrt[3]{8xy^{-3}}$.

70. $\sqrt[6]{a^8b^6}\times(a^{\frac{2}{3}}b^{-1})^{-2}$.

71. $\dfrac{7^{n+2}-35\cdot7^{n-1}}{7^n\times11}$.

72. $\dfrac{3^{2n+1}}{(3^{-2})^n}\cdot\dfrac{9^{n^2}}{(3^{2n+1})^{2n-1}}$.

SURDS

173. If a is a rational quantity which is not a perfect nth power, $\sqrt[n]{a}$ is called **a surd of the nth order.**

Thus, $\sqrt{3}$ is a surd of the second order, or a **quadratic surd**; $\sqrt[3]{5}$ is a surd of the third order. But $\sqrt{2\cdot25}$ and $\sqrt[3]{27}$ are arithmetical numbers in surd form, since $2\cdot25$ is a perfect square and 27 is a perfect cube. It should also be noted that $\sqrt{7+\sqrt{5}}$ is not a surd but a surd expression. Some surd expressions can be reduced to surds; thus, $\sqrt[5]{\sqrt{2}}$ is equal to $\sqrt[10]{2}$; in the latter form it satisfies the definition of a surd.

Surds are inexpressible either as integers or as fractions, but the value of a surd can be obtained to any degree of accuracy, i.e. a surd is an **irrational quantity.** Thus, $\sqrt{2}=1\cdot41421\ldots$; i.e. $\sqrt{2}$ lies between $1\cdot4142$ and $1\cdot4143$. The error in using either of these

quantities instead of $\sqrt{2}$ is less than 0·0001. By taking the square root to a greater number of decimal places we can obtain greater accuracy. It is therefore not *necessary* to introduce surds into numerical work, but *in practice* it will be found that, even where approximate numerical results are required, the work is considerably simplified by using surd symbols.

174. Surds may always be expressed as quantities with fractional indices. Expressions involving surds may be transformed into other expressions by using the laws stated earlier in this chapter.

Thus, **a surd of any order may be transformed into a surd of a different order.**

For $$\sqrt[5]{3}=3^{\frac{1}{5}}=3^{\frac{3}{15}}=\sqrt[15]{3^3}.$$

In particular, **surds of different orders may be transformed into surds of the same order.** This order may be any common multiple of each of the given orders, but it is usually most convenient to choose the *least* common multiple. Thus $\sqrt[4]{a^3}$, $\sqrt[3]{x^4}$, $\sqrt[8]{a^5}$ may each be expressed as surds of order 24 (the L.C.M. of 4, 3, 8) or any multiple of 24, for $\sqrt[4]{a^3}=a^{\frac{3}{4}}=a^{\frac{18}{24}}=\sqrt[24]{a^{18}}$. Similarly $\sqrt[3]{x^4}=\sqrt[24]{x^{32}}$ and $\sqrt[8]{a^5}=\sqrt[24]{a^{15}}$.

To compare surds of different orders, they must first be transformed to surds of the same order. Thus, to compare $\sqrt{3}$, $\sqrt[3]{6}$ and $\sqrt[4]{10}$, we express them in the form $\sqrt[12]{729}$, $\sqrt[12]{1296}$, $\sqrt[12]{1000}$. Thus $\sqrt[3]{6}$, $\sqrt[4]{10}$, $\sqrt{3}$ are in descending order of magnitude.

175. Since $(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}b^{\frac{1}{n}}$, **the nth root of any expression is equal to the product of the nth roots of the factors of the expression.**

Thus
$$\sqrt[3]{15}=\sqrt[3]{5}\cdot\sqrt[3]{3}, \quad\text{.........(i)}$$
$$\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}, \quad\text{.........(ii)}$$
$$\sqrt[4]{32}=\sqrt[4]{16}\cdot\sqrt[4]{2}=2\sqrt[4]{2}. \quad\text{.........(iii)}$$

From (ii) and (iii) it is clear that a surd may sometimes be expressed as the product of a rational quantity and a surd. When a surd is expressed so that the integer under the root sign is as small as possible it is said to be in its **simplest form.** Thus, $3\sqrt{2}$ is the simplest form of $\sqrt{18}$. Conversely, a coefficient of a surd may be brought under the root sign; thus, $2\sqrt{5}=\sqrt{4}\sqrt{5}=\sqrt{20}$. A surd so expressed is called an **entire surd.**

176. When surds can be expressed with the same irrational factor they are said to be **like**; otherwise they are said to be **unlike.** Thus, $4\sqrt{7}$, $9\sqrt{7}$ are like, and $\sqrt{5}$, $6\sqrt{11}$ are unlike surds.

It should be carefully noted that $\sqrt{a+b}$, $\sqrt[3]{x-y}$, etc. cannot be simplified, unless we know the numerical values of a, b, x, y, etc.

177. Addition and subtraction of surds. The sum of a number of like surds can be found when they have been expressed in their simplest form.

Example 7. *Find the sum of* $\sqrt{108}$, $\sqrt{48}$ *and* $\sqrt{75}$.

The required sum $=6\sqrt{3}+4\sqrt{3}+5\sqrt{3}=15\sqrt{3}$.

Example 8. *Simplify* $3\sqrt{147}-11\sqrt{\tfrac{1}{27}}-\tfrac{7}{3}\sqrt{\tfrac{1}{3}}$.

$$\begin{aligned}\text{The expression} &= 21\sqrt{3}-\tfrac{11}{9}\sqrt{3}-\tfrac{7}{9}\sqrt{3}\\ &=(21-\tfrac{11}{9}-\tfrac{7}{9})\sqrt{3}=19\sqrt{3}.\end{aligned}$$

Note. It is usual to express a surd with a rational denominator. Thus
$$\sqrt{\tfrac{1}{3}}=\sqrt{\tfrac{1}{9}\times 3}=\sqrt{\tfrac{1}{9}}\times\sqrt{3}=\tfrac{1}{3}\sqrt{3}.$$

Example 9. *Express* $x\sqrt[3]{27x^3y}+z\sqrt[3]{-yz^3}-\sqrt[3]{8x^6y}$ *in its simplest form.*

$$\begin{aligned}\text{The expression} &= x\,.\,3x\sqrt[3]{y}+z(-z)\sqrt[3]{y}-2x^2\sqrt[3]{y}\\ &=(3x^2-z^2-2x^2)\sqrt[3]{y}=(x^2-z^2)\sqrt[3]{y}.\end{aligned}$$

Unlike surds cannot be collected.

Thus, the sum of $3\sqrt{5}$ and $7\sqrt{2}$ is $3\sqrt{5}+7\sqrt{2}$, and can only be simplified further by substituting approximate values for $\sqrt{5}$ and $\sqrt{2}$. Such an expression is called a **compound surd.**

Example 10. *Simplify* $\sqrt{252}+2\sqrt{294}-12\sqrt{\tfrac{1}{6}}$.

The expression $=6\sqrt{7}+14\sqrt{6}-2\sqrt{6}=6\sqrt{7}+12\sqrt{6}$.

178. Multiplication and division of surds. The process is illustrated by the following examples:

Example 11. *Find the product of* $2\sqrt{32}$, $\sqrt{27}$, $\sqrt{150}$.

$$\begin{aligned}\text{The product} &= 2\,.\,4\,.\,\sqrt{2}\,.\,3\,.\,\sqrt{3}\,.\,5\,.\,\sqrt{2}\,.\,\sqrt{3}\\ &=2\,.\,4\,.\,3\,.\,5\,.\,\sqrt{2}\,.\,\sqrt{2}\,.\,\sqrt{3}\,.\,\sqrt{3}\\ &=2\,.\,4\,.\,3\,.\,5\,.\,2\,.\,3=720.\end{aligned}$$

Example 12. *Find the numerical value of* $\dfrac{3}{2\sqrt{2}}$.

$$\frac{3}{2\sqrt{2}}=\frac{3\cdot\sqrt{2}}{2\sqrt{2}\,.\,\sqrt{2}}=\frac{3\sqrt{2}}{4}=\frac{3\times 1{\cdot}414}{4}=1{\cdot}0605 \text{ approx}$$

This example illustrates a very important practical principle. Fractions with surds in the denominator should always be replaced by equivalent fractions with rational denominators. This process is called **rationalising the denominator**; the factor by which numerator and denominator are multiplied to effect this result is called a **rationalising factor.** Thus, in Ex. 12, $\sqrt{2}$ is a rationalising factor.

The advantage of this process is that the divisor is an integer instead of an inexact decimal. Not only is the working shortened, but it is much easier to obtain the answer correct to any required number of figures.

Example 13. *Evaluate* $\dfrac{3\sqrt{12}}{5\sqrt{28}} \div \dfrac{12\sqrt{21}}{\sqrt{98}}$.

$$\text{The expression} = \frac{3\cdot 2\cdot \sqrt{3}}{5\cdot 2\cdot \sqrt{7}} \times \frac{7\cdot\sqrt{2}}{12\cdot\sqrt{7}\cdot\sqrt{3}} = \frac{3\cdot 2\cdot 7\cdot\sqrt{3}\cdot\sqrt{2}}{5\cdot 2\cdot 12\cdot\sqrt{7}\cdot\sqrt{7}\cdot\sqrt{3}}$$

$$= \frac{\sqrt{2}}{20} = \frac{1{\cdot}414}{20} = 0{\cdot}0707 \text{ approx.}$$

EXERCISE 86

1. Express as surds of the twelfth order with positive indices:

(i) $\sqrt{3}$, (ii) $\sqrt[3]{2}$, (iii) $\sqrt[6]{7}$, (iv) $\sqrt[4]{a^5}$, (v) $\sqrt[6]{a^{-5}}$, (vi) $\sqrt[3]{xy^{-2}}$.

2. Arrange in descending order:

(i) $\sqrt[4]{9}$, $\sqrt[6]{24}$, $\sqrt[3]{5}$; (ii) $\sqrt[3]{6}$, $\sqrt[4]{11}$, $\sqrt{3}$.

3. Express as surds of the same lowest order:

(i) $\sqrt{x}$, $\sqrt[7]{x^6}$; (ii) $\sqrt[3]{a^4}$, $\sqrt{a}$; (iii) $\sqrt[5]{a^2}$, $\sqrt[3]{a}$; (iv) $\sqrt[4]{7}$, $\sqrt{2}$, $\sqrt[8]{9}$.

Express the following surds in their simplest form:

4. (i) $\sqrt{288}$, (ii) $\sqrt{18}$, (iii) $\sqrt{27}$, (iv) $\sqrt{16x}$, (v) $\sqrt[3]{128c^3}$, (vi) $\sqrt[3]{-16}$.

5. (i) $\sqrt{98}$, (ii) $\sqrt{242}$, (iii) $\sqrt{720}$, (iv) $\sqrt[4]{128}$,
(v) $\sqrt[3]{-375x^3y}$, (vi) $5\sqrt{768}$.

6. (i) $2\sqrt{x^2y}$, (ii) $3\sqrt[3]{x^7y^5}$, (iii) $\sqrt[5]{-x^8y^{10}}$,
(iv) $\sqrt{a^3+2a^2b+ab^2}$, (v) $3c\sqrt[3]{250c^7d^3}$.

Express as entire surds:

7. (i) $9\sqrt{2}$, (ii) $11\sqrt{5}$, (iii) $5\sqrt[3]{3}$, (iv) $2\sqrt[4]{2}$, (v) $7\sqrt{18}$, (vi) $4\sqrt[3]{5}$.

8. (i) $5\sqrt{\dfrac{2}{5}}$, (ii) $\dfrac{3}{5}\sqrt{125}$, (iii) $\dfrac{2}{7}\sqrt{\dfrac{21}{8}}$, (iv) $\dfrac{c}{d^3}\sqrt{\dfrac{5d^5}{c}}$, (v) $\dfrac{2x}{y^2}\sqrt[4]{\dfrac{y^7}{32x^3}}$

Simplify the following :

9. $4\sqrt{45} - 3\sqrt{20} + 8\sqrt{5}$.

10. $6\sqrt{63} + 5\sqrt{28} - 16\sqrt{7}$.

11. $6\sqrt{363} - 10\sqrt{243} + 6\sqrt{192}$.

12. $5\sqrt{44} - 8\sqrt{176} + 4\sqrt{99}$.

13. $3\sqrt[3]{189} + 6\sqrt[3]{875} - 11\sqrt[3]{56}$.

14. $10\sqrt[3]{81} - 12\sqrt[3]{192} + 7\sqrt[3]{648}$.

15. $6\sqrt{128} + 5\sqrt{75} - 10\sqrt{162}$.

16. $3\sqrt[3]{-54} - 5\sqrt[3]{-16} + 2\sqrt[3]{1029}$.

17. $5\sqrt{147} - \frac{11}{3}\sqrt{\frac{1}{3}} - 7\sqrt{\frac{1}{27}}$.

18. $60\sqrt{\frac{1}{6}} - 2\sqrt{54} - 12\sqrt{\frac{1}{24}}$.

19. $2x\sqrt{x^3y^2} + 3x^2y\sqrt{9x} - x^2\sqrt{25xy^2}$.

20. $x^2\sqrt[3]{27xy^3} - 3z\sqrt[3]{-8xz^6} - \sqrt[3]{125xy^6z^3}$.

Simplify the following surds, and find their numerical values, correct to two places of decimals, given

$\sqrt{2} = 1{\cdot}4142$, $\sqrt{3} = 1{\cdot}7321$, $\sqrt{5} = 2{\cdot}2361$, $\sqrt{6} = 2{\cdot}4495$, $\sqrt{7} = 2{\cdot}6458$.

21. $5\sqrt{98}$.

22. $5\sqrt{18} + \sqrt{50}$.

23. $3\sqrt{147}$.

24. $\sqrt{200} - \sqrt{128}$.

25. $2\sqrt{150} - \sqrt{243}$.

26. $8\sqrt{128} - 10\sqrt{108} + 5\sqrt{125}$.

27. $3\sqrt{10} \times \sqrt{15}$.

28. $5\sqrt{8} \times 2\sqrt{6}$.

29. $2\sqrt{70} \times \sqrt{35}$.

30. $3\sqrt{15} \times 4\sqrt{5}$.

31. $2\sqrt{33} \times 5\sqrt{22}$.

32. $3\sqrt{27} \div \sqrt{50}$.

33. $\frac{7}{\sqrt{2}}$.

34. $\frac{30}{\sqrt{18}}$.

35. $\frac{30}{\sqrt{7}}$.

36. $\frac{4\sqrt{3}}{9\sqrt{2}}$.

37. $\frac{300}{\sqrt{125}}$.

38. $\frac{10\sqrt{96}}{\sqrt{75}}$.

39. $\frac{1}{5\sqrt{8}}$.

40. $\frac{10\sqrt{3}}{7\sqrt{27}} \times \frac{\sqrt{126}}{3\sqrt{8}}$.

179. Compound surds. In simplifying expressions containing compound surds we may proceed as in dealing with rational expressions.

Example 14. *Multiply* $2\sqrt{x} - 5\sqrt{y}$ *by* $3\sqrt{x} + 2\sqrt{y}$.

$$\begin{aligned}\text{The product} &= (2\sqrt{x} - 5\sqrt{y})(3\sqrt{x} + 2\sqrt{y})\\ &= 2\sqrt{x}\,.\,3\sqrt{x} - 5\sqrt{y}\,.\,3\sqrt{x} + 2\sqrt{x}\,.\,2\sqrt{y} - 5\sqrt{y}\,.\,2\sqrt{y}\\ &= 6x - 15\sqrt{xy} + 4\sqrt{xy} - 10y\\ &= 6x - 11\sqrt{xy} - 10y.\end{aligned}$$

Example 15. *Find the value of* $(7\sqrt{2} + 5\sqrt{3})(7\sqrt{2} - 5\sqrt{3})$.

Since $(a+b)(a-b) = a^2 - b^2$, we have

$$\begin{aligned}(7\sqrt{2} + 5\sqrt{3})(7\sqrt{2} - 5\sqrt{3}) &= (7\sqrt{2})^2 - (5\sqrt{3})^2\\ &= 49 \times 2 - 25 \times 3 = 98 - 75\\ &= 23.\end{aligned}$$

This example is of great importance. If we multiply together the sum and the difference of any two surds of the second order (quadratic surds), we obtain a rational product.

180. Binomial expressions such as $7\sqrt{2}+5\sqrt{3}$, $7\sqrt{2}-5\sqrt{3}$, which differ only in the sign which connects their terms, are said to be **conjugate.** We have shown above that the product of two conjugate surds is rational. This result enables us to rationalise the denominator of a fraction, when the denominator is a binomial. The rationalising factor is the surd which is conjugate to the denominator.

Example 16. *Divide $7+3\sqrt{2}$ by $5-3\sqrt{2}$, and find the value of the quotient correct to 3 significant figures.*

$$\text{The quotient}=\frac{7+3\sqrt{2}}{5-3\sqrt{2}}=\frac{(7+3\sqrt{2})(5+3\sqrt{2})}{(5-3\sqrt{2})(5+3\sqrt{2})}$$

$$=\frac{35+15\sqrt{2}+21\sqrt{2}+18}{25-18}=\frac{53+36\sqrt{2}}{7}$$

$$=\frac{53+36\times 1\cdot414}{7}=\frac{103\cdot904}{7}=14\cdot8, \text{ correct to 3 significant figures.}$$

Example 17. *Express* $\dfrac{1}{\sqrt{5}+\sqrt{2}+1}$ *with a rational denominator.*

Here it is impossible to rationalise in one step. We have

$$\frac{1}{\sqrt{5}+\sqrt{2}+1}=\frac{\sqrt{5}+\sqrt{2}-1}{(\sqrt{5}+\sqrt{2}+1)(\sqrt{5}+\sqrt{2}-1)}=\frac{\sqrt{5}+\sqrt{2}-1}{(\sqrt{5}+\sqrt{2})^2-1}$$

$$=\frac{\sqrt{5}+\sqrt{2}-1}{5+2+2\sqrt{10}-1}=\frac{\sqrt{5}+\sqrt{2}-1}{2(\sqrt{10}+3)}=\frac{(\sqrt{5}+\sqrt{2}-1)(\sqrt{10}-3)}{2(\sqrt{10}+3)(\sqrt{10}-3)}$$

$$=\frac{5\sqrt{2}-3\sqrt{5}+2\sqrt{5}-3\sqrt{2}-\sqrt{10}+3}{2(10-9)}=\frac{2\surd\quad-\sqrt{5}-\sqrt{10}+3}{2}.$$

181. Square root of binomial quadratic surds. From the formula $(\sqrt{x}\pm\sqrt{y})^2=x+y\pm2\sqrt{xy}$ we can write down the square root of an expression $A+2\sqrt{B}$, if we can find two quantities x, y such that their sum is A and their product is B.

Example 18. *Find the square root of $10-2\sqrt{21}$.*

The expression may be written in the form $7+3-2\sqrt{7\times3}$.

It is then clear that the square root is $\sqrt{7}-\sqrt{3}$.

Example 19. *Find the square root of* $22 + 12\sqrt{2}$.

We first write the expression so that the coefficient of the surd is 2.

Thus, $\quad 22 + 12\sqrt{2} = 22 + 2\sqrt{36 \times 2} = 22 + 2\sqrt{72}.$

We must therefore find two numbers such that their sum is 22 and their product is 72. The numbers are 18 and 4;

$$\therefore \sqrt{22 + 12\sqrt{2}} = \sqrt{18} + \sqrt{4} = 3\sqrt{2} + 2.$$

If the numbers cannot be readily guessed, we solve the equations $x + y = 22$, $xy = 72$ in the usual manner.

Note. Since every quantity has two square roots, $-(\sqrt{7} - \sqrt{3})$ and $-(3\sqrt{2} + 2)$ are also square roots of the expressions $10 - 2\sqrt{21}$, $22 + 12\sqrt{2}$ respectively; but it is a convention that only the positive root should be considered.

182. Irrational equations. An **irrational equation** is one in which the unknown or unknowns occur under a root sign.

In the following examples the positive value of the square root is always taken. Thus, a term such as $\sqrt{2x - 3}$, when $x = 6$, means the positive value of $\sqrt{9}$, i.e. $+3$.

The method of solution is illustrated in the following examples.

Example 20. *Solve* $\sqrt{x + 9} + 11 = x$.

There is only one term in which the unknown occurs under a root sign. Rearrange the equation so that this term is by itself on one side of the equation. This is known as isolating the term. We then have $\sqrt{x + 9} = x - 11$. Square each side,

$$\therefore x + 9 = (x - 11)^2 = x^2 - 22x + 121,$$

$$\therefore x^2 - 23x + {}^{-}12 = 0, \quad \therefore (x - 16)(x - 7) = 0,$$

$$\therefore x = 16 \quad \text{or} \quad x = 7.$$

Check. When $x = 16$, L.H.S. $= \sqrt{25} + 11 = 5 + 11 = 16$,

R.H.S. $= 16$,

$\therefore x = 16$ is a solution.

When $x = 7$, L.H.S. $= \sqrt{16} + 11 = 4 + 11 = 15$,

R.H.S. $= 7$,

$\therefore x = 7$ is not a solution.

It should be carefully noted that the process of squaring each side of an equation is a non-reversible step. It may therefore introduce a root or roots which do not satisfy the original equation.

It is therefore essential to test the results by substituting in the original equation. The root $x=7$ is a root of the equation $-\sqrt{x+9}+11=x$. If we solve this equation by the method given above, we shall find that it leads to the same equation as before, $x^2-23x+112=0$, giving $x=16$ or $x=7$; but, on checking, it will be seen that we must now reject $x=16$ and take $x=7$.

Example 21. *Solve* $\sqrt{x+6}+\sqrt{x+1}=\sqrt{6x+7}$.

Square each side, $\therefore\ x+6+x+1+2\sqrt{(x+6)(x+1)}=6x+7$.

Isolate the term containing the root, $\therefore\ 2\sqrt{(x+6)(x+1)}=4x$.

Divide each side by 2, $\therefore\ \sqrt{(x+6)(x+1)}=2x$.

Square each side, $\therefore\ (x+6)(x+1)=4x^2$, which reduces to

$$3x^2-7x-6=0, \quad \text{or} \quad (x-3)(3x+2)=0,$$

$$\therefore\ x=3 \quad \text{or} \quad x=-\tfrac{2}{3}.$$

Check. When $x=3$, L.H.S. $=\sqrt{9}+\sqrt{4}=3+2=5$,

R.H.S. $=\sqrt{25}=5$,

$\therefore\ x=3$ is a solution.

When $x=-\frac{2}{3}$, L.H.S. $=\sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\frac{4}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}=\frac{5}{3}\sqrt{3}$,

R.H.S. $=\sqrt{3}$,

$\therefore\ x=-\frac{2}{3}$ is not a solution, but it is easily seen that $x=-\frac{2}{3}$ is a solution of $\sqrt{x+6}-\sqrt{x+1}=\sqrt{6x+7}$.

It should be noted that the check is not complete, unless it is seen that the rejected root satisfies an equation derived from the original equation by one or more changes of sign. It must not be overlooked that a root may fail to satisfy because of some error in the working.

EXERCISE 87

Find the value of:

1. $(5\sqrt{x}-3)\times 4\sqrt{x}$.

2. $(4+7\sqrt{l})\times 3\sqrt{l}$.

3. $(2\sqrt{5}-5\sqrt{2})^2$.

4. $(3\sqrt{x}-4\sqrt{y})(2\sqrt{x}+5\sqrt{y})$.

5. $(4\sqrt{5}-3\sqrt{2})(3\sqrt{5}+7\sqrt{2})$.

6. $(2\sqrt{7}+3\sqrt{5})(2\sqrt{7}-3\sqrt{5})$.

7. $(6+5\sqrt{3})^2$.

8. $(7\sqrt{5}-5\sqrt{3})^2$.

9. $(\sqrt{x+2y}-\sqrt{x-2y})\times\sqrt{x+2y}$.

10. $(\sqrt{2+x}-\sqrt{2-x})^2$.

11. $(2+\sqrt{5}-\sqrt{7})(2-\sqrt{5}+\sqrt{7})$.

12. $(3+2\sqrt{2}-\sqrt{5})(3+2\sqrt{2}+\sqrt{5})$.

Simplify and express with rational denominators:

13. $1\div(9-4\sqrt{3})$.

14. $9\div(5\sqrt{3}-6\sqrt{2})$.

15. $(13-\sqrt{5})\div(2+3\sqrt{5})$.

16. $(3+\sqrt{5})\div(\sqrt{5}+1)$.

17. $5\sqrt{14}\div(\sqrt{7}+\sqrt{2})$.

18. $(6+4\sqrt{3})\div(6-3\sqrt{3})$.

19. $(2\sqrt{3}+3\sqrt{2})\div(5-2\sqrt{6})$. **20.** $b^2\div(a-\sqrt{a^2-b^2})$.

21. $(\sqrt{16+x^2}+4)\div(\sqrt{16+x^2}-4)$. **22.** $1\div(3-\sqrt{5})^2(2+\sqrt{5})$.

23. $(1+\sqrt{3})\div(2+\sqrt{3})(3\sqrt{3}+5)$. **24.** $(3+2\sqrt{2})\div(1+\sqrt{2})^2$.

Given $\sqrt{2}=1{\cdot}4142$, $\sqrt{3}=1{\cdot}7321$, $\sqrt{5}=2{\cdot}2361$, $\sqrt{6}=2{\cdot}4495$, $\sqrt{7}=2{\cdot}6458$, find, correct to two decimal places, the value of:

25. $\dfrac{1}{3-\sqrt{7}}$. **26.** $\dfrac{5-\sqrt{6}}{2\sqrt{6}-3}$.

27. $\dfrac{1}{\sqrt{5}+\sqrt{3}}$. **28.** $\dfrac{2-\sqrt{6}}{5-2\sqrt{6}}\cdot\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}}$.

29. $\dfrac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$. **30.** $\dfrac{7\sqrt{3}+5\sqrt{2}}{4\sqrt{3}-3\sqrt{2}}$.

31. $\dfrac{\sqrt{3}-\sqrt{2}}{5-\sqrt{3}+4\sqrt{2}}$. **32.** $\dfrac{1}{7+2\sqrt{3}-5\sqrt{2}}$.

33. Verify by substitution that $3+2\sqrt{5}$ is a root of $x^2-6x=11$.

34. Verify by substitution that $\sqrt{5}+\sqrt{3}$ is a root of $x^4-16x^2+4=0$.

35. Given that $m=\sqrt{7}+\sqrt{5}$ and $n=\sqrt{7}-\sqrt{5}$, find the values of (i) mn, (ii) m^2+n^2, (iii) m^4+n^4.

36. If $x=\sqrt{7}+\sqrt{3}$, show that $x-\dfrac{4}{x}=2\sqrt{3}$.

37. Show that $\dfrac{2-3x^2}{x^3}$ is a rational number when $x=\sqrt{3}+1$.

38. If $x=3-\sqrt{5}$, calculate the value of x^3-32x.

39. Find the value of $8y-y^2$ when $y=(\sqrt{5}-\sqrt{3})\div(\sqrt{5}+\sqrt{3})$.

40. Find the value of $x^3-11x^2+27x-17$ when $x=5-2\sqrt{2}$.

Find the square root of:

41. $6+2\sqrt{5}$. **42.** $8+2\sqrt{15}$. **43.** $12-2\sqrt{32}$.

44. $9-2\sqrt{18}$. **45.** $30-12\sqrt{6}$. **46.** $67-12\sqrt{7}$.

47. $57+12\sqrt{15}$. **48.** $8+\sqrt{55}$.

49. $10-2\sqrt{(4-x)(6+x)}$. **50.** $4x-5+2\sqrt{3x^2-11x+6}$.

51. $6-\sqrt{17-12\sqrt{2}}$. **52.** $9+\sqrt{29-2\sqrt{28}}$.

Solve the equations:

53. $\sqrt{3x+4}=7$. **54.** $\sqrt{9x-5}=\sqrt{22}$.

55. $8\sqrt{x+3}=40$. **56.** $2+\sqrt[3]{x-1}=7$.

57. $\sqrt{5x+7}-\sqrt{3x+11}=0$. **58.** $3\sqrt{2x}=2\sqrt{x+5}$.

59. $\sqrt{x+6}+\sqrt{x-1}=7$. **60.** $\sqrt{4x+5}-\sqrt{x-1}=\sqrt{x+4}$.

61. $\sqrt{3x-2}+\sqrt{3x-11}=9$. **62.** $\sqrt{2x+5}-\sqrt{2x-11}=\sqrt{2x-16}$.

63. $\sqrt{5x+l}+\sqrt{5x-l}=\sqrt{l}$. **64.** $\sqrt{2x-1}+\sqrt{x+3}=3$.

65. $2\sqrt{x+1}-1=\sqrt{4x-7}$.

66. $\sqrt{2x-3}=\sqrt{8x-12}-\sqrt{x+3}$.

67. $\sqrt{x+1}+\sqrt{x+8}=\sqrt{6x+1}$.

68. $\sqrt{x+2a}+\sqrt{x}=\sqrt{12x+a}$.

69. $\sqrt{x-2}+\sqrt{x-10}-2=0$.

70. $2\sqrt{6+3x}=\sqrt{4-3x}+\sqrt{28-3x}$.

71. $\sqrt{2+x}-\sqrt{2-x}-x=0$.

72. $\sqrt{2x+11}+\sqrt{3x-12}=\sqrt{7x+15}$.

73. $\dfrac{\sqrt{x}+9}{\sqrt{x}-6}=\dfrac{\sqrt{x}-5}{\sqrt{x}-13}$.

74. $\sqrt{9+4x}-2\sqrt{x}=\dfrac{5}{\sqrt{9+4x}}$.

75. $\sqrt{4x+5}-\sqrt{4x+3}=\sqrt{6x+2}-\sqrt{6x}$.

76. $\sqrt{4x^2-4x-3}-\sqrt{2x^2+5x-12}=\sqrt{2x-3}$.

77. $3x^2-21x+26=4\sqrt{x^2-7x+10}$.

[HINT : Let $y=+\sqrt{x^2-7x+10}$.]

78. $x^2+27=10x+6\sqrt{x^2-10x+18}$.

79. $6x-2\sqrt{2x^2+12x+4}=4-x^2$.

80. $4x+5-2\sqrt{x^2-4x-3}=x^2-1$.

Find, correct to two decimal places, the value of :

81. $\dfrac{1}{2\sqrt{2}+\sqrt{3}}$.

82. $\dfrac{3+\sqrt{5}}{\sqrt{5}-2}$.

83. $\dfrac{7-2\sqrt{6}}{3+\sqrt{6}}$.

84. $\dfrac{2+\sqrt{5}}{9+4\sqrt{5}}$.

85. $\dfrac{5-2\sqrt{3}}{7+4\sqrt{3}}$.

86. $\dfrac{40}{9\sqrt{5}-5\sqrt{7}}$.

87. $\dfrac{\sqrt{2}}{2+\sqrt{3}+\sqrt{2}}$.

88. $\dfrac{1}{5-3\sqrt{3}+2\sqrt{2}}$.

89. Verify by substitution that $7-2\sqrt{5}$ is a root of $x^2-14x+29=0$.

90. Given that $m=3+2\sqrt{2}$ and $n=3-2\sqrt{2}$, find the values of

(i) mn, (ii) m^2+n^2, (iii) m^3+n^3.

91. Find the value of $x^4-8x^3+4x^2+8x-3$ when $x=4-\sqrt{11}$.

92. Find the value of $4x^3-12x^2-5x+11$ when $2x=4+\sqrt{5}$.

Solve the equations :

93. $5+2\sqrt{x-3}=13$.

94. $\sqrt[3]{x+2}+4=3$.

95. $\sqrt{3x-5}=\sqrt{4x-3}-\sqrt{x-6}$.

96. $\sqrt{2x+7}-\sqrt{x-5}=\sqrt{x}$.

97. $\dfrac{12\sqrt{x}-11}{4\sqrt{x}+1}=\dfrac{3\sqrt{x}}{\sqrt{x}+3}$.

98. $\dfrac{\sqrt{x}+\sqrt{x-8}}{\sqrt{x}-\sqrt{x-8}}=2$.

99. $\sqrt[3]{x^3+8}-x=2$.

100. $\sqrt{4x^2+17x+15}-\sqrt{2x^2+5x-3}=\sqrt{x^2+2x-3}$.

CHAPTER XXVIII

LOGARITHMS

183. It is frequently convenient to express a number **N** in the form a^x, where a is some fixed number other than zero. Thus, we may write 100 in the form 10^2, 8 in the form 2^3, $\frac{1}{36}$ in the form 6^{-2}.

Definition. If a number **N** can be expressed in the form a^x, the index x is called the **logarithm** of the **number N** to the **base** a.

Thus, since $100 = 10^2$, 2 is the logarithm of 100 to base 10,

since $\quad 8 = 2^3$, 3 ,, ,, ,, 8 ,, 2,

since $\quad \frac{1}{36} = 6^{-2}$, -2 ,, ,, $\frac{1}{36}$,, 6.

The logarithm of **N** to base a is usually written $\log_a \mathbf{N}$, so that the statements $\mathbf{N} = a^x$ and $x = \log_a \mathbf{N}$ are equivalent.

EXERCISE 88 (*Oral*)

1. Find the logarithms (or indices) to base 10 of :

1000, 0·01, 1,000,000, 0·001, 0·1, 1.

2. Find the logarithms (or indices) to base 4 of :

16, 256, $\frac{1}{64}$, 2, $\frac{1}{32}$, 8.

3. Find the logarithms (or indices) to base 27 of :

729, 3, $\frac{1}{9}$, 81, 243, $\frac{1}{27}$.

4. Find the logarithms (or indices) to base 25 of :

625, 5, $\frac{1}{25}$, 125, 15,625, 3125.

184. It is recommended that the *proofs* in this Article be omitted at a first reading.

The following general propositions are applicable to all logarithms, whatever the base.

(1) **The logarithm of 1 is 0, and the logarithm of the base is 1.**

For $a \neq 0$, $\therefore a^0 = 1$, i.e. $\log_a 1 = 0$. Also $a^1 = a$, i.e. $\log_a a = 1$.

(2) **The logarithm of a product MN is the sum of the logarithms of the factors M, N,** i.e. $\log_a \mathbf{MN} = \log_a \mathbf{M} + \log_a \mathbf{N}$.

Let $\log_a \mathbf{M} = h$, $\log_a \mathbf{N} = k$. Then $\mathbf{M} = a^h$, $\mathbf{N} = a^k$.

We may then proceed

either $\quad \mathbf{M} \,.\, \mathbf{N} = a^h a^k = a^{h+k}$;

$\therefore$ by the definition $\log_a \mathbf{MN} = h + k = \log_a \mathbf{M} + \log_a \mathbf{N}$;

or $$\mathbf{M}\,.\,\mathbf{N} = a^{\log_a \mathbf{M}}\,.\,a^{\log_a \mathbf{N}} = a^{\log_a \mathbf{M} + \log_a \mathbf{N}};$$

$$\therefore \text{ by the definition } \log_a \mathbf{MN} = \log_a \mathbf{M} + \log_a \mathbf{N}.$$

Similarly, it may be shown that

$$\log_a \mathbf{MNP} = \log_a \mathbf{M} + \log_a \mathbf{N} + \log_a \mathbf{P};$$

and so on for any number of factors.

(3) **The logarithm of a fraction $\frac{\mathbf{M}}{\mathbf{N}}$ is the logarithm of the numerator, M, minus the logarithm of the denominator, N,**

$$\text{i.e. } \log_a \frac{\mathbf{M}}{\mathbf{N}} = \log_a \mathbf{M} - \log_a \mathbf{N}.$$

As before, we have

either $$\frac{\mathbf{M}}{\mathbf{N}} = \frac{a^h}{a^k} = a^{h-k};$$

$$\therefore \text{ by the definition } \log_a \frac{\mathbf{M}}{\mathbf{N}} = h - k = \log_a \mathbf{M} - \log_a \mathbf{N};$$

or $$\frac{\mathbf{M}}{\mathbf{N}} = \frac{a^{\log_a \mathbf{M}}}{a^{\log_a \mathbf{N}}} = a^{\log_a \mathbf{M} - \log_a \mathbf{N}};$$

$$\therefore \text{ by the definition } \log_a \frac{\mathbf{M}}{\mathbf{N}} = \log_a \mathbf{M} - \log_a \mathbf{N}.$$

(4) **The logarithm of $\mathbf{M}^r$ is r times the logarithm of M,** i.e. $\log_a \mathbf{M}^r = r \log_a \mathbf{M}$, where r is any rational number.

As before, we have

either $$\mathbf{M}^r = (a^h)^r = a^{rh};$$

$$\therefore \text{ by the definition } \log_a \mathbf{M}^r = rh = r \log_a \mathbf{M};$$

or $$\mathbf{M}^r = (a^{\log_a \mathbf{M}})^r = a^{r \log_a \mathbf{M}};$$

$$\therefore \text{ by the definition } \log_a \mathbf{M}^r = r \log_a \mathbf{M}.$$

Thus, $\log_a \mathbf{M}^5 = 5 \log_a \mathbf{M}$; $\log_a \sqrt[3]{\mathbf{M}} = \log_a \mathbf{M}^{\frac{1}{3}} = \frac{1}{3} \log_a \mathbf{M}$.

185. Any number may be taken as base, and the logarithms of all positive numbers to any given base may be calculated to any required degree of accuracy. But in all practical calculations it is customary to use 10 as the base.

Logarithms to base 10 are called **Common Logarithms**.

Unless there is any danger of ambiguity, the suffix denoting the base may be omitted. Thus, we usually write $\log 2$, $\log 5$, $\cdots$ instead of $\log_{10} 2$, $\log_{10} 5$, $\cdots$. If, in any piece of work, the suffix denoting the base is omitted, it is implied that all logarithms which occur have the same base, unless otherwise stated.

186. Powers of 10. Since $10^{\frac{1}{2}} = \sqrt{10}$, we can find its value to as many places of decimals as we like by using the process for calculating square roots. We may also calculate the values of $10^{\frac{1}{8}}$, $10^{\frac{1}{4}}$, $10^{\frac{3}{8}}$, $10^{\frac{5}{8}}$, $10^{\frac{3}{4}}$, $10^{\frac{7}{8}}$. For $10^{\frac{1}{4}} = \sqrt{10^{\frac{1}{2}}}$, $10^{\frac{1}{8}} = \sqrt{10^{\frac{1}{4}}}$, $10^{\frac{3}{8}} = 10^{\frac{1}{4}} \times 10^{\frac{1}{8}}$, $10^{\frac{5}{8}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{8}}$, $10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}}$, $10^{\frac{7}{8}} = 10^{\frac{3}{4}} \times 10^{\frac{1}{8}}$.

The values correct to two decimal places are found to be: $10^{\frac{1}{8}} = 1{\cdot}33$, $10^{\frac{1}{4}} = 1{\cdot}78$, $10^{\frac{3}{8}} = 2{\cdot}37$, $10^{\frac{1}{2}} = 3{\cdot}16$, $10^{\frac{5}{8}} = 4{\cdot}22$, $10^{\frac{3}{4}} = 5{\cdot}62$, $10^{\frac{7}{8}} = 7{\cdot}50$. Also $10^0 = 1$, $10^1 = 10$.

We may therefore plot the values of 10^x for

$$x = 0, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{2}, \tfrac{5}{8}, \tfrac{3}{4}, \tfrac{7}{8}, 1.$$

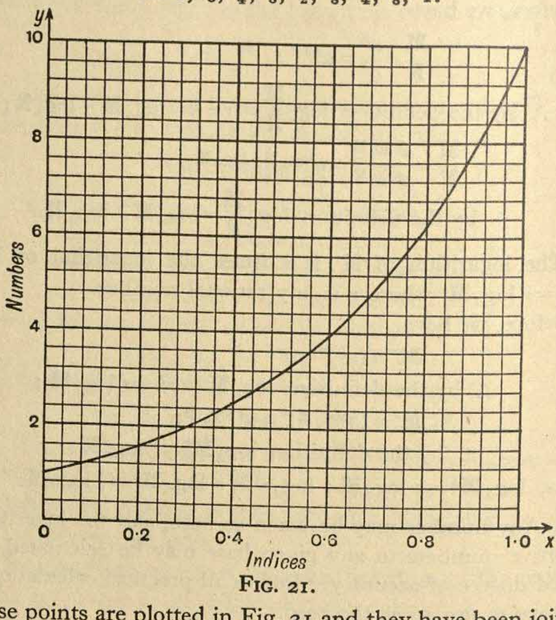


FIG. 21.

These points are plotted in Fig. 21 and they have been joined by a smooth curve. This curve is, in fact, the graph of $y = 10^x$ for values of x between 0 and 1, although we are not strictly justified in saying this until we have considered the values of 10^x for irrational values of x. The curve may be used to read off powers of x. Thus, from the curve we obtain $10^{0\cdot2} = 1{\cdot}6$, $10^{0\cdot6} = 4{\cdot}0$, in each case correct to 2 significant figures. We may also read off numbers as

powers of 10. Thus $2 = 10^{0 \cdot 30}$, $3 = 10^{0 \cdot 48}$, $5 = 10^{0 \cdot 70}$, etc. In other words, we have found from the graph that

$\log 2 = 0 \cdot 30$ approx., $\log 3 = 0 \cdot 48$ approx., $\log 5 = 0 \cdot 70$ approx.

It is clear that by using the graph we could obtain the logarithms of any number between 1 and 10, and that these logarithms all lie between 0 and 1. But the degree of accuracy obtained from the graph is insufficient for practical purposes, and it would be inconvenient to read off the logarithms of such numbers as 1·773. To obtain a higher degree of accuracy it is necessary to use the tables which are given at the end of the book. Whenever these tables are used it is implied that the base of logarithms is 10, and there is no need to write the suffix denoting the base.

USE OF FOUR-FIGURE TABLES

187. Numbers between 1 and 10. The following is an extract from the tables at the end of the book.

No.	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16

Example 1. *Find log* 2·3, *log* 2·32, *log* 2·328.

To find log 2·3 we find the number 23 in the left-hand column. Opposite to this and beneath the figure 0 we find the digits 3617, and since we know that the logarithm of a number between 1 and 10 lies between 0 and 1, we place the decimal point before the first digit. Thus, $\log 2 \cdot 3 = 0 \cdot 3617$, or $2 \cdot 3 = 10^{0 \cdot 3617}$.

To save space the decimal points are usually omitted in the table. It is understood that we always look up the logarithm of a number between 1 and 10, and that 0· is always prefixed to the number found in the table.

To find log 2·32, we find the number 23 in the left-hand column. Opposite to this and beneath the **first** figure 2 we find the digits 3655. Reasoning as above, we have

$$\log 2 \cdot 32 = 0 \cdot 3655 \quad \text{or} \quad 2 \cdot 32 = 10^{0 \cdot 3655}.$$

To find log 2·328, we proceed as before and obtain

$$\log 2 \cdot 32 = 0 \cdot 3655.$$

The increase required on account of the final figure 8 is obtained by adding to 3655 the number in the 23 row which appears under the **second** 8, i.e. under the 8 in the narrow columns on the right called difference columns. This number is 15, so that

$$\log 2{\cdot}328 = 0{\cdot}3670 \quad \text{or} \quad 2{\cdot}328 = 10^{0{\cdot}3670}.$$

Example 2. *Find the numbers whose logarithms are* 0·3856, 0·3861, 0·3862.

From the tables we see that the digits 3856 come in the 24 row under the first figure 3. But 0·3856 is a number between 0 and 1; it is therefore the logarithm of a number between 1 and 10. We conclude that $0{\cdot}3856 = \log 2{\cdot}43$.

The digits 3861 do not occur in the table, but can be obtained from 3856 by adding 5. The digit 5 occurs in the 24 row, in the difference columns beneath the figure 3. Thus $0{\cdot}3861 = \log 2{\cdot}433$.

The digits 3862 do not occur in the table, but can be obtained from 3856 by adding 6. The digit 6 does not occur in the difference columns in the row 24. But 6 is mid-way between 5 and 7 which occur in the difference columns beneath 3 and 4 respectively. It may be inferred that 0·3862 lies approximately mid-way between log 2·433 and log 2·434.

This last example serves to emphasise that the results obtained from the tables are approximate only. The numbers obtained from the tables are correct to 4 significant figures only, and we cannot always rely upon the accuracy of the fourth figure. It will be found that the results of calculations performed with the aid of 4-figure tables may *in general* be relied upon to 3 significant figures only. The pupil should therefore make a habit of giving his result to 3 significant figures whenever he uses 4-figure tables.

188. The number corresponding to a given logarithm is called its **antilogarithm**. Thus, in the last example 2·433 is the number whose logarithm is 0·3861; this is the same as saying that the antilogarithm of 0·3861 is 2·433, or $\text{antilog}\, 0{\cdot}3861 = 2{\cdot}433$.

In the last example it has been shown how to obtain from the logarithm tables the number corresponding to a given logarithm, i.e. the anti-logarithm of a given number. The beginner is strongly advised to use this method; later on he may prefer to use the antilogarithm tables given at the end of the book. But it should be

realised that antilogarithm tables are a luxury, not a necessity—a luxury which it is well to do without.

EXERCISE 89 (*Oral*)

Use tables to express as powers of 10 (i.e. find the logarithm of): Give your answer " $2 \cdot 3 = 10^{0 \cdot 3617}$, i.e. $\log 2 \cdot 3 = 0 \cdot 3617$ ".

1. 3·4.	**2.** 4·8.	**3.** 5·9.	**4.** 6·2.	**5.** 7.
6. 2·83.	**7.** 9·02.	**8.** 9·47.	**9.** 6·72.	**10.** 7·31.
11. 3·417.	**12.** 4·879.	**13.** 5·933.	**14.** 6·258.	**15.** 7·044.
16. 2·836.	**17.** 9·007.	**18.** 9·479.	**19.** 6·721.	**20.** 7·312.
21. 7·143.	**22.** 9·704.	**23.** 3·395.	**24.** 8·506.	**25.** 4·407.
26. 6·386.	**27.** 7·209.	**28.** 9·749.	**29.** 1·276.	**30.** 2·108.

Use tables to find the values of:

31. $10^{0 \cdot 1584}$.	**32.** $10^{0 \cdot 4548}$.	**33.** $10^{0 \cdot 9335}$.	**34.** $10^{0 \cdot 4239}$.	**35.** $10^{0 \cdot 8506}$.
36. $10^{0 \cdot 5326}$.	**37.** $10^{0 \cdot 6989}$.	**38.** $10^{0 \cdot 7887}$.	**39.** $10^{0 \cdot 0192}$.	**40.** $10^{0 \cdot 2548}$.

Use tables to find the numbers whose logarithms are:

41. 0·1644.	**42.** 0·4472.	**43.** 0·9304.	**44.** 0·4089.	**45.** 0·8981.
46. 0·5915.	**47.** 0·6963.	**48.** 0·8271.	**49.** 0·0224.	**50.** 0·2513.

189. Multiplication and division.

Example 3. *Find the value of* $3 \cdot 97 \times 2 \cdot 03$.

Rough estimate: $4 \times 2 = 8$.

Either $3 \cdot 97 \times 2 \cdot 03 = 10^{0 \cdot 5988} \times 10^{0 \cdot 3075} = 10^{0 \cdot 5988 + 0 \cdot 3075}$

$= 10^{0 \cdot 9063} = 8 \cdot 06$.

$$\begin{array}{r} 0 \cdot 5988 \\ \underline{0 \cdot 3075} \\ 0 \cdot 9063 \end{array}$$

Or Let $x = 3 \cdot 97 \times 2 \cdot 03$.

Then $\log x = \log 3 \cdot 97 + \log 2 \cdot 03$

$= 0 \cdot 5988 + 0 \cdot 3075 = 0 \cdot 9063$,

$\therefore \ x = 8 \cdot 06$.

Example 4. *Find the value of* $3 \cdot 97 \div 2 \cdot 03$.

Rough estimate: $3 \cdot 97 \div 2 = 1 \cdot 985$.

Either $3 \cdot 97 \div 2 \cdot 03 = 10^{0 \cdot 5988} \div 10^{0 \cdot 3075} = 10^{0 \cdot 5988 - 0 \cdot 3075}$

$= 10^{0 \cdot 2913} = 1 \cdot 956$

$= 1 \cdot 96$, correct to 3 sig. figs.

$$\begin{array}{r} 0 \cdot 5988 \\ \underline{0 \cdot 3075} \\ 0 \cdot 2913 \end{array}$$

Or Let $x = 3 \cdot 97 \div 2 \cdot 03$.

Then $\log x = \log 3 \cdot 97 - \log 2 \cdot 03$

$= 0 \cdot 5988 - 0 \cdot 3075 = 0 \cdot 2913$,

$\therefore \ x = 1 \cdot 956 = 1 \cdot 96$, correct to 3 sig. figs.

Note. The working should always show (1) the result as obtained from the tables, (2) the result correct to 3 sig. figs.

EXERCISE 90

Find the value of :

(Give the answers to 4 figures as given by the tables and also correct to 3 figures.)

1. $1{\cdot}32 \times 7{\cdot}07$. **2.** $2{\cdot}31 \times 3{\cdot}92$. **3.** $(2{\cdot}89)^2$.

4. $3{\cdot}431 \times 2{\cdot}117$. **5.** $6{\cdot}382 \times 1{\cdot}182$. **6.** $5{\cdot}934 \times 1{\cdot}343$.

7. $9{\cdot}35 \div 7{\cdot}19$. **8.** $8{\cdot}46 \div 5{\cdot}29$. **9.** $4{\cdot}117 \div 4{\cdot}039$.

10. $9{\cdot}859 \div 6{\cdot}093$. **11.** $7{\cdot}282 \div 5{\cdot}008$. **12.** $3{\cdot}223 \div 2{\cdot}112$.

13. $1{\cdot}13 \times 1{\cdot}35 \times 1{\cdot}57$. **14.** $1{\cdot}89 \times 1{\cdot}98 \times 2{\cdot}04$.

15. $3{\cdot}93 \times 1{\cdot}93 \times 1{\cdot}005$. **16.** $3{\cdot}93 \times 2{\cdot}58 \div 1{\cdot}78$.

17. $3{\cdot}77 \times 1{\cdot}87 \div 2{\cdot}19$. **18.** $9{\cdot}38 \times 1{\cdot}002 \div 6{\cdot}73$.

190. Numbers of any magnitude.

Example 5. *Find the logarithm of* $232{\cdot}8$.

In Ex. 1 above it was shown that $\log 2{\cdot}328 = 0{\cdot}3670$, or $2{\cdot}328 = 10^{0\cdot3670}$.

But $232{\cdot}8 = 2{\cdot}328 \times 100 = 10^{0\cdot3670} \times 10^2 = 10^{2\cdot3670}$,

$\therefore$ by the definition $\log 232{\cdot}8 = 2{\cdot}3670$.

Or Since $\log \mathbf{MN} = \log \mathbf{M} + \log \mathbf{N}$,

we have

$$\log 232{\cdot}8 = \log(2{\cdot}328 \times 100) = \log 2{\cdot}328 + \log 100$$
$$= 0{\cdot}3670 + 2, \text{ since } \log 100 = 2, \ = 2{\cdot}3670.$$

Similarly

$\log 23{\cdot}28 = 1{\cdot}3670$, $\log 2328 = 3{\cdot}3670$, $\log 23280 = 4{\cdot}3670$, etc.

Example 6. *Find the number whose logarithm is* $2{\cdot}3861$.

In Ex. 2 above it was shown that

$$0{\cdot}3861 = \log 2{\cdot}433 \quad \text{or} \quad 10^{0\cdot3861} = 2{\cdot}433.$$

But $10^{2\cdot3861} = 10^{0\cdot3861} \times 10^2 = 2{\cdot}433 \times 100 = 243{\cdot}3$,

$\therefore$ by the definition $2{\cdot}3861 = \log 243{\cdot}3$.

Or we have $2{\cdot}3861 = 0{\cdot}3861 + 2 = \log 2{\cdot}433 + \log 100$

$= \log(2{\cdot}433 \times 100)$, since $\log \mathbf{M} + \log \mathbf{N} = \log \mathbf{MN}$

$= \log 243{\cdot}3$.

Similarly

$1{\cdot}3861 = \log 24{\cdot}33$, $3{\cdot}3861 = \log 2433$, $4{\cdot}3861 = \log 24330$, etc.

Example 7. *Find the logarithm of* 0·02328.

In Ex. 1 above it was shown that

$$\log 2{\cdot}328 = 0{\cdot}3670 \quad \text{or} \quad 2{\cdot}328 = 10^{0\cdot3670}.$$

But $0{\cdot}02328 = 2{\cdot}328 \div 100 = 10^{0\cdot3670} \div 10^2 = 10^{0\cdot3670-2}$,

$\therefore$ by the definition $\log 0{\cdot}02328 = 0{\cdot}3670 - 2$.

Or Since $\log \dfrac{\mathbf{M}}{\mathbf{N}} = \log \mathbf{M} - \log \mathbf{N}$,

we have $\log 0{\cdot}02328 = \log(2{\cdot}328 \div 100) = \log 2{\cdot}328 - \log 100$

$$= 0{\cdot}3670 - 2.$$

191. Logarithms of numbers between 0 and 1 are negative, but it is usual to write them so that the decimal portion is positive; thus, we write $\log 0{\cdot}0238 = 0{\cdot}3670 - 2$, instead of $-1{\cdot}6330$. For brevity it is usually written $\bar{2}{\cdot}3670$, the "minus" being placed above the 2 to show that it refers only to the 2 and not to ·3670. It may be read

either "minus 2 plus point 3670",

or "bar 2 point 3670".

Similarly, $\log 0{\cdot}2328 = \bar{1}{\cdot}3670$, $\log 0{\cdot}0002328 = \bar{4}{\cdot}3670$, etc.

Example 8. *Find the number whose logarithm is* $\bar{2}{\cdot}3861$.

In Ex. 2 above it was shown that

$$0{\cdot}3861 = \log 2{\cdot}433 \quad \text{or} \quad 10^{0\cdot3861} = 2{\cdot}433.$$

But $10^{\bar{2}\cdot3861} = 10^{-2} \times 10^{0\cdot3861} = \frac{1}{100} \times 2{\cdot}433 = 0{\cdot}02433$,

$\therefore$ by the definition $\bar{2}{\cdot}3861 = \log 0{\cdot}02433$.

Or we have $\bar{2}{\cdot}3861 = 0{\cdot}3861 - 2 = \log 2{\cdot}433 - \log 100$

$= \log(2{\cdot}433 \div 100)$, since $\log \mathbf{M} - \log \mathbf{N} = \log \dfrac{\mathbf{M}}{\mathbf{N}}$

$= \log(0{\cdot}02433)$.

Similarly, $\bar{1}{\cdot}3861 = \log 0{\cdot}2433$, $\bar{3}{\cdot}3861 = \log 0{\cdot}002433$,

$\bar{4}{\cdot}3861 = \log 0{\cdot}0002433$, etc.

192. By considering Exs. 1, 5, 7 above the following results are easily seen:

(1) The logarithm of a number (or the index corresponding to a number) consists of two parts: an integral part (which may be positive, zero or negative) and a fractional part (which is usually written as a positive decimal).

The integral part is called the **characteristic**, and the fractional part, when written as a positive decimal, is called the **mantissa.**

(2) In finding the logarithm of a number (or the index corresponding to a number),

(*a*) all numbers with the same significant figures have the same mantissa,

(*b*) the position of the decimal point in the number determines the characteristic.

If the number is in standard form (i.e. if it lies between 1 and 10), the integral part of the logarithm (or index) is 0, and the 0 should be written down. If the number is not in standard form, it should be written as a number in standard form multiplied by a power of 10, e.g. 532·9 should be written $5{\cdot}329 \times 10^2$; 0·05329 should be written $5{\cdot}329 \times 10^{-2}$, etc.

(3) In finding the number which has a given logarithm (or the number corresponding to a given index),

(*a*) the fractional part of the logarithm or index determines the significant figures in the number,

(*b*) the integral part of the logarithm or index determines the position of the decimal point in the number.

Always write the number in the form (decimal part) plus or minus (integral part). This corresponds to a number in standard form multiplied or divided by a power of 10. If the integral part is 0, the corresponding number is in standard form.

EXERCISE 91 (*Oral*)

Use tables to find the logarithms of :

1. 34. **2.** 4800. **3.** 0·59. **4.** 0·062. **5.** 700.
6. 0·00283. **7.** 90200. **8.** 94·7. **9.** 0·672. **10.** 7310.
11. 34·17. **12.** 0·4879. **13.** 0·05933. **14.** 625800.
15. 704·4. **16.** 283·6. **17.** 0·9007. **18.** 0·009479.
19. 672·1. **20.** 73·12. **21.** 0·7143. **22.** 970·4.
23. 0·03395. **24.** 0·8506. **25.** 44070. **26.** 0·06386.
27. 0·7209. **28.** 974·9. **29.** 1276. **30.** 0·00002108

Use tables to find the values of :

31. $10^{2\cdot1584}$. **32.** $10^{1\cdot4548}$. **33.** $10^{\bar{2}\cdot9335}$. **34.** $10^{\bar{3}\cdot4239}$. **35.** $10^{3\cdot8506}$.
36. $10^{4\cdot5326}$. **37.** $10^{\bar{4}\cdot6989}$. **38.** $10^{5\cdot7887}$. **39.** $10^{3\cdot0192}$. **40.** $10^{\bar{1}\cdot2548}$.

Use tables to find the numbers whose logarithms are :

41. 1·1644. **42.** $\bar{2}\cdot4472$. **43.** 3·9304. **44.** $\bar{1}\cdot4089$. **45.** $\bar{3}\cdot8981$.
46. 4·5915. **47.** 2·6963. **48.** $\bar{1}\cdot8271$. **49.** $\bar{2}\cdot0224$. **50.** 3·2513.

193. Some preliminary practice in working with negative characteristics is desirable before proceeding to apply logarithms to more difficult calculations.

Example 9. *Simplify and express with the decimal portion positive :*

(i) $3{\cdot}43+\bar{6}{\cdot}82$, (ii) $\bar{2}{\cdot}383+\bar{1}{\cdot}822$.

(i)
$$\begin{array}{r} 3+\ {\cdot}43 \\ -6+\ {\cdot}82 \\ \hline -3+1{\cdot}25=\bar{2}{\cdot}25 \end{array}$$

(ii)
$$\begin{array}{r} -2+\ {\cdot}383 \\ -1+\ {\cdot}822 \\ \hline -3+1{\cdot}205=\bar{2}{\cdot}205 \end{array}$$

Example 10. *Simplify and express with the decimal portion positive :*

(i) $\bar{3}{\cdot}832-1{\cdot}912$, (ii) $\bar{1}{\cdot}412-\bar{3}{\cdot}816$.

(i)
$$\begin{array}{r} -3+{\cdot}832 \\ 1+{\cdot}912 \\ \hline -5+{\cdot}920=\bar{5}{\cdot}920 \end{array}$$

(ii)
$$\begin{array}{r} -1+{\cdot}412 \\ -3+{\cdot}816 \\ \hline 1+{\cdot}596=1{\cdot}596 \end{array}$$

Or (i) $\bar{3}{\cdot}832-1{\cdot}912=-3+{\cdot}832-(1+{\cdot}912)$
$=-3+1{\cdot}832-1-1-{\cdot}912=-5+{\cdot}920=\bar{5}{\cdot}920.$

(ii) $\bar{1}{\cdot}412-\bar{3}{\cdot}816=-1+{\cdot}412-(-3+{\cdot}816)$
$=-1+1{\cdot}412-1+3-{\cdot}816=1{\cdot}596.$

Mistakes frequently occur in such subtractions and it is essential that the pupil should make a habit of checking his subtraction by adding the result to the bottom line.

Example 11. *Simplify and express with the decimal portion positive :*

(i) $\bar{1}{\cdot}72\times 5$, (ii) $\bar{4}{\cdot}856\div 2$. (iii) $\bar{5}{\cdot}384\div 4$.

(i) $(-1+{\cdot}72)\times 5=-5+3{\cdot}6=-2+{\cdot}6=\bar{2}{\cdot}6.$

(ii) $(-4+{\cdot}856)\div 2=-2+{\cdot}428=\bar{2}{\cdot}428.$

(iii) $(-5+{\cdot}384)\div 4=(-8+3{\cdot}384)\div 4=-2+{\cdot}846=\bar{2}{\cdot}846.$

Note. In (iii) since -5 is not exactly divisible by 4, we write it in the form $-8+3$, so that after division the negative portion is an integer.

Mistakes frequently occur in such divisions, and it is essential that the student should make a habit of checking his division by multiplying the result by the divisor.

As soon as these processes are understood the work should be done mentally.

EXERCISE 92 (*Mainly oral*)

Simplify and express with the decimal portion positive:

1. $3{\cdot}22+\bar{4}{\cdot}19$. **2.** $\bar{3}{\cdot}17+\bar{1}{\cdot}44$. **3.** $\bar{2}{\cdot}83+\bar{1}{\cdot}96$.

4. $\bar{3}{\cdot}79+2{\cdot}41$. **5.** $\bar{4}{\cdot}55+2{\cdot}65$. **6.** $\bar{3}{\cdot}67+4{\cdot}33$.

7. $3{\cdot}22-\bar{4}{\cdot}19$. **8.** $\bar{3}{\cdot}17-\bar{1}{\cdot}44$. **9.** $\bar{2}{\cdot}83-\bar{1}{\cdot}96$.

10. $\bar{3}{\cdot}79-2{\cdot}41$. **11.** $2{\cdot}65-\bar{4}{\cdot}55$. **12.** $\bar{3}{\cdot}67-\bar{4}{\cdot}93$.

13. $0-1{\cdot}2$. **14.** $2{\cdot}64-5{\cdot}88$. **15.** $3{\cdot}69-3{\cdot}75$.

16. $0-\bar{3}{\cdot}28$. **17.** $\bar{1}{\cdot}8\times 3$. **18.** $\bar{2}{\cdot}3\times 5$.

19. $\bar{5}{\cdot}7\times 7$. **20.** $\bar{3}{\cdot}9\times 2$. **21.** $\bar{2}{\cdot}94\div 2$.

22. $\bar{6}{\cdot}87\div 3$. **23.** $\bar{3}{\cdot}94\div 2$. **24.** $\bar{2}{\cdot}89\div 3$.

25. $\bar{6}{\cdot}28\div 4$. **26.** $\bar{3}{\cdot}95\div 5$. **27.** $\bar{1}{\cdot}82\div 6$.

28. $\bar{9}{\cdot}44\div 4$. **29.** $\bar{8}{\cdot}26\div 3$. **30.** $\bar{2}{\cdot}38\div 5$.

31. $\bar{3}{\cdot}27\div 7$. **32.** $\bar{4}{\cdot}22\div 9$. **33.** $2{\cdot}7\times(-3)$.

34. $\bar{2}{\cdot}7\times(-3)$. **35.** $2{\cdot}7\div(-3)$. **36.** $\bar{2}{\cdot}8\div(-3)$.

194. Example 12. *Find the value of* $(0{\cdot}3937)^5$.

Rough estimate: $(0{\cdot}4)^5=0{\cdot}01024$.

$$\begin{array}{r}\bar{1}{\cdot}5952\\ \underline{5}\\ \bar{3}{\cdot}9760\end{array}$$

Either $(0{\cdot}3937)^5=(10^{\bar{1}\cdot 5952})^5=10^{\bar{1}\cdot 5952\times 5}=10^{\bar{3}\cdot 9760}$

$=9{\cdot}462\times 10^{-3}$ or $9{\cdot}463\times 10^{-3}$

$=0{\cdot}00946$, correct to 3 sig. figs.

Or Let $x=(0{\cdot}3937)^5$.

Then $\log x=5\log(3{\cdot}937\div 10)=5\times\bar{1}{\cdot}5952=\bar{3}{\cdot}9760$,

$\therefore\ x=9{\cdot}462\div 10^3$ or $9{\cdot}463\div 10^3$

$=0{\cdot}00946$, correct to 3 sig. figs.

Example 13. *Find the value of* $\sqrt[3]{0{\cdot}4326}$.

Rough estimate: $0{\cdot}4326$ lies between $(0{\cdot}7)^3$, i.e. $0{\cdot}343$ and $(0{\cdot}8)^3$ i.e. $0{\cdot}512$.

Either $\sqrt[3]{0{\cdot}4326}=(10^{\bar{1}\cdot 6361})^{\frac{1}{3}}=10^{\bar{1}\cdot 6361\times\frac{1}{3}}$

$=10^{\bar{1}\cdot 8787}=7{\cdot}563\times 10^{-1}$ or $7{\cdot}564\times 10^{-1}$

$=0{\cdot}756$, correct to 3 sig. figs.

$$3\,\underline{|\,\bar{1}{\cdot}6361}\\ \quad\ \ \bar{1}{\cdot}8787$$

Or Let $x=\sqrt[3]{0{\cdot}4326}$.

Then $\log x=\frac{1}{3}\log(4{\cdot}326\div 10)=\frac{1}{3}\times\bar{1}{\cdot}6361=\bar{1}{\cdot}8787$,

$\therefore\ x=7{\cdot}563\div 10$ or $7{\cdot}564\div 10=0{\cdot}756$, correct to 3 sig. figs.

195. When the principles have been grasped the working may be set out as in the following examples.

Example 14. *Find the value of* $\dfrac{(0{\cdot}6321)^4 \times \sqrt[3]{0{\cdot}0253}}{6\frac{1}{3} \times \sqrt[5]{0{\cdot}002037}}$.

To avoid replacing $6\frac{1}{3}$ by an inexact decimal, we write the expression in the form $\dfrac{3 \times (0{\cdot}6321)^4 \times \sqrt[3]{0{\cdot}0253}}{19 \times \sqrt[5]{0{\cdot}002037}}$.

Number	Logarithm	
3		$0{\cdot}4771$
$(0{\cdot}6321)^4$	$\bar{1}{\cdot}8008 \times 4$	$\bar{1}{\cdot}2032$
$\sqrt[3]{0{\cdot}0253}$	$\bar{2}{\cdot}4031 \times \frac{1}{3}$	$\bar{1}{\cdot}4677$
Numerator		$\bar{1}{\cdot}1480$
19		$1{\cdot}2788$
$\sqrt[5]{0{\cdot}002037}$	$\bar{3}{\cdot}3090 \times \frac{1}{5}$	$\bar{1}{\cdot}4618$
Denominator		$0{\cdot}7406$

Number	Logarithm
Numerator	$\bar{1}{\cdot}1480$
Denominator	$0{\cdot}7406$
Expression	$\bar{2}{\cdot}4074$

$\therefore$ expression $= 2{\cdot}555 \div 10^2$
$= 0{\cdot}0255$ or $0{\cdot}0256$, correct to 3 sig. figs.

Example 15. *Find the value of* $(0{\cdot}00313)^{-1{\cdot}2}$.

Number	Logarithm	
$(0{\cdot}00313)^{-1{\cdot}2}$	$\bar{3}{\cdot}4955 \times (-1{\cdot}2)$	$3{\cdot}0054$
	$= (-3 + 0{\cdot}4955)(-1{\cdot}2)$	
	$= 3{\cdot}6 - 0{\cdot}5946$	
	or	
	$= (-2{\cdot}5045)(-1{\cdot}2)$	

$\therefore$ expression $= 1{\cdot}012 \times 10^3$ or $1{\cdot}013 \times 10^3$
$= 1010$, correct to 3 sig. figs.

Example 16. *Find the value of* $(1{\cdot}938)^5 - \sqrt[7]{21{\cdot}83}$.

Here the terms must be calculated separately.

Number	Logarithm	
(1) $(1{\cdot}938)^5$	$0{\cdot}2874 \times 5$	$1{\cdot}4370$, $\therefore (1{\cdot}938)^5 = 27{\cdot}35$,
(2) $(21{\cdot}83)^{\frac{1}{7}}$	$1{\cdot}3391 \times \frac{1}{7}$	$0{\cdot}1913$, $\therefore \sqrt[7]{21{\cdot}83} = 1{\cdot}554$;

$\therefore$ the given expression $= 25{\cdot}796$
$= 25{\cdot}8$, correct to 3 sig. figs.

EXERCISE 93

Find the values of:

(Give the answers to 4 figures as given by the tables, and also correct to 3 figures.)

1. $44{\cdot}8 \times 21{\cdot}9$. **2.** $838 \times 12{\cdot}7$. **3.** $4366 \times 1{\cdot}372$.

4. $24400 \times 10{\cdot}32$. **5.** $57{\cdot}57 \div 21{\cdot}52$. **6.** $703{\cdot}9 \div 22{\cdot}91$.

7. $87650 \div 291 \cdot 9$. 8. $2468 \div 1357$. 9. $(12 \cdot 33)^3$.
10. $(14 \cdot 97)^5$. 11. $\sqrt{28 \cdot 5}$. 12. $\sqrt[3]{487 \cdot 6}$.
13. $10^{3 \cdot 4237}$. 14. $(27 \cdot 35)^{\frac{2}{3}}$. 15. $20 \div 1 \cdot 031$.
16. $(11 \cdot 47)^2$. 17. $\sqrt{2}\,\sqrt{95}$. 18. $3 \cdot 142 \times (15 \cdot 29)^2$.
19. $\dfrac{3 \cdot 1 \times 31}{23 \cdot 26}$. 20. $\dfrac{1 \cdot 483 \times 193}{21 \cdot 22 \times 10 \cdot 09}$. 21. $\dfrac{23 \times 71 \cdot 4}{4 \cdot 79}$.
22. $0 \cdot 317 \times 44 \cdot 2$. 23. $0 \cdot 819 \times 0 \cdot 234$. 24. $0 \cdot 0089 \times 21 \cdot 73$.
25. $0 \cdot 09581 \times 0 \cdot 1423$. 26. $0 \cdot 06655 \times 249$.
27. $0 \cdot 0009837 \times 52 \cdot 93$. 28. $0 \cdot 7832 \div 11 \cdot 43$.
29. $3 \cdot 684 \div 16 \cdot 87$. 30. $0 \cdot 003259 \div 0 \cdot 00083$.
31. $16 \cdot 47 \div 0 \cdot 09229$. 32. $433 \cdot 4 \div 0 \cdot 7117$. 33. $32 \cdot 23 \div 899 \cdot 8$.
34. $1 \div 0 \cdot 07654$. 35. $\dfrac{578 \times 0 \cdot 845}{23}$. 36. $\dfrac{0 \cdot 06275 \times (37 \cdot 21)^2}{2 \cdot 731}$.
37. $\sqrt[3]{0 \cdot 07896}$. 38. $\sqrt{0 \cdot 0147}$. 39. $\sqrt[3]{0 \cdot 7896}$.
40. $(0 \cdot 1846)^{\frac{2}{3}} \div 6 \cdot 206$. 41. $(0 \cdot 395)^3$. 42. $(0 \cdot 08341)^2$.
43. $1 \div (1 \cdot 025)^{30}$. 44. $(0 \cdot 5621)^{\frac{2}{3}}$. 45. $29 \cdot 57 \div 0 \cdot 084$.
46. $\dfrac{4 \cdot 143 \sqrt[3]{352}}{6 \cdot 952}$. 47. $\dfrac{\sqrt{25 \cdot 67 \times 0 \cdot 323}}{453 \cdot 1}$.
48. $(0 \cdot 3714)^{\frac{1}{3}} \times (42 \cdot 38)^{-\frac{2}{5}}$. 49. $9 \cdot 546 \times 36 \cdot 25 \times 0 \cdot 373$.
50. $(26 \cdot 54)^2 \times 0 \cdot 3377 \div 121 \cdot 8$. 51. $34 \cdot 47 \times (59 \cdot 11)^2 \div 1937$.
52. $2\frac{6}{7} \times 4744 \div (1\frac{5}{6} \times 214 \cdot 8)$. 53. $1000 \div (0 \cdot 9989)^3$.
54. $5 \cdot 871 \times 0 \cdot 005437 \div (112 \times 0 \cdot 001525)$.
55. $42 \times \sqrt{0 \cdot 338} \div (8 \cdot 87)^2$. 56. $\sqrt[3]{477 \div 3 \cdot 142}$.
57. $\sqrt{1 \div 0 \cdot 003129}$. 58. $\sqrt{\dfrac{67 \cdot 58}{9 \cdot 765}} \div (3 \cdot 57)^2$.
59. $\sqrt[3]{\dfrac{2805 \times 45 \cdot 34}{777 \times 0 \cdot 7001}}$. 60. $\sqrt[5]{3\frac{3}{7} \times 82 \cdot 37 \times 15\frac{2}{11}}$.

In the following examples, take $\log \pi = 0 \cdot 4971$.

61. The Simple Interest £I on £P for t years at r per cent. per annum is given by $I = \dfrac{Ptr}{100}$. Find I if $P = 68 \cdot 73$, $r = 2 \cdot 25$, $t = 3 \cdot 6$.

62. If a body starts from rest with uniform acceleration f ft. per sec. per sec., the velocity, v ft. per sec., acquired in passing through s ft. is given by $v^2 = 2fs$. Find (i) v if $f = 32 \cdot 2$, $s = 19 \cdot 7$, (ii) s if $f = 18 \cdot 09$, $v = 113$.

63. At a height of h ft., the distance of the horizon is d miles, where $d = \sqrt{1 \cdot 5h}$. Find (i) d when $h = 221$, (ii) h when $d = 5 \cdot 113$.

64. The area of an equilateral triangle, side a cm. long, is A sq. cm., where $A=\frac{\sqrt{3}a^2}{4}$. Find (i) A when $a=3\cdot171$, (ii) a when $A=33\cdot83$.

65. The volume of a cylinder, base-radius r cm., height h cm., is V cu. cm., where $V=\pi r^2 h$ and $r=\sqrt{V\div\pi h}$. Find (i) V if $r=7\cdot62$, $h=3\cdot15$, (ii) r if $V=164$, $h=3\cdot65$.

66. A body falls freely from rest to the ground from a height h ft. The time taken is t sec., where $t=\sqrt{2h\div g}$, and $g=32\cdot2$. Find (i) t when $h=75$, (ii) h when $t=2\cdot37$.

67. The volume of a cylinder l ft. long, outer radius r in., thickness t in. is V cu. in., where $V=12\pi lt(2r-t)$. Find (i) V if $l=0\cdot65$, $r=3$, $t=0\cdot2$, (ii) l if $V=260$, $r=4\cdot7$, $t=0\cdot35$.

68. If a sum of money, £P, is lent at Compound Interest at r per cent. per annum, it amounts in n years to £$P\left(1+\frac{r}{100}\right)^n$. Find the amount if (i) $P=500$, $r=3\cdot5$, $n=4$, (ii) $P=282\cdot5$, $r=4$, $n=3$.

EXERCISE 93. c

Find the values of :

(Give the answers to 4 figures as given by the tables, and also correct to 3 figures, unless otherwise stated.)

1. $\frac{3\cdot75}{(1\cdot63)^2}$.

2. $\frac{17\cdot65\times0\cdot0437}{7\cdot46}$.

3. $\frac{4\cdot732\times(0\cdot1785)^2}{\sqrt{276\cdot9}}$.

4. $\frac{1\cdot097\times0\cdot0823}{0\cdot9036}$.

5. $\sqrt{\frac{9\cdot347\times10\cdot73}{55\cdot09\times0\cdot173}}$.

6. $\frac{13\cdot93-7\cdot27}{13\cdot93\times7\cdot27}$.

7. $2^{1\cdot133}$.

8. $\sqrt[7]{77}$.

9. $(0\cdot08009)^{0\cdot6}$.

10. $\frac{2\cdot83\times\sqrt{472}}{5\cdot16\times1\cdot03}$.

11. $\left(\frac{13\cdot93-7\cdot27}{13\cdot93\times7\cdot27}\right)^2$.

12. $\frac{2731\times(0\cdot0354)^3}{\sqrt[5]{0\cdot224}}$.

13. $(0\cdot3149)^{0\cdot23}$.

14. $1\cdot713\times2\cdot718^{3\cdot142}$.

15. $\sqrt[5]{6\cdot3}-\sqrt[3]{0\cdot63}$.

16. $0\cdot55\left[\frac{175}{11}(1+\log_{10}11)-18\right]$.

17. $\log_{10}\left[\frac{1}{55}\left(\frac{76\cdot2}{304\cdot3}+\frac{4\cdot5}{5\cdot69}\right)\right]$.

18. $\left(\frac{3\cdot721}{3}\right)^{-2}+(3\cdot721)^{\frac{4}{3}}$.

19. $\sqrt[4]{2\cdot56}\div\sqrt[5]{0\cdot658}$.

20. $\frac{(1\cdot246)^3-1}{(1\cdot246)^3+1}$.

21. $\dfrac{0{\cdot}15+(0{\cdot}15)^{\frac{2}{3}}}{1{\cdot}15}\times 3{\cdot}046.$ **22.** $\dfrac{(19{\cdot}37)^2-(9{\cdot}14)^2}{1273}.$

23. $\sqrt[3]{\dfrac{59{\cdot}26\times(1{\cdot}414)^2}{0{\cdot}022\times 365}}.$ **24.** $\sqrt{78{\cdot}96\times 0{\cdot}2895-0{\cdot}7896\times 2{\cdot}895}.$

25. $\dfrac{(0{\cdot}8714)^3\times(2{\cdot}051)^{\frac{2}{3}}}{\sqrt[4]{0{\cdot}06333}}.$ **26.** $\dfrac{(0{\cdot}3456)^{\frac{2}{3}}\times 27{\cdot}81}{(0{\cdot}02459)^{\frac{1}{5}}}.$

27. $(5{\cdot}62)^{-0{\cdot}3}.$ **28.** $(0{\cdot}704)^{-1{\cdot}4}.$ **29.** $(740{\cdot}2)^{-3{\cdot}5}.$

30. $(0{\cdot}0065)^{-\frac{3}{4}}.$ **31.** $(0{\cdot}04)^{2{\cdot}7}.$ **32.** $(0{\cdot}2)^{0\ 2}.$

In the following examples take $\log\pi=0{\cdot}4971$.

33. The volume V cu. cm. of a sphere is given by the formula $V=\dfrac{4\pi r^3}{3}$, where r cm. is the radius. Find (i) the volume of a sphere of radius $2{\cdot}51$ cm., (ii) the radius of a sphere whose volume is 7296 cm.

34. Given the formula $s=\dfrac{1{\cdot}27M}{d^2h}$, find d when $s=8{\cdot}9$, $M=2{\cdot}51$, $h=15$.

35. Evaluate $\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}$, when $a=351$, $b=41$, $c=320$, $2s=a+b+c$.

36. If $a^2=b^2+c^2$, find b if $a=23{\cdot}81$, $c=6{\cdot}19$.

37. Evaluate $\sqrt{a^3\div b}$, where $a=0{\cdot}702$, $b=0{\cdot}189$.

38. Evaluate $a^2b^3c^{-\frac{1}{2}}$, where $a=24{\cdot}67$, $b=0{\cdot}0426$, $c=0{\cdot}007843$.

39. Evaluate $Mv^2\div gr$, where $M=35$, $v=5{\cdot}82$, $g=32{\cdot}2$, $r=1{\cdot}843$.

40. Evaluate $\pi\sqrt{l\div g}$, where $l=3{\cdot}25$, $g=32{\cdot}2$.

41. Calculate A from the formula $A=P\,.\,\dfrac{R^t-1}{R-1}$, given that $P=26{\cdot}67$, $R=1{\cdot}055$, $t=20$.

42. Evaluate $a\sqrt{\dfrac{b}{c^3}}$, where $a=47{\cdot}2$, $b=0{\cdot}3413$, $c=0{\cdot}265$.

43. Evaluate $\sqrt{\dfrac{h}{r}}+\sqrt{\dfrac{r}{h}}$, where $h=0{\cdot}5491$, $r=0{\cdot}05972$.

44. Evaluate $\pi d(d+2h)\div 4$, where $d=0{\cdot}967$, $h=4{\cdot}162$.

45. Evaluate $e^{-\frac{rt}{l}}$, where $e=2{\cdot}718$, $r=1{\cdot}7$, $l=0{\cdot}98$, $t=1{\cdot}8$.

46. If $y=\sqrt{\dfrac{1{\cdot}63x^5}{(P+0{\cdot}4x)^3}}$, find y when $x=23{\cdot}2$, $P=15{\cdot}68$.

47. A formula for F, the frictional resistance of an aeroplane of plane area A, moving with velocity v, is $F=kA^{0{\cdot}93}v^{1{\cdot}86}$. Find F, to the nearest unit, when $k=8{\cdot}2\times 10^{-6}$, $A=2500$, $v=132$.

48. If $\frac{4}{3}\pi e r^3 = 24{\cdot}2$, and $e = 7{\cdot}7$, calculate the value of $4\pi r^2$.

49. Evaluate N where

$$N = \frac{2\pi^2 e^4 m}{h^3},\ h = 6{\cdot}545 \times 10^{-27},\ e = 4{\cdot}774 \times 10^{-10},\ \frac{e}{m} = 1{\cdot}767 \times 10^7.$$

50. If $W = C^2R + \frac{t^2}{R}$, find the value of W when

$$R = 0{\cdot}044,\ C = 41{\cdot}1,\ t = 1{\cdot}8.$$

51. If $n = 500\left(\frac{h}{6}\right)^{1{\cdot}25} p^{-0{\cdot}5}$, find the value of n, when $h = 20$, $p = 75$.

52. The area of the curved surface of a right circular cone of height h and radius of base r is given by $\pi r\sqrt{h^2 + r^2}$. Find the area of the curved surface of such a cone, if the height is 20·3 cm., and the base-radius is 12·7 cm.

53. If $P = cd^{5{\cdot}5}n^{3{\cdot}5}$, find c when $d = 0{\cdot}6$, $n = 50$, $P = 0{\cdot}15$.

54. If $D = 5Wl^3 \div 384EI$, find D when $W = 1500$, $l = 240$, $E = 600 \times 2240$, $I = 54$.

55. Find V from the formula $V = 0{\cdot}56\sqrt{2g(H-h)}$, when $g = 32{\cdot}2$, $H = 125{\cdot}3$, $h = 12$.

56. Evaluate $\sqrt[3]{a^2b^5c^{\frac{1}{2}}}$, where $a = 2{\cdot}53$, $b = 34{\cdot}7$, $c = 0{\cdot}018$.

196. We now proceed to consider further examples illustrating the general propositions proved in Art. 184.

Example 17. *Without using tables, find the value of:*

(i) $\log_{10}35 + \log_{10}\frac{6}{7} + \log_{10}3\frac{1}{3}$,

(ii) $\log 2\frac{2}{5} + \log 2\frac{1}{7} - \log 5\frac{1}{7}$.

(i) Since $\log a + \log b + \log c = \log(a \times b \times c)$,

the expression $= \log_{10}(35 \times \frac{6}{7} \times \frac{10}{3}) = \log_{10}100 = 2$.

(ii) Since $\log a + \log b - \log c = \log(a \times b) - \log c = \log\left(\frac{a \times b}{c}\right)$,

the expression $= \log(\frac{12}{5} \times \frac{15}{7} \div \frac{36}{7}) = \log(\frac{12}{5} \times \frac{15}{7} \times \frac{7}{36}) = \log 1 = 0$.

This result is true whatever base is chosen.

Example 18. *Without using tables, find the values of:*

(i) $\frac{\log 625}{\log 125}$, (ii) $10^{\log_{10}7}$.

(i) $\frac{\log 625}{\log 125} = \frac{\log 5^4}{\log 5^3} = \frac{4\log 5}{3\log 5} = \frac{4}{3} = 1\frac{1}{3}$.

(ii) Since $N = 10^{\log_{10}N}$ by definition, it follows that $10^{\log_{10}7} = 7$.

But beginners often find difficulty in realising that N and $10^{\log_{10}N}$ are identical and they may prefer to proceed as follows :

Let $$x = 10^{\log_{10}7}.$$

Take logarithms of each side to base 10.

Then $\log_{10}x = \log_{10}7 \times \log_{10}10 = \log_{10}7 \times 1 = \log_{10}7$,

$$\therefore\ x = 7.$$

Note. If the base is not specified, as in Ex. 17 (ii) and Ex. 18 (i), it is implied that all logarithms which occur have the same base, unless otherwise stated. The expressions in those examples are abbreviations for $\log_a 2\frac{2}{5} + \log_a 2\frac{1}{7} - \log_a 5\frac{1}{7}$ and $\dfrac{\log_a 625}{\log_a 125}$ respectively.

It should also be noted that examples such as Ex. 17 (i) have frequently been set in examinations in the form : "without using tables, find the value of $\log 35 + \log \frac{6}{7} + \log 3\frac{1}{3}$".

It is then implied that the base of logarithms is 10, i.e. that the logarithms are common logarithms. It is necessary for the pupil to be acquainted with these conventions.

Example 19. *Solve* (i) $4^x = 32$, (ii) $4^x = 21$.

(i) We have $$4^x = 32, \quad \therefore\ (2^2)^x = 2^5,$$

$$\therefore\ 2^{2x} = 2^5, \quad \therefore\ 2x = 5, \quad \therefore\ x = 2\tfrac{1}{2}.$$

(ii) We have $$4^x = 21.$$

Take logarithms of each side to base 10.

Then $x \log_{10}4 = \log_{10}21, \quad \therefore\ x(0{\cdot}6021) = 1{\cdot}3222.$ ....(i)

$\therefore\ x = \dfrac{1{\cdot}322}{0{\cdot}6021} = 2{\cdot}20$, by ordinary division, correct to 3 sig. figs.

It should be carefully noted that after taking logarithms we obtain a simple equation containing x, i.e. (i) is an ordinary simple equation. The beginner is sometimes confused by this process, because the equations previously obtained in this chapter have given the value of $\log x$, and it has then been necessary to obtain x from the tables.

The value of $\dfrac{1{\cdot}322}{0{\cdot}6021}$ may of course be obtained by logarithms, if desired, but the beginner is advised to avoid the double use of logarithms, except possibly to check the result obtained by ordinary division.

Example 20. *Solve*:

(i) $(0\cdot93)^x=1\cdot832$, (ii) $6^{3x+1}\,.\,3^{15x-2}=21$, (iii) $4^{x+y}=5$, $3^{x-y}=7$.

(i) Taking logarithms of each side to base 10, we have

$$x\log(0\cdot93)=\log 1\cdot832,\quad \therefore\ x(\bar{1}\cdot9685)=0\cdot2630,$$

$\therefore\ x(-0\cdot0315)=0\cdot2630$, $\therefore\ x=-\dfrac{0\cdot2630}{0\cdot0315}=-8\cdot35$, by ordinary division.

If it is desired to do the final step by logarithms, care must be exercised, for **we cannot find the logarithm to base 10 of a negative quantity.** But we may proceed as follows:

$$\log(-x)=\log\frac{0\cdot2630}{0\cdot0315}=\log 0\cdot2630-\log 0\cdot0315=0\cdot9217,\qquad \begin{array}{r}\bar{1}\cdot4200\\ \underline{\bar{2}\cdot4983}\\ 0\cdot9217\end{array}$$

$\therefore\ -x=8\cdot350$, $\therefore\ x=-8\cdot35$, correct to 3 sig. figs.

(ii) Taking logarithms of each side to base 10, we have

$$(3x+1)\log 6+(15x-2)\log 3=\log 21,$$
$$\therefore\ x(3\log 6+15\log 3)=\log 21-\log 6+2\log 3,$$
$$\therefore\ x(2\cdot3346+7\cdot1565)=1\cdot3222-0\cdot7782+0\cdot9542,$$
$$\therefore\ x(9\cdot4911)=1\cdot4982,\quad \therefore\ x=0\cdot16 \text{ approx.}$$

(iii) Taking logarithms to base 10 of each side of each equation, we have

$$(x+y)\log 4=\log 5\ ;\quad (x-y)\log 3=\log 7,$$
$$\therefore\ x+y=\frac{\log 5}{\log 4}=\frac{0\cdot699}{0\cdot6021}=1\cdot161 \text{ approx.,}$$
$$x-y=\frac{\log 7}{\log 3}=\frac{0\cdot8451}{0\cdot4771}=1\cdot771 \text{ approx.,}$$
$$\therefore\ 2x=2\cdot932,\ 2y=-0\cdot61\ ;$$
$$\therefore\ x=1\cdot466=1\cdot5,\quad y=-0\cdot305=-0\cdot3,$$

correct to 1 dec. place in each case.

197. Example 21. *Given* $\log_{10}2=0\cdot30103$, $\log_{10}3=0\cdot47712$, $\log_{10}7=0\cdot84510$, *find, without using tables, the value of* (i) $\log_{10}5$, (ii) $\log_{10}\sqrt[3]{1176}$.

(i) $\log_{10}5=\log_{10}(10\div2)=\log_{10}10-\log_{10}2$
$$=1-0\cdot30103=0\cdot69897.$$

(ii) $\log_{10}\sqrt[3]{1176}=\frac{1}{3}\log_{10}(2^3.\,3\,.\,7^2)$
$$=\tfrac{1}{3}[3\log_{10}2+\log_{10}3+2\log_{10}7]$$
$$=\tfrac{1}{3}[0\cdot90309+0\cdot47712+1\cdot69020]$$
$$=\tfrac{1}{3}(3\cdot07041)=1\cdot02347.$$

Example 22. *If* $3\log_{10}\sqrt{y}+2\log_{10}x=3$, *express* y *in terms of* x.

We have $3\log_{10}y^{\frac{1}{2}}+2\log_{10}x=3$, $\therefore\ \log_{10}y^{\frac{3}{2}}+\log_{10}x^2=3$,

$$\therefore\ \log_{10}y^{\frac{3}{2}}x^2=\log_{10}1000,\quad \therefore\ y^{\frac{3}{2}}x^2=1000,$$

$$\therefore\ y^{\frac{3}{2}}=1000x^{-2},\quad \therefore\ y=(1000x^{-2})^{\frac{2}{3}}=100x^{-\frac{4}{3}}.$$

198. If we have a table of logarithms calculated for some given base a, it is possible to deduce the logarithms for any other base b.

Suppose that it is required to find the logarithm of x to base b.

Let $\log_b x=k$, then by definition $b^k=x$.

Taking logarithms of each side to base a, we have

$$k\log_a b=\log_a x,\quad \therefore\ k=\frac{\log_a x}{\log_a b},$$

$$\text{i.e. } \mathbf{\log_b x=\frac{\log_a x}{\log_a b}}.$$

Or By definition, $x=b^{\log_b x}$.

Taking logarithms of each side to base a, we have

$$\log_a x=\log_a(b^{\log_b x})=\log_b x\ .\ \log_a b, \text{ etc., as above.}$$

Note. If $x=a$, the above result becomes $\log_b a=1\div\log_a b$.

In practice it is unwise to rely upon this formula; it is wiser to work from first principles in each case.

Example 23. *Find* $\log_4 21$.

Let $x=\log_4 21$. Then by definition $4^x=21$,

$\therefore\ x=2{\cdot}20$, the working being as in Ex. 19 (ii).

199. Determination of laws from experimental data (continued). The law $\mathbf{y=kx^n}$. In Chapter XVIII we have shown that in certain cases a series of plotted points may be used to determine a linear equation between two variables whose values have been found experimentally. If the graph is not linear, the problem is not always easily solved, but there is one very important case, which occurs frequently, in which the use of logarithms reduces the problem to that already dealt with.

If x and y satisfy an equation of the form $y=kx^n$, where k, n are constants, we have, by taking logarithms,

$$\log y=\log k+n\log x \text{ or } Y=K+nX,$$

where $Y=\log y$, $K=\log k$, $X=\log x$.

This is a linear equation in X and Y. If, therefore, the values of X, Y, i.e. of $\log x$, $\log y$, are plotted against each other, the points lie on a straight line and the values of the constants can be found.

We conclude that, if difficulty is found in deducing an equation from plotted points, the logarithms of the variables should be plotted. If, after allowing for small experimental errors, the points lie on a line, we can deduce an equation of the type $y = kx^n$.

Example 24. *The table gives corresponding values of x and y found by experiment. Test whether the law is of the type $y = kx^n$, and if so, find the values of k and n.*

x ……	20	24	28	32	36	40
y ……	108	60	40	24	16·8	12·8

From the tables, we have

$\log x$ …	1·3010	1·3802	1·4472	1·5051	1·5563	1·6021
$\log y$ …	2·0334	1·7782	1·6021	1·3802	1·2253	1·1072

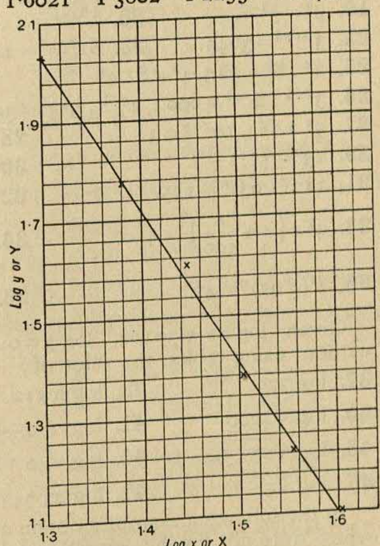


FIG. 22.

After allowing for experimental errors, the graph is as shown in Fig. 22.

The equation of the graph is $Y = K + nX$.

To obtain K and n, choose two suitable points *on the line*, say (1·4, 1·72), (1·55, 1·25). Since these lie on the line, we have

$$1{\cdot}72 = K + 1{\cdot}4n$$

and $$1{\cdot}25 = K + 1{\cdot}55n,$$

$$\therefore\ 0{\cdot}47 = -0{\cdot}15n,$$

$$\therefore\ n = -3{\cdot}1 \text{ approx.}$$

Also $K = 1{\cdot}25 + 1{\cdot}55 \times \frac{47}{15}$
$= 6{\cdot}1067$ approx

$\therefore\ \log k = 6{\cdot}1067$ approx.,

$\therefore\ k = 1{\cdot}28 \times 10^6$ approx.,

$\therefore$ the required equation is $x^{3\cdot1}y = 1{\cdot}28 \times 10^6$ approx.

EXERCISE 94

(*In this exercise log x means $\log_{10} x$, unless otherwise stated*)

Without using tables, find the value of :

1. $\log_6 12 + \log_6 3$. **2.** $\log_4 128 - \log_4 8$. **3.** $8^{\log_8 3}$.

4. $\log_2 36 - \frac{1}{2}\log_2 81$. **5.** $\dfrac{\log 27}{\log 3}$. **6.** $\dfrac{\log_a 64}{\log_a 32}$.

7. $\dfrac{\log 216}{\log \frac{1}{6}}$. **8.** $\log 11 + \log \dfrac{1}{11}$.

9. $\log 42 + \log 1\frac{1}{3} + \log 17\frac{6}{7}$. **10.** $\log 35 - \log 1\frac{3}{4} + \log 5$.

11. $\log \frac{3}{5} + 2\log 2\frac{1}{2} - \log \frac{5}{36} - \log 27$.

12. $\log 2\frac{17}{32} + 3\log 1\frac{2}{3} + \log \frac{1}{9} + \log 768$.

13. $\log\sqrt{84} + \log\sqrt{650} - \log\sqrt{546}$.

14. $\log 5 + \log\sqrt{7} + \frac{1}{2}\log 12 - \frac{1}{2}\log 21$. **15.** $\log \dfrac{1000}{\sqrt{10}}$.

16. $\log_{16} 8$. **17.** $10^{\log 9}$. **18.** $\log 10^{\bar{1}\cdot 7}$.

Solve (if possible, without using tables) :

19. $9^x = 3^{2-x}$. **20.** $2^{3x-2} = 4 \cdot 3^{x-1}$. **21.** $3 \cdot 9^x = 17{\cdot}4$.

22. $3^{5x+4} = 3645$. **23.** $81^{2x+3} = 243^{5-x}$. **24.** $5^{2x} = 121^{x+1} \cdot 7^{1-2x}$.

25. $2^x \cdot 8^y = 32$; $4^x = 8 \cdot 2^y$.

26. $3^{x+1} \cdot 2^{y+1} = 600$; $2^{x+1} \cdot 3^{y+1} = 300$.

27. $5^x = 7^y$; $10^{x-y} = 3$. **28.** $2^{3x} = 8^{2y+1}$; $9^{2y} = 3^{3x-9}$.

29. $25^x = 5^{x+1} - 6$. **30.** $3^{2x} - 6 \cdot 3^x = 567$.

31. $112(1{\cdot}03)^n = 135$. **32.** $430(1{\cdot}025)^n = 600$.

33. $412\left(1 + \dfrac{r}{100}\right)^3 = 450$. **34.** $800\left(1 + \dfrac{r}{100}\right)^5 = 940$.

35. $P(1{\cdot}05)^n = 4P$. **36.** $\dfrac{31}{40}\left(1 + \dfrac{r}{100}\right)^4 = 1$.

Given $\log 2 = 0{\cdot}30103$, $\log 3 = 0{\cdot}47712$, $\log 7 = 0{\cdot}84510$, find, without using tables, the value of :

37. $\log 729$. **38.** $\log 0{\cdot}512$. **39.** $\log \sqrt[5]{392}$.

40. $\log \sqrt[3]{4200}$. **41.** $\log(270 \div 64)$. **42.** $\log \sqrt[3]{540}$.

43. $\log \sqrt{27 \div 64}$. **44.** $\log \sqrt{40 \div 81}$. **45.** $\log(1{\cdot}44)\sqrt{1{\cdot}44}$.

46. $\log \sqrt[3]{5 \div 6}$. **47.** $\log 0{\cdot}0343$. **48.** $\log(122{\cdot}5)^{-\frac{3}{4}}$.

In the following examples obtain an equation connecting x and y which does not involve logarithms :

49. $\log x + \log y^2 = \log 7$. **50.** $\log x^2 - \log y^3 = \log 11$.

51. $\log x + \log\sqrt{y} = 1$. **52.** $4\log x = 3\log y$.

53. $\log 2x + 3\log y = \log 5$.

54. $x \log 5 = 2y \log 6$.

55. $\log x = y \log 2 + \log 3$.

56. $2 \log x = y \log 5 + 2$.

57. $\log y = 1{\cdot}5 \log x + 0{\cdot}30103$.

58. $\log x = \log(a - by) - \log a$.

59. $3 \log \sqrt[3]{x^2 + y^2} = 2 \log y + 2$.

60. $2 \log(x + y) = \log(x - y) + 3$.

Evaluate :

61. $\log_3 4$. **62.** $\log_9 10$. **63.** $\log_{2\cdot 34} 15{\cdot}67$. **64.** $\log_{0\cdot 37} 3{\cdot}7$.

In 65–70 the observations given are thought to obey laws of the type $y = kx^n$; by graphing logarithms of the variables find the most probable laws.

65.	x	1	2	3	4		
	y	0·47	4·1	13·5	32		
66.	x	10	9	7	6	5	4
	y	6	6·22	6·7	7·11	7·56	8·14
67.	x	0·3	0·4	0·5	0·6	0·7	
	y	15·1	12·5	10·8	9·5	8·6	
68.	x	1	2	3	4	5	
	y	41	22·8	16	12·6	10·4	
69.	x	1·68	2·43	3·18	3·60	4·03	
	y	152	320	480	640	740	
70.	x	2·3	1·7	1·4	1·2	1	
	y	138	216	288	362	480	

EXERCISE 94. c

(*In this exercise log x means* $\log_{10} x$, *unless otherwise stated*)

Find (if possible, without using tables) the value of :

1. $\log(\sqrt{10} \times 100)$.

2. $\log_{0\cdot 25} 8$.

3. $\log\sqrt{28} + \log\sqrt{325} - \log\sqrt{91}$.

4. $10^{-2\cdot 4}$.

5. $\log 32 \div \log 16$.

6. $\log 125 + 2 \log 27 - 3 \log 4{\cdot}5$.

7. Prove that $2[\log\sqrt{125} + \log 27 - \log\sqrt{1000}] = \log 729 - 3 \log 2$.

Solve the equations :

8. $\log x = \bar{1}{\cdot}2480$.

9. $\log 3x = \bar{1}{\cdot}7442$.

10. $(3{\cdot}981)^{2x-5} = (7{\cdot}943)^x$.

11. $2^{x+y} = 5^{3y}$; $x + 4y = 4$.

12. $2^{2x} \,.\, 3^{-y} = 8$; $x - 2y = 2$.

13. $\log_3(x + y) + \log_3(x - y) = 2$; $\log_{10} x + \log_{10} y = 1 + \log_{10} 2$.

14. Find the least integral value of n such that $(1{\cdot}01)^n$ will exceed $2{\cdot}5$.

15. The equation $y = ax^n$ is satisfied by the pair of values $x = 2$, $y = 3$, and also by the pair of values $x = 4$, $y = 7$. Prove that $7a = 9$, and find (i) the value of n (correct to one decimal place), (ii) the value of y (correct to one decimal place), when $x = 3{\cdot}40$.

16. Find the least integral value of n which will make $(0{\cdot}863)^n < 0{\cdot}1$.

17. Prove that $3 \log_6 1296 = 2 \log_4 4096$, and find tneir common value.

18. How many integers between a million and a million millions are exact fifth powers?

19. Express $\log \frac{5}{6}$ and $\log(0{\cdot}018)$ in terms of $\log 2$ and $\log 3$.

20. If $\log 7 = a$, find $\log \sqrt{34{\cdot}3}$ in terms of a.

21. Simplify $\log\left(\frac{5}{4}\right) + \log 14 - \log\left(\frac{7a}{3}\right)$ and, without using tables, find the value of a so that the expression is equal to -1.

22. Find y, if $\log y - \log \sqrt{9} = 1$.

23. If $x^2 + y^2 = 3xy$, show that $\log(x-y) = \frac{1}{2}(\log x + \log y)$.

24. Prove that $3^{\log 2} = 2^{\log 3}$.

25. Express $2^{\frac{1}{3}}$ as a power of 10 and $10^{\frac{1}{3}}$ as a power of 2.

26. Prove that $3 \log_{1000} x = \log_{10} x$.

27. If $\log_{10} e = 0{\cdot}4343$, find $\log_e 100$, correct to 2 places of decimals.

28. Evaluate $(2{\cdot}34)^{4{\cdot}32} - (4{\cdot}32)^{2{\cdot}34}$.

29. If $\log_a 12 = 1{\cdot}832$, find (i) $\log_a 15$, (ii) a.

30. Given $\log_{10} 6{\cdot}934 = 0{\cdot}8410$, find $\log_{0{\cdot}1} 69{\cdot}34$.

Solve (if possible, without using tables):

31. $2^{x+1} = 3$. **32.** $16^{x-1} = 8^{2x+5}$. **33.** $5^{3x+2} = 8^{6x-1}$.

34. $5^{2x-3} = 8$. **35.** $20^{5-3x} = 2^{12-5x}$. **36.** $343^{2x-5} = 49^{5x+2}$.

37. $10^x \cdot 14^{x-1} = 2^{2x-1} \cdot 11^{x+2}$. **38.** $5^{3x-4} \cdot 3^{2x} = 7^{x-1} \cdot 11^{2-x}$.

39. $3^x \cdot 27^{2y} = 81$; $9^{2x} = 3 \cdot 81^y$. **40.** $9^{8-4x} = 27^{x+y}$; $25^{3x} = 125^y$.

41. $3^{2x} - 5 \cdot 3^x + 6 = 0$. **42.** $4^x - 4^{x-1} = 192$.

In the following examples obtain an equation connecting x and y which does not involve logarithms:

43. $\log \sqrt{x} + \log y^2 = \log 5$. **44.** $4 \log x - \log y = \log 13$.

45. $\log x\sqrt{x} + \log y = 2$. **46.** $3 \log x = 5 \log y$.

47. $x \log 3 = 3y \log 2$. **48.** $\log 5x - 2 \log y = \log 2$.

49. $x \log 4 = 3 \log y - 1$. **50.** $\log x = 0{\cdot}5 \log y + 0{\cdot}47712$.

51. $\log(x^2 - y^2) = 2 \log x - 3 \log y$.

52. $2 \log \sqrt{x^3 + y^3} = 3 \log x + 1$.

Evaluate:

53. $\log_2 7$. **54.** $\log_{11} 5$. **55.** $\log_{2{\cdot}718} 2$. **56.** $\log_{99} 100$.

CHAPTER XXIX

RATIO AND PROPORTION

200. Ratio. If the areas of two triangles are 40 sq. cm. and 60 sq. cm., we say that the first is $\frac{40}{60}$ or $\frac{2}{3}$ of the second. This fact is also expressed by saying that the ratio of the areas of the two triangles is 40 to 60, or 2 to 3. This is sometimes written 2 : 3.

More generally, if two quantities contain respectively x units and y units **of the same kind**, we may say that their ratio is $\frac{x}{y}$ or $x : y$. The ratio of the two quantities is thus a comparison of their magnitudes. The quantities must of course be of the same kind; we cannot compare £2 with 6 yards, but we can compare £2 with 4 shillings, since they may both be expressed in shillings.

Similarly, if three quantities contain respectively x units, y units and z units **of the same kind**, we may say that they are in the ratio $x : y : z$. The statement $x : y : z = 5 : 8 : 17$ is equivalent to the three statements $x : y = 5 : 8$; $y : z = 8 : 17$; $z : x = 17 : 5$, only two of which are independent. For if $x : y = 5 : 8$ and $y : z = 8 : 17$,

we have $\quad \frac{x}{y} = \frac{5}{8}$ and $\frac{y}{z} = \frac{8}{17}$, $\quad \therefore \frac{x}{y} \times \frac{y}{z} = \frac{5}{8} \times \frac{8}{17}$, $\quad \therefore \frac{x}{z} = \frac{5}{17}$,

$$\therefore \frac{z}{x} = \frac{17}{5}, \quad \text{i.e. } z : x = 17 : 5.$$

This relationship is more usually expressed in the form

$$\frac{x}{5} = \frac{y}{8} = \frac{z}{17}.$$

Similarly, $a : b : c : d : e = k : l : m : n : r$ means that

$$\frac{a}{k} = \frac{b}{l} = \frac{c}{m} = \frac{d}{n} = \frac{e}{r}.$$

201. The ratio $a : b$ is the same as the ratio $ka : kb$ $(k \neq 0)$; for $\frac{a}{b} = \frac{ka}{kb}$. Similarly $a : b : c = ka : kb : kc$, etc.

If we have to work with two numbers in the ratio $a : b$, it is often convenient to take them as ak, bk respectively.

Example 1. *If* $a : b = 7 : 3$, *find the ratio* $(7a - 9b) : (2a + 3b)$.

We may take the numbers a, b as $7k$, $3k$ respectively;

$$\therefore \frac{7a-9b}{2a+3b} = \frac{49k-27k}{14k+9k} = \frac{22k}{23k} = \frac{22}{23} \text{ (since } k \neq 0\text{)}.$$

The required ratio is therefore $22 : 23$.

Example 2. *A's age is to B's in the ratio of* $4 : 7$. *In* 5 *years' time the ratio of their ages will be* $17 : 26$. *Find their ages.*

Let A's and B's ages be $4x$ years and $7x$ years respectively; then

$$\frac{4x+5}{7x+5} = \frac{17}{26}, \quad \therefore 104x + 130 = 119x + 85, \text{ giving } x = 3.$$

Hence the ages are 12 years and 21 years.

Example 3. *If* $a : b = 2 : 3$ *and* $b : c = 4 : 5$, *find*:

(i) $a : b : c$, (ii) $(a + 2b + 3c) : (3a - b + 2c)$.

(i) We have $\frac{a}{2} = \frac{b}{3}$ and $\frac{b}{4} = \frac{c}{5}$, $\therefore \frac{a}{8} = \frac{b}{12}$ and $\frac{b}{12} = \frac{c}{15}$,

$$\therefore \frac{a}{8} = \frac{b}{12} = \frac{c}{15}, \text{ i.e. } a : b : c = 8 : 12 : 15.$$

(ii) We may write $a = 8k$, $b = 12k$, $c = 15k$;

$$\therefore \frac{a+2b+3c}{3a-b+2c} = \frac{8k+24k+45k}{24k-12k+30k} = \frac{77k}{42k} = \frac{11}{6},$$

$$\therefore (a + 2b + 3c) : (3a - b + 2c) = 11 : 6.$$

202. The following technical terms were formerly much used in connection with ratio, but have now become almost obsolete.

(1) In the ratio $a : b$, a is called the antecedent and b the consequent.

(2) The ratio $a^2 : b^2$ is called the duplicate ratio of $a : b$.

(3) The ratio $a^3 : b^3$ is called the triplicate ratio of $a : b$.

(4) The ratio $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ is called the ratio compounded by (or "of") the ratios $a : b$ and $c : d$. But it is now more usual to refer to the product of the ratios $a : b$ and $c : d$.

203. Homogeneous expressions and equations. Consider the homogeneous expression $5x^2 + 4xy + 15y^2$. This may be written

$$x^2\left(5 + 4\frac{y}{x} + 15\frac{y^2}{x^2}\right), \quad \text{or} \quad y^2\left(5\frac{x^2}{y^2} + 4\frac{x}{y} + 15\right).$$

The expression in the first bracket is a function of the ratio $\frac{y}{x}$, and the expression in the second bracket is a function of the ratio $\frac{x}{y}$.

In other words, a homogeneous expression of the second degree in x and y may be written in either of the forms

$$x^2 f\left(\frac{y}{x}\right) \text{ or } y^2 F\left(\frac{x}{y}\right),$$

where $f\left(\frac{y}{x}\right)$ and $F\left(\frac{x}{y}\right)$ are functions of the second degree.

Similarly, it may be shown that a homogeneous expression of the nth degree may be written either in the form

$$x^n f\left(\frac{y}{x}\right) \text{ or } y^n F\left(\frac{x}{y}\right),$$

where $f\left(\frac{y}{x}\right)$ and $F\left(\frac{x}{y}\right)$ are functions of the nth degree.

It follows that if a homogeneous expression is equated to zero, we can deduce the possible values of the ratio $x : y$, or $y : x$.

Example 4. *If $x^2 - xy - 12y^2 = 0$, find the ratio $x : y$.*

Let $x : y = a$, then we have

$$y^2\left(\frac{x^2}{y^2} - \frac{x}{y} - 12\right) = 0, \quad \text{i.e. } y^2(a^2 - a - 12) = 0;$$

$$\therefore \text{ unless } y = 0, \ a^2 - a - 12 = 0, \text{ i.e. } (a-4)(a+3) = 0,$$

$$\therefore a = 4 \text{ or } -3.$$

We conclude that the ratio $x : y$ must be either $4 : 1$ or $-3 : 1$; unless $y = 0$. If $y = 0$, x also $= 0$, and the ratio $x : y$ cannot be determined.

204. We may also deduce the ratio of two homogeneous expressions of the same degree in x and y, given the ratio $x : y$.

Example 5. *If $x : y = 5 : 2$, find the ratio of $(x^3 + y^3) : (x + y)^3$.*

We have $$\frac{x^3 + y^3}{(x+y)^3} = \frac{y^3\left(\frac{x^3}{y^3} + 1\right)}{y^3\left(\frac{x}{y} + 1\right)^3} = \frac{y^3\left(\frac{125}{8} + 1\right)}{y^3\left(\frac{5}{2} + 1\right)^3} = \frac{y^3(133)}{y^3(343)}$$

$$= \frac{133}{343} \text{ (since } y \neq 0) = \frac{19}{49}.$$

EXERCISE 95

1. What is the ratio of :

(i) 8 in. to 3 ft. ; (ii) 2 cm. 2·5 mm. to 4 m. ;
(iii) 122° to 2 rt. ∠s ; (iv) 6 min. 30 sec. to $2\frac{1}{4}$ hr. ;
(v) $9a^2$ sq. in. to $2b^2$ sq. ft. ; (vi) $3x$ hours to $\frac{x}{7}$ weeks?

2. If $a : b = 2 : 5$ and $c : d = 3 : 7$, find the following ratios :

(i) $2a : 9b$; (ii) $3a^2 : 7b^2$; (iii) $\frac{4}{a} : \frac{3}{b}$;
(iv) $5ac : 3bd$; (v) $\frac{a}{2c} : \frac{b}{3d}$; (vi) $2ad^2 : 7bc^2$.

3. Find the ratio of $a : b$,

(i) if $5a = 4b$; (ii) if $3a - 4b = 5b - a$; (iii) if $16a^2 = 25b^2$;
(iv) if a exceeds b by 15 per cent., by x per cent.

4. Find $a : b : c$,

(i) if $a : b = 8 : 15$ and $b : c = 9 : 4$;
(ii) if $2a = 3b$ and $4b = 3c$.

5. If $x : y = 4 : 3$, find the following ratios :

(i) $\frac{3x - 2y}{3x + 2y}$; (ii) $\frac{3x^2 - 2y^2}{3x^2 + 2y^2}$; (iii) $\frac{x^2 - 3xy + 2y^2}{x^2 - xy - 3y^2}$.

6. If $a : b : c = 2 : 5 : 7$, find the following ratios :

(i) $\frac{3a}{2b + 3c}$; (ii) $\frac{3a - 4c}{2a - b + 5c}$; (iii) $\frac{a^2 + b^2}{b^2 + c^2}$.

7. The ratio of $(x + 2y)$ to $(x - 2y)$ is equal to the ratio of a to 1. Find in terms of a the ratio of $(5x + 4y)$ to $(3x + 2y)$. What is the value of a if $x : y = 7 : 3$?

8. If $3x - 2y : 3x + 2y = 1 : 3$, find the ratio of $x^2 + y^2$ to $7x^2 - xy$.

9. If $12x^2 + 7xy - 10y^2 = 0$, find values for $x : y$.

10. If $15(2x^2 - y^2) = 7xy$, and if x and y are both positive, find the ratio of x to y.

11. Find the value of $(2x^2 + 2xy + 3y^2) : (4x^2 - 6xy + 5y^2)$, if $2x = 3y$.

12. Find the ratio in which the following expressions are altered, if a and b are each increased in the ratio 2 to 1 :

(i) $\frac{a}{3b}$; (ii) $\frac{2a^2}{5b^2}$; (iii) $2a - 7b$; (iv) $\frac{2a - 7b}{2a + 7b}$;
(v) $\frac{5a^2}{b}$; (vi) $\frac{2a^2}{3b^3}$; (vii) $\frac{a^2 - 2ab + 3b^2}{a^2 + 2ab + 3b^2}$; (viii) $\frac{a^2 + b^2}{a^3 + b^3}$.

13. If $4x : y = 5 : 4$, find the value of $(8x - 3y) : (16x + 5y)$.

14. Two numbers are in the ratio $l : m$. The second is x. What is the first?

15. Two men's ages are in the ratio $3 : 4$. In 4 years' time they will be in the ratio $7 : 9$. Find their ages.

16. When 1 is added to each of two numbers their ratio becomes $1 : 2$, and when 5 is subtracted from each of them their ratio becomes $5 : 11$. Find the numbers.

17. Two men, A and B, engage to mow certain fields in 12 days, but after 8 days they have to engage a third man C, with whose help the work is just finished in time. The rates of working of A, B, C are in the ratios $5 : 4 : 3$. Find how long the work would have taken if all three had started together.

18. Equal volumes are taken of two alloys of copper and tin. In the first, the volumes of copper and tin are in the ratio $9 : 2$ and in the second they are in the ratio $19 : 3$. If the two alloys are fused together without diminution of volume, find the ratios of the volumes of copper and tin in the resulting alloy.

19. The weights of three lumps of metal are as $5 : 6 : 7$. By what fractions of themselves must the first two be increased so that the ratio of the weights may be changed to $7 : 6 : 5$?

20. The volumes of two cubes are in the ratio of $3{\cdot}52 : 1$. Find the ratio of their surfaces.

205. Proportion. If we take four quantities a, b, c, d such that $a : b = c : d$, i.e. $\frac{a}{b} = \frac{c}{d}$, the quantities are said to be **in proportion,** and d is said to be the **fourth proportional** to a, b, c. The first and fourth quantities are called the **extremes,** the second and third are called the **means.**

If we take three quantities a, b, c such that a, b, b, c are in proportion, i.e. $\frac{a}{b} = \frac{b}{c}$, the three quantities are said to be in **continued proportion.**. Also c is said to be the **third proportional** to a and b, and b is said to be the **mean proportional** between a and c.

If we take a number of quantities $a, b, c, d, e, \ldots$, such that $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \ldots$, the quantities are said to be in **continued proportion.**

In the following paragraph it is assumed that no zero quantities occur.

206. Some important results. (1) If $\frac{a}{b}=\frac{c}{d}$ we have, by multiplying each side by bd, $ad=bc$, i.e. when four quantities are in proportion, the product of the extremes is equal to the product of the means.

(2) If $\frac{a}{b}=\frac{c}{d}$ we have, as above, $ad=bc$. .... (i)

Dividing each side of (i) by ac, we have $\frac{d}{c}=\frac{b}{a}$, or $\frac{b}{a}=\frac{d}{c}$.

This result is sometimes called "*Invertendo*".

Dividing each side of (i) by cd, we have $\frac{a}{c}=\frac{b}{d}$.

This result is sometimes called "*Alternando*".

(3) If $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a}{b}+1=\frac{c}{d}+1$, $\therefore \frac{a+b}{b}=\frac{c+d}{d}$.

This result is sometimes called "*Componendo*".

Also $\frac{a}{b}-1=\frac{c}{d}-1$, $\therefore \frac{a-b}{b}=\frac{c-d}{d}$.

This result is sometimes called "*Dividendo*".

(4) If $\frac{a}{b}=\frac{c}{d}$, we have, by combining the results of (3),

$$\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d}, \quad \text{i.e. } \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

This result is sometimes called "*Componendo et Dividendo*".

207. There is no need for the pupil to remember the names of these operations, but he must be familiar with the results (1) and (2). Results (3) and (4) are not frequently used, except to shorten the working of equations and identities.

Most results arising out of equal ratios may be proved by putting each equal ratio equal to some other symbol, usually k.

Thus if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots$, we may put each ratio equal to k. It then follows that $a=bk$, $c=dk$, $e=fk$, etc. The pupil should verify that the results of the preceding Article may be proved by this method. These results are all special cases of a fundamental theorem that if $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios, any expression containing a and b which

can be expressed as a function of $\frac{a}{b}$ only is equal to the corresponding expression containing c and d.

Example 6. *If* $\frac{a}{b}=\frac{c}{d}$, *prove that* $\frac{a^2+3b^2}{a^2-ab+b^2}=\frac{c^2+3d^2}{c^2-cd+d^2}$.

Let $\frac{a}{b}=\frac{c}{d}=k$; then $a=bk$, $c=dk$.

We then have $\frac{a^2+3b^2}{a^2-ab+b^2}=\frac{b^2(k^2+3)}{b^2(k^2-k+1)}=\frac{k^2+3}{k^2-k+1}$;

$$\frac{c^2+3d^2}{c^2-cd+d^2}=\frac{d^2(k^2+3)}{d^2(k^2-k+1)}=\frac{k^2+3}{k^2-k+1};$$

$$\therefore \frac{a^2+3b^2}{a^2-ab+b^2}=\frac{c^2+3d^2}{c^2-cd+d^2}.$$

208. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, we may put each ratio equal to k, whence $c=dk$, $b=ck=dk^2$, $a=bk=dk^3$.

It follows that a sequence of numbers in continued proportion can be expressed in terms of k and the last number. This is a very important result.

Example 7. *If* $a:b=b:c=c:d$, *prove that*

$$(a+b)(c+d)=(b+c)^2.$$

As above, we have $c=dk$, $b=dk^2$, $a=dk^3$.

$$\therefore (a+b)(c+d)-(b+c)^2=(dk^3+dk^2)(dk+d)-(dk^2+dk)^2$$
$$=dk^2(k+1)\,.\,d(k+1)-[dk(k+1)]^2=d^2k^2(k+1)^2-d^2k^2(k+1)^2=0,$$
$$\therefore (a+b)(c+d)=(b+c)^2.$$

209. An important property of equal ratios is:

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots$, then each ratio $=\frac{la+mc+ne+\dots}{lb+md+nf+\dots}$, where l, m, n are any quantities.

Let each ratio $=k$. Then $a=bk$, $c=dk$, $e=fk$, etc.;

$$\therefore \frac{la+mc+ne+\dots}{lb+md+nf+\dots}=\frac{lbk+mdk+nfk+\dots}{lb+md+nf+\dots}$$
$$=\frac{k(lb+md+nf+\dots)}{lb+md+nf+\dots}=k.$$

Example 8. *Find the value of a in the following expression:*

$$\frac{x}{2}=\frac{y}{7}=\frac{z}{3}=\frac{3x-5y+2z}{a}.$$

Since $\frac{x}{2}=\frac{y}{7}=\frac{z}{3}$, each fraction is equal to $\frac{lx+my+nz}{2l+7m+3n}$. Now put $l=3$, $m=-5$, $n=2$.

Each fraction is therefore equal to $\frac{3x-5y+2z}{6-35+6}$,

$$\therefore\ a=6-35+6=-23.$$

***Example 9.** *If* $\frac{2m+2n-3l}{x}=\frac{2n+6l-m}{y}=\frac{6l+2m-n}{z}$, *prove that*

$$\frac{3l}{2y+2z-x}=\frac{m}{2z+2x-y}=\frac{n}{2x+2y-z}.$$

The clue is given by the denominators in the result.

We form a new fraction equal to each of the original fractions by use of the multipliers -1, 2, 2, as in Art. 209.

Thus, each of the given fractions is equal to

$$\frac{(-1)(2m+2n-3l)+2(2n+6l-m)+2(6l+2m-n)}{-x+2y+2z}=\frac{27l}{2y+2z-x}.$$

Similarly, by use of the multipliers 2, -1, 2, each of the given fractions is equal to

$$\frac{2(2m+2n-3l)+(-1)(2n+6l-m)+2(6l+2m-n)}{2x-y+2z}=\frac{9m}{2z+2x-y}.$$

Similarly, by use of the multipliers 2, 2, -1, each of the given fractions is equal to

$$\frac{2(2m+2n-3l)+2(2n+6l-m)+(-1)(6l+2m-n)}{2x+2y-z}=\frac{9n}{2x+2y-z}.$$

Hence, $$\frac{27l}{2y+2z-x}=\frac{9m}{2z+2x-y}=\frac{9n}{2x+2y-z},$$

$$\therefore\ \frac{3l}{2y+2z-x}=\frac{m}{2z+2x-y}=\frac{n}{2x+2y-z},$$

dividing each fraction by 9.

Example 10. *Solve the equation* $\frac{9x^2-3x+2}{3x-2}=\frac{12x^2-5x+2}{5x-2}$.

Either $(9x^2-3x+2)(5x-2)=(12x^2-5x+2)(3x-2)$,

$$\therefore\ 45x^3-33x^2+16x-4=36x^3-39x^2+16x-4,$$

$$\therefore\ 9x^3+6x^2=0;\quad \therefore\ 3x^2(3x+2)=0,$$

$$\therefore\ x=0\ \text{ or }\ -\tfrac{2}{3}.$$

Or Using "*Componendo*" we have $\dfrac{9x^2}{3x-2}=\dfrac{12x^2}{5x-2}$,

$$\therefore\ 3x^2=0 \text{ or } \frac{3}{3x-2}=\frac{4}{5x-2}, \text{ i.e. } 15x-6=12x-8\,;$$

$$\therefore\ x=0 \text{ or } 3x+2=0, \text{ i.e. } x=-\tfrac{2}{3}.$$

It is never *necessary* to use "*Componendo*", etc., but as seen above, the working of a question is sometimes made considerably shorter thereby.

210.* The rule of cross multiplication.

If $$a_1x+b_1y+c_1z=0, \quad \text{.......................(i)}$$

and $$a_2x+b_2y+c_2z=0, \quad \text{.......................(ii)}$$

we have, by multiplying (i) by a_2, (ii) by a_1 and subtracting,

$$y(a_2b_1-a_1b_2)+z(a_2c_1-a_1c_2)=0,$$

$$\text{i.e. } y(a_1b_2-a_2b_1)=z(c_1a_2-c_2a_1),$$

$$\text{or } \frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Similarly, by eliminating z between (i) and (ii), we obtain

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1},$$

$$\therefore\ \frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

This result is known as the Rule of Cross Multiplication.

The result may be easily remembered as follows :

Write down the coefficients of x, y, z in order, beginning with those of y ; repeat these last, as shown below.

$$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

Multiply the coefficients across in the way indicated by the arrows ; any product formed in descending is positive, and any formed in ascending is negative. Then the three results $b_1c_2-b_2c_1$, $c_1a_2-c_2a_1$, $a_1b_2-a_2b_1$ are the denominators for x, y, z respectively.

Note. If $z=1$, we obtain a formula for solving the linear simultaneous equations $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.

Example 11. *Find* (i) *the ratios of* $x : y : z$ *from the equations* $x - 2y + z = 0$, $3x + 2y - 3z = 0$; (ii) *the values of* x, y, z *which satisfy these equations and also the equation* $3x^2 - 4y^2 + 2z^2 = 8$.

(i) Write down the coefficients in the equations according to the rule; thus

$$\begin{array}{ccccccc} -2 & & 1 & & 1 & & -2 \\ & \times & & \times & & \times & \\ 2 & & -3 & & 3 & & 2 \end{array},$$

whence we obtain the products

$$(-2)\times(-3)-(2)\times(1),\ (1)\times(3)-(-3)\times(1),\ (1)\times(2)-(3)\times(-2),$$

or $4, 6, 8$;

$$\therefore\ \frac{x}{4}=\frac{y}{6}=\frac{z}{8},\ \text{i.e. } \frac{x}{2}=\frac{y}{3}=\frac{z}{4},$$

$$\therefore\ x : y : z = 2 : 3 : 4.$$

(ii) Since x, y, z satisfy the first two equations, we have, as above,

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{4}, \quad \therefore \text{ we may write } x = 2k,\ y = 3k,\ z = 4k.$$

Substitute these values for x, y, z in $3x^2 - 4y^2 + 2z^2 = 8$;

$$\therefore\ 3(2k)^2 - 4(3k)^2 + 2(4k)^2 = 8;$$

$$\therefore\ 8k^2 = 8, \quad \therefore\ k^2 = 1, \quad \therefore\ k = 1 \text{ or } -1;$$

and the required values are $x = 2,\quad y = 3,\quad z = 4,$

or $x = -2,\ y = -3,\ z = -4.$

EXERCISE 96

1. Find the fourth proportional to:

(i) $12, 9, 32$; (ii) ab, bc, cd; (iii) $12a^3, 9a^2b, 6ab^2$.

2. Find the third proportional to:

(i) $4, 16$; (ii) a^2b^4, a^4b^2; (iii) $12b^2, 4ab$.

3. Find the mean proportional between:

(i) $8, 32$; (ii) a^2b^4, a^4b^2; (iii) $9a^2b, 25b^3$.

4. If $a : b = c : d = e : f$, prove that:

(i) $\dfrac{a+3c}{b+3d}=\dfrac{5a-2c}{5b-2d}$; (ii) $\dfrac{2a^2-7ac}{2b^2-7bd}=\dfrac{ac+4c^2}{bd+4d^2}$;

(iii) $\dfrac{a}{b}=\sqrt{\dfrac{2a^2-5c^2+3e^2}{2b^2-5d^2+3f^2}}$; (iv) $\left(\dfrac{7a-5c}{7b-5d}\right)^2=\dfrac{4c^2+9e^2}{4d^2+9f^2}$;

(v) $\dfrac{a}{d}=\dfrac{(a+b)(a-c)}{(b-d)(c+d)}$; (vi) $\sqrt{\dfrac{a^2-ac}{b^2-bd}}=\dfrac{c+e}{d+f}$.

5. If $\dfrac{x}{5}=\dfrac{y}{2}=\dfrac{z}{3}$, and $4x - 5y + 2z = 8$, find x, y and z.

6. Fill in the blanks in :

(i) $\frac{a}{7}=\frac{b}{3}=\frac{a+3b}{\quad}=\frac{4a-5b}{\quad}$;

(ii) $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}=\frac{x-y+z}{\quad}=\frac{7x-8y+3z}{\quad}$.

7. If $a : b = b : c = c : d$, prove that :

(i) $\frac{a}{a+2b}=\frac{a-2b}{a-4c}$; (ii) $\sqrt{\frac{ab+bc-ca}{bc+cd-db}}=\sqrt[3]{\frac{a}{d}}$.

8. If a, b, c, d are in continued proportion, prove that :

(i) $b+c : c-a = d : d-c$;

(ii) $b+c$ is a mean proportional to $a+b$ and $c+d$.

9. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each of these ratios is equal to

(i) $\sqrt[3]{\frac{2c^3-5ce^3+7ac^2e}{2d^3-5def^2+7bcdf}}$; (ii) $\sqrt[4]{\frac{5a^4-2a^2c^2-de^4}{5b^4-2b^2d^2-df^4}}$.

10. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that (i) each of these ratios is equal to $\frac{ab+3bc-2cd}{b^2+3c^2-2d^2}$; (ii) $\frac{a^3+b^3}{b^3+c^3}=\frac{b^3+c^3}{c^3+d^3}$.

11. If $(a-b):(c+a)=b^2:c^2$, and $(b+c)$, $(a-b)$ are not zero, express c in terms of a and b.

12. If $a : b = b : c$, prove that $\sqrt{ab}+\sqrt{bc}$ is a mean proportional between $a+b$ and $b+c$.

*EXERCISE 96. c

In Nos. 1–6, find the ratios of $x : y : z$.

1. $2x+3y-4z=0$,
$5x-2y+3z=0$.

2. $5x-y-3z=0$,
$4x+2y+z=0$.

3. $2x-5y=3z$,
$7x-z=4y$.

4. $4x-3y+6z=0$,
$+y+z=0$.

5. $2x=3(y+z)$,
$5(x-y)=2z$.

6. $ax+by+cz=0$,
$px+qy+rz=0$.

Solve the equations :

7. $2x-6y-7z=0$,
$5x+4y-8z=0$,
$3x^2-16y^2+z^2=9$.

8. $3x-y+4z=0$,
$-5x+2y+7z=0$,
$4x-y+9z=10$.

9. $7x+5y+6z=0$,
$3x=4(y-z)$,
$5x-7y+6z=16$.

10. $3x-4y-14z=0$,
$2x-y+4z=0$,
$3x^3-y^3+8z^3=16$.

11. $x+y+z=1,$
$ax+by+cz=0,$
$a^2x+b^2y+c^2z=0.$

12. $x+y+z=0,$
$ax+by+cz=0,$
$5(x^3+y^3+z^3)=(b-c)(c-a)(a-b).$

13. $\dfrac{x^2+x-2}{x^2-x+2}=\dfrac{4x^2+5x-6}{4x^2-5x+6}.$

14. $\dfrac{9x^2+3x-2}{9x^2-3x-2}=\dfrac{3x+2}{3x-2}.$

15. $\dfrac{2x-5}{x^2-2x+5}=\dfrac{3x+2}{x^2-3x-2}.$

16. $\dfrac{2x^2-3x+7}{2x^2+3x+7}=\dfrac{5x^2-3x-20}{5x^2+3x-20}.$

17. If $\dfrac{x}{3m+4n-2l}=\dfrac{y}{4n+2l-3m}=\dfrac{z}{2l+3m-4n}$, prove that

$$x(3m-4n)+y(4n-2l)+z(2l-3m)=0.$$

18. If $x=\dfrac{a}{3b+2c}=\dfrac{3b}{2c+a}=\dfrac{2c}{a+3b}$, show that x must equal either $\frac{1}{2}$ or -1.

19. Prove that if $\dfrac{b+c}{a+2b+2c}=\dfrac{c+a}{2a+b+2c}=\dfrac{a+b}{2a+2b+c}$, then either a, b, c are all equal or their sum is zero.

20. If $\dfrac{l}{m+n-l}=\dfrac{m}{n+l-m}=\dfrac{n}{l+m-n}$, prove that either each ratio equals 1 or $l+m+n=0$. What is the value of the ratio in the latter case?

21. If $\dfrac{b+c-a}{l}=\dfrac{c+a-b}{m}=\dfrac{a+b-c}{n}$, prove that

$$\frac{a}{m+n}=\frac{b}{n+l}=\frac{c}{l+m}.$$

22. If the ratios $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \cdots$ are all equal, prove that each is equal to $\sqrt[n]{\dfrac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}}$, for all values of $n, p, q, r, \ldots$.

23. Prove that (i) if $a>b$, $\dfrac{a}{b}>\dfrac{a+x}{b+x}$, (ii) if $a<b$, $\dfrac{a}{b}<\dfrac{a+x}{b+x}$, where a, b, x are all positive quantities. How must the result be modified if x is negative?

24. If x, y, z, u are all positive quantities, show that $\dfrac{x}{y}, \dfrac{x+z}{y+u}, \dfrac{z}{u}$ are in either ascending or descending order of magnitude.

25. If $a, b, c, d, e, f \ldots$ are all positive quantities, show that the fraction $\dfrac{a+c+e+\ldots}{b+d+f+\ldots}$ lies between the greatest and least of the fractions $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \cdots$, provided that these fractions are not all equal.

CHAPTER XXX

VARIATION

211. Up to the present, we have always regarded letters as standing for numbers. We can, however, also use letters to represent quantities. Thus, we may use X to denote the weight of a mass of metal and Y to denote its cost. When letters are used to represent magnitudes it will be found convenient to use capital letters and to reserve small letters to represent variables whose values are purely numerical. Thus, in the above instance we should say that the weight X is x tons and the cost Y is £y.

Direct variation. One quantity Y is said to **vary directly as**, or to **vary as**, or **to be proportional to** another quantity X, if $Y_1 : Y_2 = X_1 : X_2$, where X_1, X_2 denote any two particular values of X and Y_1, Y_2 are the corresponding values of Y. This is usually expressed by writing $Y \propto X$, where $\propto$ is an abbreviation for " varies directly as ", or " varies as ".

It should be particularly noted that X and Y may denote magnitudes which are not necessarily of the same kind, or they may represent variables whose values are purely numerical. Thus, if Y (or £y) be the cost of X (or x tons of metal), we may write either $Y \propto X$, or $y \propto x$.

212. If we wish to decide from first principles whether y varies as x, the simplest test to apply is to ask the questions (1) is y doubled, if x is doubled? (2) Is y halved, if x is halved? (3) Is y multiplied by 5, if x is multiplied by 5? If the answer to any of these questions is " no ", then we know that y does not vary as x; if the answer to all the questions is " yes ", then it is probable that $y \propto x$.

It should be particularly noted (1) that if $y \propto x$, then $y = 0$ if $x = 0$, (2) that y does not necessarily vary as x if, when x is increased or diminished, y is also increased or diminished. The truth of this is easily realised by consideration of the functions $y = x + 5$, $y = x^2$, $y = \sqrt{x}$, etc.

213. Questions involving variation may often be answered by direct application of the definition, as in Ex. 1, Method 1, but it is

usually more convenient to make use of the following theorem: **If $y \propto x$, then $y = kx$, where k is some constant.**

Let x and y stand for *any* corresponding values of the variables and let x_1 and y_1 be particular values of x and y respectively. Then, by definition,

$$y : y_1 = x : x_1, \quad \therefore\ y : x = y_1 : x_1, \text{ or } \frac{y}{x} = \frac{y_1}{x_1}.$$

But $\frac{y_1}{x_1}$ is a fixed number, which may be denoted by k.

Hence $\frac{y}{x} = k$, or $y = kx$.

The symbol k is called the **variation constant**, and its value can be found when we know one pair of corresponding values of the connected variables.

Example 1. *If $y \propto x$, and $y = 40$ when $x = 16$, find the value of y when $x = 24$.*

Method 1. By definition $y_1 : y_2 = x_1 : x_2$, where x_1, x_2 and y_1, y_2 are *any* two particular values of x and y respectively.

Therefore, if y_1 is the required number, we have

$$y_1 : 40 = 24 : 16, \text{ whence } y = 60.$$

Method 2. Since $y \propto x$, we have $y = kx$, where k is a constant.

But when $x = 16$, $y = 40$, $\therefore\ 40 = k \cdot 16$, $\therefore\ k = \frac{5}{2}$,

$\therefore$ all corresponding values of x and y satisfy the equation

$$y = \tfrac{5}{2} x; \quad \therefore \text{ when } x = 24, \quad y = \tfrac{5}{2} \times 24 = 60.$$

Method 3. Since $y \propto x$, we have $y = kx$, where k is a constant, for all corresponding values of x. Therefore, if y_1 is the required number, we have

$$y_1 = k \cdot 24, \quad \text{.............................(i)}$$

and

$$40 = k \cdot 16. \quad \text{.............................(ii)}$$

Dividing (i) by (ii), we obtain

$$\frac{y_1}{40} = \frac{k \cdot 24}{k \cdot 16} = \frac{3}{2}, \quad \therefore\ y_1 = 60.$$

In simple cases there is very little to choose between these methods. Methods 2 and 3 are of more general application than Method 1, and the introduction of one or more variation constants is in some cases essential. The beginner will find it better to use Method 2 until he is really familiar with the subject. It should, however, be noted that in questions involving only one variation

constant it is never *necessary* to calculate the value of the constant, although in simpler cases it is convenient to do so.

Method 3 is of great value when k is an awkward number ; in such cases it is mere waste of time to calculate its value.

Example 2. *The time of one swing of a simple pendulum* $\propto$ *the square root of its length. If a pendulum* 39·14 *in. long swings once in a second, find the length of the pendulum which swings* 56 *times in one minute.*

Let the time T be t secs., and the length L be l in. Then $t=k\sqrt{l}$, and $t=1$ when $l=39{\cdot}14$;

$$\therefore\ 1=k\sqrt{39{\cdot}14},\quad \therefore\ k=\frac{1}{\sqrt{39{\cdot}14}}.$$

(*N.B.*—**Do not** work out the value of k.)

The exact relationship between t and l is therefore $t=\frac{\sqrt{l}}{\sqrt{39{\cdot}14}}$.

When $t=\frac{60}{56}$, we have

$$\frac{60}{56}=\frac{\sqrt{l}}{\sqrt{39{\cdot}14}},\quad \therefore\ \frac{60^2}{56^2}=\frac{l}{39{\cdot}14},$$

$$\therefore\ l=\frac{39{\cdot}14\times 225}{196},\text{ after cancelling,}$$

$$=45\text{ approx.}$$

The length required is therefore 45 in.

Great care must be taken with units. Thus, in Ex. 2, the time must be expressed in terms of the same unit throughout, either seconds or minutes or any other convenient unit.

Similarly, the length of the pendulum must be expressed throughout either in inches or in feet. It does not matter what unit is used provided that, for each variable, the same unit is retained throughout the question. The value of k is different for different units.

In working any example it is best to start by choosing the units and to keep to that set of units throughout. It should, however, be noted that if, e.g., both variables represent lengths, we need not use the same unit for each variable. Thus, the circumference of a circle $\propto$ the radius, and we may, if we wish, use one foot as the unit for the circumference and one inch as the unit for the radius. But we must keep the same unit for the circumference throughout and the same unit for the radius throughout.

214. In Ex. 2, we have $t = k\sqrt{l}$ and at first sight it might appear that for each value of l there are two values of t, equal in magnitude but opposite in sign. This is not the case, for in the question t is essentially a positive number. But, in general, it is worth noting that if $x^2 = ky^2$, we are not entitled to say that $x \propto y$; there are two values of x corresponding to each value of y, and our definition is not satisfied. This point is of theoretical importance only; in most practical applications of variation the letters represent real positive numbers or quantities and to each value of one letter there corresponds one and only one value of the other letter. In all such cases we can say that if $x^2 \propto y^2$, then $x \propto y$.

215. Inverse Variation. One quantity Y is said to **vary inversely as** another quantity X, if $Y_1 : Y_2 = X_2 : X_1$,
where X_1, X_2 denote any two particular values of X and Y_1, Y_2 are the corresponding values of Y.

As above, X and Y may denote magnitudes which are not necessarily of the same kind, or they may represent variables whose values are purely numerical.

It may be shown that if y varies inversely as x, then $y = \frac{k}{x}$, where k is some constant.

For, let x and y stand for *any* corresponding values of the variables and let x_1, y_1 be *particular* values of x, y respectively. Then, by definition,

$$y : y_1 = x_1 : x; \quad \therefore\ xy = x_1y_1.$$

But x_1y_1 is a fixed number which may be denoted by k,

$$\therefore\ xy = k \quad \text{or} \quad y = \frac{k}{x}.$$

216. In questions involving inverse variation, as in questions involving variation, we may either apply the definition or make use of the relationship $y = \frac{k}{x}$.

Example 3. *If y varies inversely as x, and $y = 80$ when $x = 4$, find the value of y when $x = 32$.*

Method 1. By definition,

$$y_1 : y_2 = x_2 : x_1,$$

where x_1, x_2 and y_1, y_2 are *any* two particular values of x and y

respectively. Therefore if y_1 is the required number, we have

$$y_1 : 80 = 4 : 32, \text{ whence } y_1 = 10.$$

Method 2. Since y varies inversely as x, we have $y = \frac{k}{x}$.

But when $x = 4$, $y = 80$,

$$\therefore\ 80 = \frac{k}{4}, \quad \therefore\ k = 320,$$

$\therefore$ for all corresponding values of x and y, $y = \frac{320}{x}$;

$$\therefore \text{ when } x = 32, \quad y = \frac{320}{32} = 10.$$

Method 3. As above, $y = \frac{k}{x}$ for all corresponding values of x and y.

Therefore if y_1 is the required number, we have

$$y_1 = \frac{k}{32}, \quad \text{..............................(i)}$$

$$80 = \frac{k}{4}. \quad \text{..............................(ii)}$$

Dividing (i) by (ii), we obtain

$$\frac{y_1}{80} = \frac{k}{32} \times \frac{4}{k} = \frac{1}{8}, \quad \therefore\ y_1 = 10.$$

The beginner is recommended to use Method 2 until he is really familiar with the subject, when it may be replaced by Method 3.

Example 4. *If* $a \propto b$, $b \propto \frac{1}{c}$, *and* $c \propto d^2$, *prove that* a *varies inversely as* d^2.

We have

$$a = kb, \quad \text{..............................(i)}$$

$$b = l \div c, \quad \text{..............................(ii)}$$

$$c = m d^2, \quad \text{..............................(iii)}$$

where k, l, m are the variation constants.

(It should be carefully noted that we must use different letters to represent the variation constant in (i), (ii) and (iii). In particular cases they may be the same, but we have no right to assume that this will be so.)

From (i) and (ii), we have $a = \frac{kl}{c}$, and, combining this with (iii), we obtain $a = \frac{kl}{m d^2}$.

But since k, l, m are constants, $\frac{kl}{m}$ is a constant, say n,

$$\therefore\ a=\frac{n}{d^2},\ \text{i.e. } a \text{ varies inversely as } d^2.$$

EXERCISE 97 (*Oral*)

Express each of the following statements algebraically, as an equation :

1. $a \propto b$. **2.** $c \propto d^2$. **3.** $z \propto y^3$. **4.** $a \propto \frac{1}{b}$.

5. $c \propto \frac{1}{d^2}$. **6.** $x^2 \propto \frac{1}{y^2}$. **7.** $(a+b) \propto \frac{1}{c^2}$. **8.** $t \propto \frac{1}{\sqrt{l}}$.

9. The circumference of a circle (c ft.) varies as the radius (r in.).

10. The amount of work a man does (w units) varies as the time he works (t hours).

11. t^2 varies inversely as $\sqrt{s}$.

12. (l^2+m^2) varies inversely as $\frac{1}{n^2}$.

13. The simple interest (£i) on a given sum at a given rate varies as the number (n) of years for which the money is lent.

14. The volume (v c. ft.) of a given quantity of gas at a constant temperature varies inversely as the pressure (p lb. per sq. ft.) on it.

15. The value of a diamond (£x) varies as the square of its weight (w grains).

16. x^2y^3 varies inversely as z^5.

17. The horse-power (h) of the engines of a given ship is proportional to the cube of the speed (v m.p.h.).

18. When a stone is let fall, the time (t secs.) it takes to fall any distance (d ft.) varies as the square root of the distance.

19. The square of the time (t days) taken by a planet to go round the sun varies as the cube of its mean distance (d million miles) from the sun.

Assuming that the law of variation in the following cases is the simplest that would give the pairs of values supplied in the tables, state the law in words, give the algebraic equation connecting the variables, and fill in the gaps :

20. x	0	2	4	6		9		−11	
y	0	10		30	35		−12·5	−55	−20
21. p	1	2	3		5		2·5		
v	40	20		10	8	1·6			

22.

x......	0	4	8	12	-6	
y......	0	4	16	36		6·25

23.

x......	1	2	3	4		10
y......	288	72	32		11·52	

24.

c......	2	4	6	8		12	20
d......	50	25		12·5	10		5

EXERCISE 98

1. If y varies as x^2, and $y=36$ when $x=4$, find the value of (i) y when $x=6$, (ii) x when $y=\frac{1}{4}$.

2. If y varies inversely as x, and $y=6$ when $x=3{\cdot}6$, find the value of y when $x=5{\cdot}76$.

3. If $y \propto x$, and $y=15$ when $x=18$, find the equation between x and y. Find also (i) y when $x=4{\cdot}5$, and (ii) x when $y=2{\cdot}5$.

4. If $x \propto \frac{1}{y}$, and $y=4{\cdot}5$ when $x=4$, find x when $y=4{\cdot}2$.

5. If $y \propto \frac{1}{x^2}$, and $y=48$ when $x=3$, find y when $x=4$.

6. If $y \propto x^3$, and $y=81$ when $x=3$, find y when $x=2$.

7. It is known that $y \propto (ax-2)$. If $x=3$ when $y=2$ and $x=4$ when $y=4$, find a. What is the value of y when $x=5$?

8. The breaking strain (s tons) of a steel wire varies as the square of its circumference (c in.). If $s=19{\cdot}6$ when $c=3{\cdot}5$, find s when $c=1{\cdot}5$.

9. The volume of a given mass of gas at a constant temperature is inversely proportional to the pressure on it. At a pressure of 20 lb. per sq. ft. a certain mass of gas occupies 18 c. ft. Express in c. ft. the volume of the same mass of gas at a pressure of 7·2 lb. per sq. ft.

10. The distance through which a heavy body falls from rest varies as the square of the time taken. A body falls through 1600 ft. in 10 sec.; how long would it take to fall through 576 ft.?

11. The intensity of the illumination given by a certain lamp varies inversely as the square of the distance from the lamp. A surface is illuminated by a certain light at a distance of 3 feet. Where must it be placed to receive three times the illumination?

12. If the distances of an object and of its image, formed by a mirror, are measured from a certain point, it is found that the sum of the distances varies as the product. If the distance of the image is 120 cm. when the distance of the object is 300 cm., calculate the distance of the image when the distance of the object is 540 cm.

13. The tension of an elastic string varies as the extension. A string whose unstretched length is 6 in. is stretched to 7·5 in. by a pull of 8 lb. wt. What pull will stretch it to 8·4 in.?

14. The value of a diamond $\propto$ the square of its weight, and a diamond of 4 carats is worth £14 ; find the value of one of the same quality weighing 5 carats.

15. A railway engine without coaches can travel at 80 m.p.h., and its speed is diminished by a quantity which varies as the square of the number of coaches attached. With 8 coaches its speed is 48 m.p.h. Find its speed when there are 10 coaches and the greatest number of coaches the engine can move.

16. The volume of water in an inverted vessel, in the shape of a right circular cone with vertical axis, varies as the cube of the depth of the water in the vessel. The depth of the water is 5 in. when the volume is 12·5 gallons ; find the volume when the depth is 2 ft.

17. If the time of a beat of a pendulum of length l in. is t sec., it is known that $l \propto t^2$. A pendulum with a beat of 2 sec. is 156·56 in. long. What is the time of beat of a pendulum 2 in. long?

18. The horse-power of the engines of a ship is proportional to the cube of the speed ; if the horse-power is 80 at a speed of 8 knots, what is the horse-power when the speed is 10 knots?

19. The square of the time taken by a planet to go round the sun varies as the cube of its mean distance from the sun. If the mean distances of Jupiter and the earth from the sun are 483 millions and 93 millions of miles respectively, find to the nearest year how long it takes Jupiter to travel round the sun.

20. The square of the velocity of a particle varies as the cube of its distance from a certain fixed point P. If this distance is increased by 1·2 per cent., find the approximate percentage increase in the velocity.

21. The receipts for railway trains vary as the excess of the average speed above 20 miles an hour, while the expenses vary as the square of that excess. When the average speed is 40 m.p.h. there is neither profit nor loss ; what percentage of the receipts is the profit, when the average speed is 35 m.p.h.?

22. Weight on and above the earth's surface varies inversely as the square of the distance from the earth's centre ; on and below the surface it varies as the distance from the centre.

The earth's radius being taken as 4000 miles, at what distance below the surface is the weight of an object the same as at 50 miles above it?

23. The number of trees required to plant an acre varies inversely as the square of the distance between the trees. If 432 trees are required when the distance is 10 ft., how many are required when the distance is 15 ft.? What is the distance when 108 trees are required?

24. A clock keeps accurate time at 60° F. but gains as the temperature falls and loses as it rises, the rate of gain or loss varying as the square of the number of degrees between the actual temperature and 60° F. If it gains 2 sec. per day when the temperature is 47° F., how much does it lose (to the nearest sec.) in 3 days when the temperature is 75° F.?

25. The volume of a sphere varies as the cube of its radius. Prove that three spheres of radii $1\cdot 8''$, $2\cdot 4''$, $3''$ are together equal in volume to one of radius $3\cdot 6''$.

26. If $(x+y)\propto(x-y)$, prove that $x\propto y$.

27. If $(x+y)\propto(x-y)$, prove that $(x^2+xy+y^2)\propto(x^2-xy+y^2)$.

28. If $y\propto x$, and $x\propto\frac{1}{z}$, prove that z varies inversely as y.

29. If $a\propto\frac{c}{b^2}$, and $c^2\propto\frac{b}{a}$, prove that $b\propto c\propto\frac{1}{a}$, assuming that a, b, c are all real and positive.

30. If $(c+d)$ varies inversely as $\left(\frac{1}{c}+\frac{1}{d}\right)$, prove that $(c^2+d^2)\propto cd$.

217. It sometimes happens that a quantity depends on the variation of two or more other quantities which may vary independently of each other. Thus, in Geometry, if v is the volume of a right circular cylinder of height h on a base whose radius is r, we know that $v\propto h$ when r is constant,

and $v\propto r^2$, when h is constant.

We also know that v is given by the formula $v=\pi r^2 h$, and since π is constant, this is the same thing as saying that v varies as the product r^2h when both r and h vary.

This is a particular case of a general proposition, viz.:

If a varies as b when c is constant, and if a varies as c when b is constant, then a varies as bc when b and c both vary.

If c has any definite fixed value c_1, $a\propto b$, i.e. there exists a definite constant k such that $a=kb$. Hence, to every pair of values b_1, c_1 of b and c, there corresponds one definite value a_1 of a.

Let three such sets of values be

$$a_1,\ b_1,\ c_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$a',\ b_1,\ c_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$a_2,\ b_2,\ c_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Now b has the same value in (i) and (ii), and $a \propto c$ when b is constant,

$$\therefore\ \frac{a_1}{a'} = \frac{c_1}{c_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Again c has the same value in (ii) and (iii), and $a \propto b$ when c is constant,

$$\therefore\ \frac{a'}{a_2} = \frac{b_1}{b_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

From (iv) and (v) by multiplication

$$\frac{a_1}{a_2} = \frac{b_1 c_1}{b_2 c_2},$$

and (b_1c_1), (b_2c_2) denote *any* values of (bc), $\therefore\ a \propto bc$.

This theorem is of considerable theoretical importance, as it frequently enables us to deduce the law of variation, but in practice it is rarely used, since, in most of the cases in which it can be applied, either the law of variation is well known or the question can be done by compound proportion.

It will be easier for the beginner to learn the result from the consideration of special cases such as the area of a rectangle, the volume of a cone, etc. than from the general proof.

218. Joint variation. One quantity is said to **vary jointly as** a number of others when it varies directly as their product. The word " jointly " is sometimes omitted.

Thus, a varies jointly as b and c when $a = kbc$, where k is a constant. This phrase often causes considerable difficulty to beginners who have been accustomed to replace " and " by " + ". It is most important that they should realise that in this context " and " is equivalent to " × ". They will most easily master the use of the phrase, if they notice that, since $a = kbc$, $a \propto b$ when c is constant and $a \propto c$ when b is constant. This will enable them to check their work and avoid the mistake of writing a as the sum of two terms.

Similarly, if a varies directly as b and inversely as c, we have $a = \frac{kb}{c}$; also, if a varies directly as b and c and inversely as d, we have $a = \frac{kbc}{d}$, where k is in each case a constant.

In all cases of joint variation, the different variables occur as factors, not as terms.

219. The above phrases should be carefully distinguished from the following :

a varies partly as b and partly as c.

Here a consists of two parts, one of which varies as b, and the other of which varies as c. We therefore have $a = kb + lc$.

It should be carefully noted that we must use different variation constants for b and c.

Example 5. *The electrical resistance of a wire* (r *ohms*) *varies as its length* (l *metres*) *and inversely as the square of its diameter* (d *millimetres*). *Compare the resistances of two wires of the same material, one of which has a diameter of* 2 *mm. and is* 8 *m. long, while the other has a diameter of* 3 *mm. and is* 12 *m. long.*

We have $$r = \frac{kl}{d^2},$$

$$\therefore \text{ the resistance } r_1 \text{ of the first wire} = k \cdot \frac{8}{2^2},$$

$$\text{and the resistance } r_2 \text{ of the second wire} = k \cdot \frac{12}{3^2},$$

$$\therefore\ r_1 : r_2 = k \cdot \frac{8}{4} : k \cdot \frac{12}{9} = 3 : 2.$$

Example 6. *The expenses of a ball are partly constant, and partly vary as the number of guests. For* 100 *guests the cost is* £175, *and for* 250 *guests the cost is* £362 10*s. Find the cost per head for* 200 *guests.*

Let £c be the total cost, and let n be the number of guests.

Then $$c = k + ln.$$

Also $$175 = k + 100l, \quad \text{.............................(i)}$$

$$362\tfrac{1}{2} = k + 250l. \quad \text{.............................(ii)}$$

From (i) and (ii) we obtain $k = 50$, $l = 1\tfrac{1}{4}$.

Thus for all values of c and n, $c = 50 + \dfrac{5n}{4}$,

$$\therefore \text{ when } n = 200,\ c = 50 + 250 = 300,$$

and the cost per head is £$300 \div 200$, i.e. £1 10s.

EXERCISE 99 (*Oral*)

Express each of the following statements algebraically, as an equation:

1. A varies directly as B and inversely as C.

2. X varies jointly as Y and the square of Z.

3. z varies as x^2 and inversely as y^2.

4. z partly varies as x and partly as x^3.

5. The reciprocal of z varies jointly as x and y^2.

6. a is partly constant, partly varies as b, and partly varies inversely as c^2.

7. E is partly constant and partly varies inversely as n.

8. H varies directly as t and V^2 and inversely as R.

9. p varies jointly as a and the square of v.

10. h varies as a and the cube of v.

11. C partly varies as A and partly varies inversely as the square of D.

12. t varies directly as l and inversely as $\sqrt{h}$.

13. z varies as x when y is constant and as y^2 when x is constant.

14. C varies as r when h is constant and as h when r is constant.

15. a varies as $\sqrt{b}$ when c is constant and as c^3 when b is constant.

16. The time (t secs.) during which a body will slide down a smooth inclined plane varies directly as the length (l ft.) and inversely as the square root of the vertical height (h in.) of the plane.

17. The electrical resistance (R) of a copper wire varies as the length (L) of the wire and inversely as the area (A) of its cross-section.

18. z varies as the sum of x and y, and $y \propto x^3$.

EXERCISE 100

1. Given that y varies jointly as x^2 and z, and that $y=12$ when $x=3$, $z=9$, find (i) the value of y when $x=2$, $z=8$, (ii) the value of z when $y=18$, $x=0{\cdot}25$.

2. Given that p varies jointly as a and v^2, and that $p=18$ when $a=9$ and $v=15$, find v when $a=81$ and $p=32$.

3. If y is equal to the sum of two quantities, of which one is constant and the other varies as x; and if $y=19$ when $x=\frac{1}{2}$, and $y=47$ when $x=4$, find the value of y when $x=-\frac{1}{4}$.

4. If x is partly constant and partly varies as t^3; and if $x=19$ when $t=2$, and $x=-44$ when $t=8$, find the value of t when $x=-7$.

5. If z varies as x^2 and inversely as y^2, and if $z=4$ when $x=8$ and $y=-\frac{1}{2}$. find z when $x=-2$ and $y=\frac{1}{6}$.

6. If h varies directly as t and v^2 and inversely as r, and if $h=6$ when $t=\frac{1}{4}$, $v=-2$, $r=5$, find t when $h=10$, $v=\frac{1}{2}$, $r=3$.

7. If z varies as the sum of two quantities, one of which varies as x and the other inversely as $\sqrt{x}$, and if $z=42$ when $x=4$, and $z=87$ when $x=25$, find z when $x=9$.

8. If z varies as x when y is constant and as y^2 when x is constant, and if $z=5$ when $x=\frac{1}{3}$ and $y=\frac{1}{2}$, find x when $y=-\frac{1}{6}$, $z=10$.

9. If x varies as y when z is constant, and as z when y is constant, prove that when $y \propto z$, then $x \propto y^2$.

10. If a varies as $\sqrt{b}$ when c is constant, and inversely as c^2 when b is constant, prove that when b varies inversely as c, then $a^2 \propto b^5$.

11. If d varies directly as s and inversely as the square of t, and if $d=3$ when $s=16$ and $t=4$, find s when $t=15$ and $d=0{\cdot}2$.

12. The volume V of a solid varies jointly as the height h and the area A of the base. If $V=30$ when $h=3$ and $A=15$, find (i) V in terms of A and h, (ii) the value of h when $V=50$, $A=8$.

13. A number y is the algebraic sum of two terms, one of which varies as x and the other inversely as x^2. When $x=-2$, $y=5$, and when $x=\frac{1}{2}$, $y=15$; find the value of y when $x=-\frac{2}{3}$.

14. The value of silver coins varies jointly as the thickness and the square of the diameter. If two such coins have values in the ratio 20 : 27 and thicknesses in the ratio 3 : 5, find the diameter of the second, if that of the first is $1\frac{1}{3}$ in.

15. If y varies as the algebraic sum of two terms, one of which varies as x and the other as the square of x; and if $y=-9$ when $x=3$, and $y=2\frac{1}{4}$ when $x=\frac{1}{2}$, find the values of x which make $y=-24$.

16. S is the sum of two parts, one of which varies as x and the other inversely as x. When $x=4$, $S=22$; when $x=-2$, $S=-14$. Find x when $S=80\frac{1}{2}$.

17. The weight of a sphere varies as the cube of its radius and also as the specific gravity of the material of which it is made. The specific gravity of gold is 19·25 and of silver 10·5. Find the radius of a sphere of silver equal in weight to three times that of a sphere of gold of radius 2 cm.

18. The cost of making a spherical ball varies as the cube of its radius, and the cost of painting the ball varies as the square of its

radius. If a painted ball of radius 6 in. costs £1 17s. 6d., and one of radius 4 in. costs 13s. 4d., find the cost of one whose radius is 2 in.

19. The illumination of a small object by a lamp varies directly as the candle-power of the lamp and inversely as the square of the distance of the object from the lamp. If an electric lamp of 10 candle-power, fixed 5 ft. above a table, is replaced by a 6 candle-power lamp, how much must the new lamp be lowered to give the same illumination as before at the point of the table directly under the lamp?

20. The cost of making a spoon of given material and given shape is the sum of two parts, which vary as the cube and square respectively of the length. If a shillings, $6a$ shillings are the costs for two spoons, the latter being twice as long as the former, find the cost of a spoon $1\frac{1}{2}$ times as long as the latter.

21. The kinetic energy T of a falling body varies directly as the product of its mass m and the square of the time t during which it has fallen; the momentum M varies as the product of m and t. Show that, if T is expressed as a function of M and m, T varies directly as the square of M and inversely as m.

22. The cost of making an overcoat is assumed to consist of a fixed sum together with an additional sum which varies inversely as the number of coats made at the factory in a day. When the number made is 40, the cost of each is £3 3s., and when the number is 100, the cost of each is £3. Find the cost of an overcoat when the daily production is 80.

23. The total *daily* cost of running a ship is made up of a fixed sum for wages, etc. and a sum which varies as the square of the ship's speed, which is assumed to be constant throughout a trip. When a certain trip takes 5 days the total cost is £1060, and when it takes 6 days the total cost is £1250; prove that the total cost of the trip, when it takes n days, is £$100 \times \left(\frac{2n^2+3}{n}\right)$.

24. The force of gravity on the surface of different planets varies jointly as the density and the radius of the planet. The radius of Jupiter is 10 times that of the earth; the density of the earth is 5·67 and that of Jupiter 1·75. If a man can jump 5 ft. high on the earth, how high can he jump on Jupiter, assuming that the height of his jump is inversely proportional to the force of gravity?

25. The sag at the centre of a plank of given width varies as the fourth power of its length and inversely as the square of its thickness. If a plank 5 ft. long and $\frac{1}{2}$ in. thick sags $\frac{1}{4}$ in., what will be the sag in a similar plank of the same width but $7\frac{1}{2}$ ft. long and $\frac{3}{4}$ in. thick?

26. The illumination of a screen varies as the strength of a source of light and inversely as the square of its distance from the screen. A source of light distant 3 ft. from the screen produces the same illumination as would three standard candles, one 1 ft., one 2 ft. and the other 4 ft. from the screen. Compare the illumination with that given by a single standard candle 3 ft. from the screen.

27. The distance in feet that a body moves from rest is the difference between two terms, one of which varies as the square and the other as the cube of the time in seconds from the start. If in 2 sec. the body moved 36 ft. and in the 3rd sec. it moved 108 ft., find how far it moved in the 1st and 4th seconds.

28. The average daily cost per head of providing a school dinner is partly constant and partly varies inversely as the number of pupils who take dinner. When 144 pupils take dinner the average cost per head is $8\frac{1}{2}$d. per day; if the number rises to 168 the average daily cost per head falls to 8d. By how much more must the number rise before the average daily cost per head falls to 7d.?

29. The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same. The volume is 396 c.c. when the height is 14 cm. and the radius of the base is 3 cm.; what will be the height of a cylinder on a base of radius 7 cm. when the volume is 770 c.c.?

30. The coal consumption of a vessel varies as the distance steamed when the speed is constant, and as the square of the speed when the distance is constant. Show that the coal consumption also varies jointly as the time taken for the voyage and the cube of the speed.

TEST PAPERS VIII

A

1. If $p(1-b)=1+b$ and $q(1-c)=1+c$, express $\dfrac{p-q}{1+pq}$ as simply as possible in terms of b and c.

2. Find the first 3 terms and the last 3 terms in the quotient of $x^{11}-11x+10$ when divided by x^2-2x+1.

3. (i) Simplify $(16b^{\frac{3}{2}}a^{-2})^{\frac{1}{4}} \times (\sqrt{2}a^{\frac{1}{4}}c^{-3})^4 \div (a^{-\frac{1}{2}}bc^{-6})^2$.

(ii) Without using tables, find the value of

$$(a)\ (1\tfrac{25}{144})^{-\frac{3}{2}}, \quad (b)\ 10^{\bar{1}\cdot 7} \times 10^{\bar{1}\cdot 3}.$$

(iii) Express $(2^{\frac{1}{3}}+2^{\frac{2}{3}}+2)(2^{\frac{1}{3}}-1)$ as a power of 2.

4. Evaluate (i) $10^{\bar{2}\cdot 345}$,

$$\text{(ii)}\ \sqrt[3]{\frac{1297 \times 3}{4 \times 3\cdot 142}}, \quad \text{(iii)}\ \sqrt[4]{2\cdot 56}+\sqrt[5]{0\cdot 658}.$$

5. (i) If $x : a = y : b = z : c$, prove that each fraction is equal to

$$(x+2y+3z) \div (a+2b+3c).$$

(ii) If $2x^2 - 13xy - 24y^2 = 0$, find the possible values of $x : y$.

6. The following table shows the theoretical time, y sec., taken to cover 1 mile at maximum speed by a yacht whose rating is x metres :

x	5	7·5	10	12·5	15	20	25	30
y	576	470	408	365	333	288	258	235

Plot y against x, taking 1″ to represent 100 sec. and 1″ to represent 5 metres.

In a race of k miles between two yachts of different ratings, the time-handicap in favour of the smaller yacht is k times the difference of their theoretical times for 1 mile. From your graph deduce the time handicap for a 40 mile race between two yachts whose ratings are 19·75 and 23·25 metres.

B

1. The annual incomes of two persons are in the ratio $x : y$ and their annual expenditures are in the ratio $x-1 : y+1$; if each person saves £20 per annum, find their annual incomes.

2. (i) Multiply together x^2, $\frac{1}{\sqrt{x}}$, x^{-4} and $x^{\frac{3}{2}}$.

(ii) If $p = ab^l$, $q = ab^m$, $r = ab^n$, evaluate $p^{m-n} \,.\, q^{n-l} \,.\, r^{l-m}$.

3. (i) Simplify $\frac{2x-1}{x-1} - \frac{x-1}{x^2-5x+6} + \frac{x+1}{x^2-4x+3}$.

(ii) If $x = p^2 + 2pq - q^2$, $y = q^2 + 2pq - p^2$, $z = p^2 + q^2$, find the value of $x^2 + y^2 - 2z^2$.

4. The square of the velocity of a particle varies inversely as the distance it has travelled from a fixed point. If the distance is increased by 1·4 per cent., find the approximate percentage decrease in the velocity.

5. Solve the equations

$$\text{(i) } \sqrt{2x-1} - \sqrt{x+3} = 1, \qquad \text{(ii) } 3^{\frac{2}{x}} = \tfrac{1}{729}.$$

6. A polynomial $f(x)$, when divided by $x-1$, leaves a remainder 3, and, when divided by $x-2$, leaves a remainder 1. Show that, when divided by $(x-1)(x-2)$, it leaves a remainder $-2x+5$.

C

1. If $17x + 4y = 21z$, and $43x - 21y = 22z$, find the value of

$$\frac{x^3+y^3+z^3}{3xyz}.$$

2. Express in the simplest form

$$\text{(i) } 3\sqrt{63}-4\sqrt{\frac{1}{5}}-2\sqrt{\frac{9}{7}}+\frac{3}{5}\sqrt{45},$$

$$\text{(ii) } \sqrt{\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}}.$$

3. Simplify $1-\dfrac{(x+a)(y+a)}{a(a-b)}-\dfrac{(x+b)(y+b)}{b(b-a)}.$

4. Simplify (i) $\dfrac{(x^{\frac{1}{2}}-y^{\frac{1}{2}})(x^2-y^2)}{(x+y+2x^{\frac{1}{2}}y^{\frac{1}{2}})(x+y-2x^{\frac{1}{2}}y^{\frac{1}{2}})},$

$$\text{(ii) } a^{\frac{1}{4}}b^{\frac{2}{3}}\left(\frac{4a^{\frac{1}{4}}}{9b^{-1}}\right)^{\frac{1}{2}}\div\frac{a^{\frac{5}{24}}}{b^{\frac{5}{6}}}.$$

5. Evaluate (i) $(50{\cdot}31)^{\frac{2}{3}}\div 7{\cdot}862,$

(ii) $(5{\cdot}67)^2\div\sqrt{0{\cdot}0765},$

(iii) $\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}},$

where $a=120{\cdot}5,\ b=91{\cdot}7,\ c=127{\cdot}4,\ 2s=a+b+c.$

6. The longer side of a rectangle is increased by k per cent. of itself and the shorter side is decreased by the same percentage. Show that the area of the rectangle is decreased, and find k, if the area decreases from 2400 to 2394 sq. units.

D

1. (i) Express as powers of 2 : $0{\cdot}0625,\ \sqrt{\frac{1}{8}},\ 1\div\sqrt{\frac{1}{32}}.$

(ii) Without using tables, find the value of

$$9^{\frac{1}{2}}\times 16^{\frac{3}{4}}+\left(4^{2\cdot 5}\times\frac{1}{16^{-\frac{3}{2}}}\right).$$

2. Prove that the equation $\dfrac{32}{5+x}-\dfrac{21}{4+x}=1-\dfrac{(x-3)(x+1)}{(5+x)(4+x)}$ is satisfied by any value of x, except $-4,\ -5$, but that the equation $\dfrac{32}{5+x}-\dfrac{21}{4+x}=1$ is satisfied by only two values of x. Find these values.

3. Express in the simplest form :

$$\text{(i) } (2\sqrt{2}-\sqrt{3})(3\sqrt{8}+\sqrt{3})(\sqrt{27}-\sqrt{2}),$$

$$\text{(ii) } \frac{\sqrt{12-6\sqrt{3}}}{\sqrt{3}-1}.$$

4. Without using tables,

(i) Calculate $\log_{10}(\sqrt[3]{10} \times 100 \div \sqrt{10})$,

(ii) Prove that $\log_{10}(\frac{4}{25}) + \log_{10}(\frac{125}{7}) - \log_{10}(\frac{2}{7}) = 1$.

5. It is given that x varies as $2 + y^3$ and that $x = 4$ when $y = -1$; also that y varies as $\sqrt[3]{3 - z}$ and that $y = 2$ when $z = -5$. Obtain the expression for x in terms of z.

6. Solve the equations:

$$\text{(i) } \sqrt{x - 1} + \sqrt{x + 4} = \sqrt{3x + 10},$$
$$\text{(ii) } 2(8^x + 8^{-x}) = 5.$$

E

1. Find $x : y$, (i) if $\dfrac{x + 2y}{6x - y} = \dfrac{11}{14}$,

(ii) if $6x^2 + 11xy = 35y^2$.

2. (i) Multiply $x^{\frac{2}{5}} + 2x^{\frac{1}{5}} - 1 + 2x^{-\frac{1}{5}} - x^{-\frac{2}{5}}$ by $x^{\frac{1}{5}} - x^{-\frac{1}{5}}$.

(ii) Express $\dfrac{a^{\frac{1}{3}}b^{\frac{5}{2}}}{c^3}$ as a cube root.

3. Without using tables, find the value of:

(i) $10^{2\cdot1} \times \sqrt{10^{1\cdot8}}$, (ii) $(12)^{\frac{1}{4}} \times (8)^{\frac{3}{8}} \div (\sqrt{18})^{-\frac{3}{4}}$,

(iii) $\log_{10}(\sqrt[3]{10} \div \sqrt[5]{10})$.

4. The cost (c) of boring a well varies partly as the depth (d) and partly as the square of the depth. A well 35 ft. deep costs £113 15s., and a well 50 ft. deep costs £200. How deep a well could be bored for £168 15s.?

5. Evaluate (i) $10^{\bar{1}\cdot7820}$,

$$\text{(ii) } \frac{3\cdot246 \times \sqrt[3]{0\cdot01638} \times 0\cdot0097}{(96\cdot51)^2 - (42\cdot87)^2}.$$

6. Solve (i) $(x + 1)(x - 2)(x - 3) = (x - 4)(x + 5)(x - 5)$;

(ii) $9x + 4y - 7z = 0$; $x + 2y = 2z$; $3x^2 + 4yz = 181$.

F

1. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}$, prove that each ratio is equal to $\sqrt{\dfrac{2ac + 3eg}{2bd + 3fh}}$ and to $\sqrt[3]{\dfrac{5a^3 - 8ceg}{5b^3 - 8dfh}}$.

2. Simplify $\dfrac{1}{a - b} - \dfrac{2b}{a^2 - b^2} - \dfrac{a + b}{a^2 + b^2}$.

3. Simplify (i) $(a+2b)^{\frac{1}{2}}(a-2b)^{-\frac{1}{3}}(a^2-4b^2)^{-\frac{1}{6}}$,

(ii) $\sqrt{\dfrac{\sqrt{5}+1}{\sqrt{5}-1}}-\sqrt{\dfrac{\sqrt{5}-1}{\sqrt{5}+1}}$.

4. Evaluate (i) $0{\cdot}9434\times\sqrt[3]{22{\cdot}73}$,

(ii) $\dfrac{613[(1{\cdot}2)^{15}-1]}{8917}$.

5. Solve the equations:

(i) $\sqrt{x-2}+\sqrt{x+46}=12$,

(ii) $2^{x+1}+2^x-2^{x-1}=40$.

6. It is known that x and y are connected by the relation $y-4=ax^n$, and the following corresponding values of x and y are obtained experimentally:

$x=$	10	20	30	40	50
$y=$	10·27	14·2	17·5	20·5	23 3

Draw a graph showing the relation between $\log_{10}x$ and $\log_{10}(y-4)$, and by means of the graph obtain approximate values for a and n.

G

1. Find the value of $\dfrac{t+c}{t-c}+\dfrac{t+d}{t-d}$, when $\dfrac{2}{t}=\dfrac{1}{c}+\dfrac{1}{d}$.

2. If $\dfrac{x+y}{2a+b}=\dfrac{y+z}{2b+c}=\dfrac{z+x}{2c+a}$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{(b+c)x+(c+a)y+(a+b)z}{2(ab+bc+ca)}.$$

3. Without using tables, find the value of:

(i) $27^{\frac{2}{3}}-4^{\frac{3}{2}}+2^{-2}$, (ii) $48\times(2^2\times3^2)^{-\frac{5}{3}}\times2^{-\frac{2}{3}}\times3^{\frac{7}{3}}$.

If $\dfrac{4^x\times2^y}{2^{x+2y}}=2^p$, find p in terms of x and y.

4. Evaluate (i) $(0{\cdot}324)^{\frac{2}{3}}\times\sqrt{3(19{\cdot}7)^3}\div530{\cdot}7$,

(ii) $\log 4{\cdot}393\times\log 2{\cdot}198+\log 5{\cdot}834\times\log 8{\cdot}704\times\log 6{\cdot}593$.

5. The total cost of making a certain article consists partly of a constant sum independent of the size of the article, and partly of a sum which varies as the cube of the length of the article. If the cost is 6s. 3d. when the length is 6 in., and 12s. 9d. when the length is $1\frac{1}{2}$ ft., what will it be when the length is 4 ft.?

6. Given $\log_{10}y=1+\frac{5}{2}\log_{10}x$, express y in terms of x, and find the value of y when $x=9$, without using tables.

H

1. Prove that, if $x+y=z$, and none of these quantities is zero, the expression

$$\frac{1}{x^2+y^2-z^2}+\frac{1}{y^2+z^2-x^2}+\frac{1}{z^2+x^2-y^2}$$

is equal to zero.

2. Simplify (i) $\dfrac{x^{\frac{1}{n}}+x^{\frac{1}{n+1}}}{x^{\frac{1}{n}}-x^{\frac{1}{n+1}}}$, where $x=\left(\dfrac{a+1}{a-1}\right)^{n^2+n}$;

(ii) $\dfrac{7+\sqrt{5}}{7-\sqrt{5}}+\dfrac{\sqrt{11}-3}{\sqrt{11}+3}$.

3. (i) Divide $a(b^{\frac{1}{3}}+c^{-1})+b(c^{-1}+a^{\frac{1}{2}})-c^{-2}(a^{\frac{1}{2}}+b^{\frac{1}{2}})+a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-1}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{-1}$.

(ii) Find the square root of

$$x^2+4x^{\frac{3}{2}}y^{\frac{1}{3}}-2xy^{\frac{2}{3}}-12x^{\frac{1}{2}}y+9y^{\frac{4}{3}}.$$

4. The distance in feet that a body moves from rest is the algebraic sum of two terms, one of which varies as the time in seconds from the start, and the other as the square of the time. If the body moves 36 ft. in the first second and 68 ft. in the next second, find how far it will have moved in the third second.

5. If the population of a town has been increasing by R per 1000 each year and was P ten years ago, show that it is now

$$P\times\left(\frac{1000+R}{1000}\right)^{10}.$$

If it is now 155,000 and was 129,000 ten years ago, find the value of R, correct to the nearest unit, by means of logarithms.

6. Solve (i) $x^4-(3x^2+6x+11)=(x^2-4x-1)(x^2+4x+1)$, correct to 2 decimal places;

(ii) $\dfrac{x^2-5x+6}{x^2+5x-6}=\dfrac{x^3-5x^2-x-7}{x^3+5x^2+x+7}$.

I

1. A shopkeeper spends a certain sum in stocking an article, and later, when the cost price of the article has risen 50 per cent., spends an equal sum on a further supply. He sells the whole stock at the mean of the two prices he paid, viz. 10d., and makes a profit of £4. Find the prices he paid and the sum spent on each occasion.

Prove that always, if he spends, as above, equal sums on the two occasions, and sells at the mean price, he necessarily makes a profit.

2. What is the smallest number of factors which must be taken in the product $2 \times 2 \times 2 \ldots$ in order to give a number greater than 10^{90}?

3. Simplify (i) $2^n \times 15^{n+1} \times 6^{-n+2} \times 5^{-n+1}$,

(ii) $\dfrac{\sqrt{3}+1}{1+\sqrt{2}+\sqrt{3}+\sqrt{6}} - \dfrac{\sqrt{5}-1}{1-\sqrt{2}-\sqrt{5}+\sqrt{10}}$.

4. Evaluate (i) $\dfrac{(78 \cdot 51)^2 \times 0 \cdot 004374}{68 \cdot 68}$,

(ii) $(0 \cdot 38)^{-1 \cdot 2} + (2 \cdot 37)^{-0 \cdot 6}$.

5. Solve the equations :

(i) $\sqrt{4x+5} + \sqrt{4x-3} - 4\sqrt{x} = 0$,

(ii) $x^2(\sqrt{3}-1) + 2x = \sqrt{3}+1$.

6. If $\dfrac{m+n}{am+bn} = \dfrac{n+l}{an+bl} = \dfrac{l+m}{al+bm}$, prove that either each ratio $= \dfrac{2}{a+b}$, or $l+m+n=0$.

J

1. (i) If $\dfrac{y+z-x}{3} = \dfrac{z+x-y}{4} = \dfrac{x+y-z}{5}$, prove that $\dfrac{x}{9} = \dfrac{y}{8} = \dfrac{z}{7}$.

(ii) If $\dfrac{a^2-bc}{l} = \dfrac{b^2-ca}{m} = \dfrac{c^2-ab}{n}$, and a, b, c are unequal, prove that $(a+b+c)(l+m+n) = al+bm+cn$.

2. Find the square root of $4x^{\frac{2}{7}} + x^{-\frac{2}{7}} - 4x^{\frac{1}{7}} + 2x^{-\frac{1}{7}} - 3$.

3. Simplify $\dfrac{x(t-x)}{(x-y)(x-z)} + \dfrac{y(t-y)}{(y-z)(y-x)} + \dfrac{z(t-z)}{(z-x)(z-y)}$.

4. The deflection at the centre of a girder of given material with fixed ends under a uniformly distributed load varies directly as W, the load, and the cube of L, the length of the girder, and inversely as I, its moment of inertia. If the load is increased by 10 per cent. and the moment of inertia by 5 per cent., find the percentage change in the length of the girder that the deflection may be unaltered.

5. (i) If $e = 2 \cdot 718$, find $\log_e 25$;

(ii) Evaluate $\dfrac{(4 \cdot 65)^{\frac{4}{5}} \log_e 25}{(1 \cdot 612)^3}$.

6. From the equations $4x - 2y - 7z = 0$ and $3x + 8y = 29z$, find $x : y : z$, and prove that $x^2 + 4y^2 = 34z^2$.

K

1. If l, $2m$, n are three consecutive integers, find the value of $l^2 + n^2 - 8m^2$.

2. If $\dfrac{2y+2z-x}{l}=\dfrac{2z+2x-y}{m}=\dfrac{2x+2y-z}{n}$, prove that

$$\frac{x}{2m+2n-l}=\frac{y}{2n+2l-m}=\frac{z}{2l+2m-n}.$$

3. (i) Simplify $(1+x^{\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{2}{3}}y^{-\frac{2}{3}})\times(x^{\frac{2}{3}}-y^{-\frac{2}{3}})\div x^{-\frac{2}{3}}y^{\frac{2}{3}}$.

(ii) Find the square root of $a^{\frac{8}{9}}+4a^{\frac{5}{9}}-2a^{\frac{1}{3}}+4a^{\frac{2}{9}}-4+a^{-\frac{2}{9}}$.

4. Evaluate (i) $(0\cdot5673)\times\sqrt[3]{187}\div(63\cdot41)^2$,

(ii) $(0\cdot792)^{2\cdot367}$.

5. Solve $9x^2-3xy-2yz=12$; $51x-4y=42z$; $129x+21y=44z$.

6. If the relation $y=ax^n$ is satisfied by $x=2$, $y=10\cdot6$, and by $x=3$, $y=6\cdot2$, find a and n.

L

1. (i) If $2s=a+b+c$, express

$$s^2+(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)$$

as simply as possible in terms of a, b, c.

(ii) Divide $4x^{\frac{4}{5}}+7x^{\frac{3}{5}}-x^{\frac{2}{5}}+5x^{\frac{1}{5}}+6$ by $x^{\frac{2}{5}}-x^{\frac{1}{5}}+1$.

2. Reduce $\dfrac{5-\sqrt{5}}{(7+\sqrt{5})(\sqrt{5}-2)}$ to its simplest form, and calculate its value correct to 2 decimal places, given $\sqrt{5}=2\cdot2361$.

3. Simplify $\dfrac{ac-bc}{(b-c)(c-a)}+\dfrac{bc}{(b-c)(a+b)}+\dfrac{ac(a-b)}{(b-c)(a-c)(a+b)}$.

4. Prove that the expression

$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$$

can be written as the sum of three squares; and that, if the expression equals zero and all the quantities involved are real, $x:a=y:b=z:c$.

5. The amount of coal per hour burnt by a steamer on a voyage consists of the sum of two parts, of which one part $\propto$ the speed in knots and the other part $\propto$ the cube of the speed. If a steamer burns 2 tons of coal per hour when travelling at 10 knots and $4\frac{1}{2}$ tons per hour when travelling at 15 knots, find how many tons per hour it will burn when travelling at 20 knots.

6. The velocity in ft. per sec. of sound in air at temp. $t°$ C. is given by the formula

$$v=\sqrt{\left\{45\cdot08\times\frac{p\times144}{d}\times\left(1+\frac{t}{273}\right)\right\}},$$

where p is the pressure of the air in lb. per sq. in., and d is the density in lb. per cu. ft. of air at 0° C. and atmospheric pressure. Change this formula into one giving t in terms of p, d, v. Find t when $v=1118\cdot9$, $p=14\cdot7$, $d=0\cdot0809$.

CHAPTER XXXI

SERIES. THE PROGRESSIONS

220. The formation of a sequence or series. We have previously considered algebraic functions of x, e.g. x^2-3x+7, and we have studied the change in the value of the function as the value of x passed continuously through all values within a given range; such change in value was usually illustrated by a graph. We now proceed to consider functions in which the variable may represent only one of the natural numbers 1, 2, 3, ... To avoid confusion, such functions are usually written $f(n)$, where n may stand only for one of the natural numbers. Thus, if we consider the function n^2 and give n the values 1, 2, 3, ... in succession, we obtain the numbers $1^2, 2^2, 3^2$ ..., i.e. 1, 4, 9, ... respectively. A succession of numbers obtained in this manner, in accordance with some given law, is called a **sequence** or **series.**

The value of any particular term in the sequence is obtained by substituting the value of n. Thus, in the sequence given by the function n^2+n, the 5th term is 5^2+5, i.e. 30; the rth term is r^2+r; the nth term is n^2+n, etc.

221. Conversely, it is sometimes possible to deduce the function from a knowledge of the sequence. Thus, consider the sequence 4, 11, 18, 25, 32, ...

It is clear that we start with 4 and add 7 each time.

To obtain the 5th term, we add 7 four times and obtain $4+7\times4=32$; to obtain the nth term, we add 7 $(n-1)$ times and obtain $4+7\times(n-1)=7n-3$. Thus, the above sequence represents the function $7n-3$, where n stands for a natural number.

EXERCISE 101

(Many of these examples may be taken orally)

1. Write down the first three terms, the 8th term and the rth term of the series given by the functions:

(i) $3n-2$, (ii) $3n+8$, (iii) $\frac{1}{n^2}$, (iv) $\frac{2n+3}{3n+4}$,

(v) $n^2 - n$, (vi) $\dfrac{1}{2n-1}$, (vii) $7 \times 2^{n-1}$, (viii) $(-1)^n$,

(ix) $18 - 4n$, (x) $3^{n-1} - (-\frac{1}{2})^{n-1}$.

2. Write down 4 terms of the sequence obtained as below. Also write down the nth term, leaving the answer in unsimplified form, e.g. $9 \times 3^{n-1}$.

(i) Write down 10 and add 3 each time.

(ii) Write down 10 and subtract 4 each time.

(iii) Write down 256 and divide by 2 each time.

(iv) Write down $\frac{1}{27}$ and multiply by -3 each time.

3. Write down the next two terms of the following sequences:

(i) $64, -32, 16, -8, 4, \dots$, (ii) $20, 18, 16, 14, 12, \dots$,

(iii) $\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \dots$, (iv) $2 \times 5, 4 \times 7, 6 \times 9, 8 \times 11, \dots$.

4. Find the third term in each of the following series :

(i) $35, 40, —, 50, 55, 60, \dots$, (ii) $162, 54, —, 6, 2, \dots$,

(iii) $\frac{3}{5}, \frac{4}{15}, —, \frac{6}{135}, \frac{7}{405}, \dots$, (iv) $16, 25, —, 49, 64, 81, \dots$.

5 Find the nth term in each of the following series, and check by substituting $n = 3$ in your result :

(i) $22, 25, 28, 31, \dots$, (ii) $1\frac{1}{3}, 2\frac{1}{4}, 3\frac{1}{5}, 4\frac{1}{6}, \dots$,

(iii) $\frac{1}{4}, -\frac{1}{2}, 1, -2, 4, \dots$, (iv) $0{\cdot}1, 0{\cdot}02, 0{\cdot}003, 0{\cdot}0004, \dots$.

6. The nth term of a series is $5n + 4$.

(i) Is (*a*) 64, (*b*) 73, a term of the series? If so, which term?

(ii) Which is the first term of the series which is greater than 40?

7. The nth term of a series is $2n - 11$.

(i) How many terms of the series are negative?

(ii) How many terms of the series are positive and less than 80?

8. The nth term of a series is $3n + 8$. What is the difference between (i) the rth term and the $(r-1)$th term, (ii) the rth term and the $(r+1)$th term?

Arithmetical Progressions

222. A series in which each term is formed from the preceding by adding to it a constant quantity is called an **Arithmetical Progression.** The constant quantity is called the **common difference,** and it may be found by subtracting any term from the term which follows it. The abbreviation A.P. is usually used for the words *arithmetical progression*, and the abbreviation C.D. for the common difference.

223. The most general form of an A.P. is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,$$

in which the first term is a, and the common difference is d. It is easy to see that any term may be expressed in terms of a and d; thus, the 8th term is obtained by adding d seven times to the first term; the nth term is obtained by adding d $(n-1)$ times to the first term. These terms are therefore, respectively,

$$a+7d,\ a+(n-1)d.$$

Example 1. *Find the 9th, 26th and nth terms of the series* $62, 57, 52, \cdots$.

The series is an A.P. with first term 62 and C.D. -5;

$$\therefore \text{ the 9th term} = 62+8(-5) = 22;$$

$$\text{the 26th term} = 62+25(-5) = -63;$$

$$\text{the } n\text{th term} = 62+(n-1)(-5) = 67-5n.$$

Example 2. *The 7th and 21st terms of an A.P. are 6 and* -22 *respectively; find the series.*

With the usual notation, $a+6d=6$, $a+20d=-22$; whence, by subtraction, $14d=-28$, $\therefore d=-2$, and from the first equation $a-12=6$, $\therefore a=18$.

Thus the series is $18, 16, 14, \cdots$.

It should be noted that an A.P. is completely determined when any two terms are known, for we can then write down two independent equations connecting a and d.

By solving these equations the values of a and d can be found and the series determined.

224. Arithmetic means. (1) When three numbers are in A.P. the middle term is called the **arithmetic mean** of the other two.

Thus, since 12, 16, 20 are in A.P., 16 is the arithmetic mean of 12 and 20. It will be noted that it is the average of the two numbers. This is proved more generally in the following example.

Example 3. *Find the arithmetic mean of x and y.*

Let $\mathbf{A}$ be the required mean; then, since x, $\mathbf{A}$, y are in A.P., the C.D. $= \mathbf{A}-x = y-\mathbf{A}$;

$$\therefore 2\mathbf{A} = x+y, \quad \therefore \mathbf{A} = \frac{x+y}{2}.$$

This is an important result and should be committed to memory.

(2) When any number of numbers are in A.P., the terms intermediate between the first and the last are called the **arithmetic means** between those two terms. It is always possible to insert any required number of arithmetic means between two numbers.

Example 4. *Insert n arithmetic means between x and y.*

After insertion of the n numbers there will be $n+2$ numbers in A.P. It follows that y is the $(n+2)$th term of an A.P. of which x is the first, $\therefore\ y = x + (n+1)d$, if d is the common difference ;

$$\therefore\ d = \frac{y-x}{n+1}, \text{ and the required means are}$$

$$x + \frac{y-x}{n+1},\quad x + \frac{2(y-x)}{n+1},\quad \ldots,\quad x + \frac{n(y-x)}{n+1}.$$

This result should not be committed to memory. All cases which arise may be done from first principles, as above.

225. In problems involving numbers in A.P. the following devices are useful.

(1) If the number of terms is odd, the *middle* number may be denoted by a. Thus, if we have three numbers, we may denote them by $a-d$, a, $a+d$; if we have five numbers, we may denote them by $a-2d$, $a-d$, a, $a+d$, $a+2d$, and so on.

In each case the middle term is a, and the C.D. is d.

(2) If the number of terms is even, there is no middle number, but the two middle numbers may be denoted by $a-d$, $a+d$; the common difference is therefore $2d$. If we have four numbers they may be denoted by $a-3d$, $a-d$, $a+d$, $a+3d$.

Example 5. *The sum of 4 numbers in A.P. is 70, and the difference between the 2nd and 4th is 14. Find the numbers.*

Let the numbers be denoted by $a-3d$, $a-d$, $a+d$, $a+3d$ respectively. Then

$$4a = 70, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$a + 3d - (a-d) = 14, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

or

$$a + 3d - (a-d) = -14. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Taking (i) and (ii), we obtain $a = 17\frac{1}{2}$, $d = 3\frac{1}{2}$, and the numbers are 7, 14, 21, 28.

Taking (i) and (iii), we obtain $a = 17\frac{1}{2}$, $d = -3\frac{1}{2}$, and the numbers are 28, 21, 14, 7, i.e. the first set of numbers written in the reverse order.

EXERCISE 102

(Many of these examples may be taken orally)

1. Which of the following series are A.P.'s? What is the C.D.?

(i) $6, 10, 14, 18, \dots$, (ii) $24, 21, 18, 15, \dots$,
(iii) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$, (iv) $3, -6, 12, -18, \dots$,
(v) $4x, 7x, 10x, 13x, \dots$, (vi) $3a, 3a-2, 3a-4, 3a-6, \dots$
(vii) $2c+7d, 3c+6d, 4c+5d, \dots$, (viii) $x, x^2, x^3, x^4, \dots$.

2. Find the 7th, 12th, and nth terms of the following A.P.'s:

(i) $11, 19, 27, \dots$, (ii) $100, 87, 74, \dots$,
(iii) $7, 0, -7, \dots$, (iv) $22, 6, -10, \dots$,
(v) $11p, 5p, -p, \dots$, (vi) $a^2+b^2, 5b^2, -a^2+9b^2, \dots$,
(vii) $7x+8y, 10y, -7x+12y, \dots$,
(viii) $3l, 2l+3m, l+6m, \dots$.

3. Find the C.D., the 5th, and nth terms of the following A.P.'s:

(i) $4, -8, -20, \dots$, (ii) $-33, -21, -9, \dots$,
(iii) $5p^2, 11p^2, 17p^2, \dots$, (iv) $-3y, 4x-y, 8x+y, \dots$.

4. Find the first 3 terms of the A.P.'s determined by the following data:

(i) The 4th term is 75, and the 10th term is 117;
(ii) The 7th term is 62, and the 19th term is 2;
(iii) The 9th term is -38, and the 33rd term is 58;
(iv) The 12th term is -65, and the 100th term is -329.

5. Find the number of terms in the following A.P.'s:

(i) $34, 37, 40, \dots 109$; (ii) $152, 145, 138, \dots 47$;
(iii) $51, 62, 73, \dots 271$; (iv) $5, 4\frac{1}{4}, 3\frac{1}{2}, \dots -25$;
(v) $42, 47, 52, \dots 102$; (vi) $11, 10{\cdot}7, 10{\cdot}4, \dots -10$.

6. Find the arithmetic means of:

(i) 202 and 268, (ii) -11 and -6,
(iii) $4a^2-5b^2$ and $2a^2-3b^2$, (iv) $x-7y$ and $7y-x$.

7. Insert 5 arithmetic means between 11 and 29.

8. Insert 17 arithmetic means between 35 and -37.

9. Insert 8 arithmetic means between -42 and -15.

10. Insert $2k$ arithmetic means between a and $a-b-2kb$.

11. The 3rd and 6th terms of an A.P. are 28 and 37 respectively; find the nth term.

12. If $3a$, $5a-6$, $6a+2$ are in A.P., find a and the nth term.

EXERCISE 102. c

1. What is the value of n, if the nth term of

(i) 12, 15, 18, ... is 132 ; (ii) $\frac{1}{8}, \frac{3}{16}, \frac{1}{4}, \ldots$ is $1\frac{1}{2}$;
(iii) $\frac{1}{6}, \frac{1}{12}, 0, \ldots$ is $-2\frac{1}{2}$; (iv) 84, 78, 72, ... is 0?

2. The 12th term of an A.P. is four times the 5th ; the 10th term is greater by 2 than five times the 4th. Find the 1st term and the C.D.

3. The sum of four numbers in A.P. is 44. The product of the 2nd and 3rd numbers exceeds twice the product of the first and last by 32. Find the numbers.

4. The 16th term of an A.P. is four times the 36th term, and exceeds it by 12. Find the 1st term and the C.D.

5. If $2x$ is the ath term and $3y$ is the bth term of an A.P., find the cth term.

6. A clerk receives £250 for his first year, and each year his salary is increased by £4 per annum. The total sum he receives for his rth, $(r+1)$th and $(r+2)$th years is £834. Find the value of r.

7. If l, m, n are the 2nd, 9th and 11th terms respectively of an A.P., prove that $2l+7n=9m$.

8. Find the intermediate terms of the A.P. of which $5(a-b)$ is the first and $5b$ the sixth term.

9. The sum of five numbers in A.P. is 30, and the product of the 2nd, 3rd and 4th is 120. Find the numbers.

10. The sum of the first four terms of an A.P. is 122, and the sum of four terms from the 11th to the 14th inclusive is 2. Find the 1st term and the C.D., and verify the result.

11. If a and b are the first and last terms of an A.P. of $r+2$ terms, find the second and $(r+1)$th terms.

12. Find the missing terms in the following A.P.'s :

(i) 5 ? 11 ? ? ; (ii) -7 ? ? 5 ? ;
(iii) ? ? -3 ? -11 ? ; (iv) ? ? ? 12 ? ? 0.

13. If the 6th term of an A.P. is double the 11th term, show that the 2nd term is double the 9th.

14. Find 5 numbers in A.P. such that their sum is 310, and the difference between the first and fourth is 27.

15. The sum of 3 numbers in A.P. is 81, and the difference of the squares of the greatest and least is 1296. Find the numbers.

16. The sum of 4 numbers in A.P. is 24, and the sum of their squares is 164. Find the numbers.

Summation of an Arithmetical Progression

226. To illustrate the general principle, we first work out in full a simple numerical case.

Example 6. *Find the sum,* **S**, *of* 12 *terms of the series*

$$48 + 46 + 44 + \dots .$$

$$\mathbf{S} = 48 + 46 + 44 + 42 + 40 + 38 + 36 + 34 + 32 + 30 + 28 + 26.$$

But the sum is unaltered, if the numbers are written down in the reverse order,

$$\therefore\ \mathbf{S} = 26 + 28 + 30 + 32 + 34 + 36 + 38 + 40 + 42 + 44 + 46 + 48.$$

Whence, by addition,

$$2\mathbf{S} = 74 + 74 + 74 + 74 + 74 + 74 + 74 + 74 + 74 + 74 + 74 + 74$$

$$= 12 \times 74\ ; \quad \therefore\ \mathbf{S} = \frac{12 \times 74}{2} = 444.$$

227. More generally, we proceed as in the following example.

Example 7. *Find the sum,* **S**, *of n terms of the series, a, $a + d$, ...*

Writing l for the last term, we have

$$\mathbf{S} = a + (a + d) + (a + 2d) + \dots + (l - 2d) + (l - d) + l.$$

But the sum is unaltered, if the numbers are written down in the reverse order,

$$\therefore\ \mathbf{S} = l + (l - d) + (l - 2d) + \dots + (a + 2d) + (a + d) + a.$$

Whence, by addition,

$$2\mathbf{S} = (a + l) + (a + l) + (a + l) + \dots + (a + l) + (a + l) + (a + l)$$
$$= n(a + l)\ ;$$

$$\therefore\ \mathbf{S} = \frac{n(a + l)}{2}. \quad \text{.............................(i)}$$

But $l = a + (n - 1)d$, $\therefore\ a + l = 2a + (n - 1)d$,

$$\therefore\ \mathbf{S} = \frac{n}{2}[2a + (n - 1)d]. \quad \text{......................(ii)}$$

228. The above formulae for **S** are both useful.

The formula (i) is more useful for direct summation; if the numbers are simple, the whole work may be done mentally.

This formula is easily remembered, if it is noticed that $\frac{a + l}{2}$ is the average of the first and last term and also the average of all the terms.

The formula (ii) is very useful in problems in which the value of n has to be found, as in Example 8.

Example 8. *How many terms of the A.P.* 56, 52, 48, ... *must be taken that the sum may be* 416?

It is more convenient to use the formula (ii). Here $\mathbf{S}=416$, $a=56$, $d=-4$,

$$\therefore\ 416=\frac{n}{2}[112+(n-1)(-4)], \quad \ldots\ldots\ldots\ldots\text{(i)}$$

or

$$832=n[116-4n],$$

whence $n^2-29n+208=0$, i.e. $(n-13)(n-16)=0$,

$$\therefore\ n=13 \text{ or } 16.$$

Both values of n satisfy the conditions of the question, for the 14th, 15th and 16th terms are respectively 4, 0, -4, and their sum is 0. Thus the sum of 16 terms is the same as that of 13 terms.

If the pupil does not wish to rely upon the formula, the line (i) may easily be obtained from first principles.

Thus, if n be the number of terms, the last term

$$=56+(n-1)(-4)=60-4n,$$

$\therefore$ the average of the first and last term is $\dfrac{56+60-4n}{2}=58-2n$,

$$\therefore \text{ the average of all the terms is } 58-2n,$$

$$\therefore \text{ the sum is } n(58-2n),$$

$$\therefore\ 416=n(58-2n), \text{ etc., as above.}$$

229. The following notation is sometimes convenient:

The successive terms of a series may be denoted by

$$\mathbf{T}_1,\ \mathbf{T}_2,\ \mathbf{T}_3,\ \ldots\ \mathbf{T}_{n-1},\ \mathbf{T}_n,$$

where the suffix denotes the *number* of the term in the series. Similarly, the sum of any assigned number of terms may be denoted by the letter **S** with a suitable suffix number. Thus, in any examples the symbols $\mathbf{S}_{40}$, $\mathbf{S}_n$ may be used instead of the words "sum to 40 terms", "sum to n terms" respectively.

Most problems connected with arithmetical progressions may be solved by expressing the data in terms of two unknowns, usually a (the first term) and d (the common difference). Occasionally, however, other devices, such as that already mentioned in Art. 225, have to be used.

Example 9. *In an A.P.,* $\mathbf{S}_{50} = 200$, $\mathbf{S}_{100} = 2900$; find a, d, $\mathbf{T}_{200}$ *and* $\mathbf{S}_{200}$.

With the usual notation,

$$\mathbf{S}_{50} = 25[2a + 49d] = 200, \quad \therefore\ 2a + 49d = 8, \quad \ldots\ldots\ldots\text{(i)}$$
$$\mathbf{S}_{100} = 50[2a + 99d] = 2900, \quad \therefore\ 2a + 99d = 58. \quad \ldots\ldots\text{(ii)}$$

Solving (i) and (ii), we obtain $d = 1$, $a = -20\frac{1}{2}$.

Also $\qquad \mathbf{T}_{200} = a + 199d = -20\frac{1}{2} + 199 = 178\frac{1}{2}$,

$$\mathbf{S}_{200} = 100(-20\tfrac{1}{2} + 178\tfrac{1}{2}) = 15800.$$

Harmonical Progression

230. If a series of numbers is such that their reciprocals are in A.P., the series is called a **Harmonical Progression.** The abbreviation H.P. is usually used for the words *harmonical progression*.

Thus, $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \ldots$ are in H.P., because $3, 7, 11, 15, \ldots$ are in A.P.

Examples in H.P. are usually solved by inverting the terms and using the properties of the corresponding A.P. There is no general formula for the sum of a number of terms in H.P.

231. Harmonic mean. If x, H, y are in H.P., H is said to be the harmonic mean of x and y. To calculate H, we use the fact that $\frac{1}{x}, \frac{1}{H}, \frac{1}{y}$ are in A.P.,

$$\therefore\ \frac{1}{H} - \frac{1}{x} = \frac{1}{y} - \frac{1}{H}, \quad \therefore\ \frac{2}{H} = \frac{1}{x} + \frac{1}{y},$$

$$\therefore\ H = \frac{2xy}{x+y}.$$

Example 10. *Find the H.P. in which* $\mathbf{T}_{12} = \frac{1}{50}$, $\mathbf{T}_{20} = \frac{1}{82}$.

Let a be the 1st term, and d the C.D. of the corresponding A.P.

Then $\qquad a + 11d = 50, \quad a + 19d = 82,$

whence $\qquad a = 6, \quad d = 4.$

Hence the A.P. is $\qquad 6, 10, 14, \cdots;$

$\therefore$ the H.P. is $\qquad \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \cdots.$

EXERCISE 103

Find the last term and the sum of:

1. $2 + 4\frac{1}{2} + 7 + \ldots$ to 21 terms. **2.** $4 + 7 + 10 + \ldots$ to 83 terms.

3. $1\frac{1}{3}+3+4\frac{2}{3}+\ldots$ to 25 terms. **4.** $\frac{1}{4}+\frac{3}{8}+\frac{1}{2}+\ldots$ to 65 terms.

5. $1+0{\cdot}9+0{\cdot}8+\ldots$ to 51 terms. **6.** $\frac{1}{4}+\frac{3}{10}+\frac{7}{20}+\ldots$ to n terms.

Find the sum of:

7. $42+34+26+\ldots$ to 24 terms.

8. $-11-14-17-\ldots$ to 40 terms.

9. $4\frac{1}{6}+4\frac{3}{4}+5\frac{1}{3}+\ldots$ to 21 terms.

10. $2+2\frac{1}{2}+3+\ldots$ to 100 terms.

11. $2\frac{1}{2}+3\frac{1}{4}+4+\ldots$ to 20 terms.

12. $-3\frac{1}{2}-1\frac{1}{4}+1+\ldots$ to n terms.

13. $a+(a+2x)+(a+4x)+\ldots$ to n terms.

14. $(2b-3)+(2b-1)+(2b+1)+\ldots$ to 24 terms.

How many terms must be taken of the series:

15. $24+20+16+\ldots$ to make 84?

16. $-9, -7, -5, \ldots$ to make 875?

17. $13+14{\cdot}2+15{\cdot}4+\ldots$ to make $4102{\cdot}6$?

18. $100+93\frac{3}{4}+87\frac{1}{2}+\ldots$ to make $812\frac{1}{2}$?

19. In an A.P., the ratio of the 6th term to the 9th is $-1:5$, and the sum of these two terms is -12. Find the sum of the first 100 terms.

20. A man's annual income has increased by the same fixed amount every year since 1925; if this income was £264 in 1927 and £339 in 1932, find his total income for the years 1925 to 1935 inclusive.

21. A besieged fortress is held by 5700 men, who have provisions for 66 days. If the garrison loses 20 men each day, for how many days can the provisions hold out?

22. Find the sum of $3x$ terms of the A.P. $12a+10b$, $13a+6b$, $14a+2b$, ...

23. The first and last terms of an A.P. are -3 and 25 and the sum of the series is 1837. Find the number of terms and the C.D.

24. Find the sum of all the numbers from 1 to 111 inclusive which are divisible by 3.

25. Find the sum to n terms of the series whose rth term is $5r-3$.

26. The 6th term of an A.P. is 102, and the 16th term is 78; find the 1st term, the C.D., and the sum of all the terms which are positive.

27. A parent puts in a box for a child on every birthday a half-crown for every year of its age. How old will the child be when the value of the money in the box is £17?

28. The side wall of a lean-to-shed slopes from a height of 5 ft. to a height of 20 ft., and 17 upright slips of wood are fastened to the wall at equal intervals, each reaching from the ground to the top of the wall. Find the total length of the wood.

29. Find the sum of all even numbers from 4 to 100 inclusive, excluding those which are multiples of 3.

30. An A.P. has 13 terms whose sum is 143. The third term is 5. Find the first term.

31. A series of fractions is written down as follows :

$$\frac{1}{1}, \frac{1+3}{1+4}, \frac{1+3+5}{1+4+7}, \frac{1+3+5+7}{1+4+7+10}, \dots .$$

Find a simple expression for the nth term of the series.

32. In an A.P. of n terms, the sum of the first two terms is a, and the sum of the last two terms is b. Find the sum of the n terms.

33. Find the harmonic mean between (i) 4 and 6, (ii) -3 and -8, (iii) $\frac{1}{a}$ and $\frac{1}{b}$.

34. Insert 3 harmonic means between (i) $-\frac{1}{6}$ and $\frac{1}{14}$, (ii) $\frac{2}{7}$ and $\frac{6}{13}$.

35. Insert 4 harmonic means between 2·5 and 15.

36. Find the 6th and nth terms of the following series in H.P. :

$$\text{(i) } 1, \tfrac{8}{9}, \tfrac{4}{5}, \dots, \quad \text{(ii) } \tfrac{1}{6}, \tfrac{7}{34}, \tfrac{7}{26}, \dots .$$

37. In a H.P., $\mathbf{T}_5 = \frac{1}{7}$, $\mathbf{T}_{16} = \frac{3}{65}$, find $\mathbf{T}_7$.

38. In a H.P., $\mathbf{T}_4 = -\frac{5}{7}$, $\mathbf{T}_7 = \frac{1}{4}$, find $\mathbf{T}_{12}$.

39. In a H.P., $\mathbf{T}_7 = 7$, $\mathbf{T}_{10} = 10$, find $\mathbf{T}_{16}$.

40. In a H.P., $\mathbf{T}_1 = 1$, $\mathbf{T}_3 = -2$; write down the first five terms of this progression. Find the sum of 13 terms of the A.P., the reciprocals of the terms of which form the above H.P.

EXERCISE 103. c

1. How many terms are there in the series 13, 16, 19, ⋯ 139? Find the middle term of the series and its sum.

2. Find the sum of all the multiples of 13 between 750 and 1000.

3. In an A.P., $\mathbf{T}_1 = 8$ and the average value of n terms is 62. Find $\mathbf{T}_n$.

4. A certain progression contains the following terms, the dots indicating that some terms have been omitted :

$$3\tfrac{1}{3}, 4\tfrac{2}{3}, 6, \dots, 32\tfrac{2}{3}, 34, 35\tfrac{1}{3}.$$

Find the sum of the missing terms.

5. In an A.P., $\mathbf{T}_1 = 12$, and the sum of the first 15 terms exceeds by 33 twice the sum of the next 7 terms. Find $\mathbf{T}_{15}$ and $\mathbf{S}_{15}$.

6. In the A.P. $1+4+7+10+\ldots$, find $\mathbf{S}_{10}$ and show that $\mathbf{S}_n = \frac{3}{2}\{(n-\frac{1}{6})^2 - \frac{1}{36}\}$. For what values of n does $\mathbf{S}_n$ lie between 300 and 400?

7. A street has its houses numbered with the consecutive numbers 1, 2, 3, etc. up to 288. Show that the sum of the numbers on the houses *before* No. 204 equals the sum of the numbers on the houses *after* No. 204.

8. In an A.P., $\mathbf{T}_3 = 15$, $\mathbf{T}_{18} = 90$. Find $\mathbf{T}_{100}$ and $\mathbf{S}_{100}$.

9. Find the sum of the whole numbers from 1 to 200 inclusive which are not divisible by 7.

10. In an A.P., $\mathbf{T}_5 = 9$, $\mathbf{S}_{11} = 115\frac{1}{2}$. Find $\mathbf{T}_1$ and the C.D.

11. 50 arithmetic means are inserted between 20 and 120. Find their sum.

12. A certain County Council pays its Grade B clerks according to the following scale: the commencing salary is £80 per annum, rising after *two* years' service by *annual* increments of £7 10s. per annum to a maximum salary of £230 per annum. How much will a Grade *B* clerk receive during 45 years' service?

13. If $\dfrac{1+3+5+7+\ldots \text{ to } n \text{ terms}}{1+2+3+4+\ldots \text{ to } n \text{ terms}} = 1{\cdot}9$, find n.

14. Prove that the sum of the odd numbers from 1 to 55 inclusive is equal to the sum of the odd numbers from 91 to 105 inclusive.

15. A debt of £54,000 is to be paid off by annual instalments. If the first instalment is £100 and each instalment after the first is £40 more than the preceding one, in how many years after the first instalment will the debt be cancelled?

16. A man has charge of 23 machines, each of which when started goes on working automatically and can produce 6·5 yards of material per hour. The man starts the first machine at 9 a.m. and the others at intervals of 5 minutes. What will be the total length of material produced at 1 p.m.?

17. There are 2457 plants in a strawberry bed; in each row the number of plants exceeds the number in the preceding row by a fixed amount; in the first row there are 77, and in the last row 157 plants. How many rows are there?

18. There are 48 terms in an A.P. and the two middle terms are $2\frac{1}{4}$ and $2\frac{3}{4}$. Find the sum.

19. The sum of n terms of an A.P. is 2718. If the C.D. is 5 and the $(n+1)$th term is 168, find n.

20. Two men start work at the rate of £200 a year ; salary is to be paid quarterly, each receiving £50 at the end of the first quarter. One is to receive an increase of £1 per quarter and the other £15 per year. Find the amount received by each in 12 years. Which is the better bargain and by how much in n years?

21. If the nth term of the series 8, 8·6, 9·2, 9·8, ... is the first which is greater than 21, find n. Sum the series as far as this term and find also the sum of the next n terms.

22. In a "potato race" 15 potatoes are arranged in a straight line at intervals of 2 yards, and in the same straight line, 5 yards from the end potato, and away from the potatoes, is placed a bucket. The competitor has to start from the bucket, bring the potatoes singly, and place them in the bucket. How far has he to run altogether?

23. In an A.P., the first term is 4 and the C.D. is 7. How many terms of the progression are required in order that the sum of these terms should exceed 500?

24. A and B begin work together. A's initial salary is £200 a year and he has an annual increment of £20. B is paid at first at the rate of £40 a half-year, and each half-year he has an increase of £8 in his half-yearly salary. At the end of how many years will B have received more than A?

25. Find the nth term and the sum to n terms of the A.P. whose 1st and 5th terms are 3 and 19 respectively. Find which term of the series is most nearly equal to 1000 and by how much it differs from 1000.

26. Find the sum of $2n+1$ terms of an A.P. whose 1st term is a and C.D. b. If the sum of the first n terms is equal to the sum of the remaining terms, prove that the progression, if continued, must contain one zero term.

27. Show that if $\dfrac{a}{1+ak}, \dfrac{b}{1+bk}, \dfrac{c}{1+ck}$ are in H.P. for any one value of k, they are so for all values of k.

28. If u, v, x, y are in H.P., find v and x in terms of u and y, and prove that $uv+vx+xy=3uy$.

29. If a, b, c are in H.P. and b, c, d are in A.P., prove that $ad=bc$.

30. If $y+z,\ z+x,\ x+y$ are in H.P., prove that x^2, y^2, z^2 are in A.P.

Geometrical Progressions

232. A series in which each term is formed from the preceding by multiplying it by a constant factor is called a **Geometrical Progression.** The constant factor is called the common ratio, and it may be found by dividing any term by the term which precedes it. The abbreviation G.P. is usually used for the words *geometrical progression*, and the abbreviation C.R. for the common ratio.

Thus, $2, 6, 18, 54, 162, \dots$ is a G.P. with C.R. $= 3$.

233. The most general form of a G.P. is

$$a,\ ar,\ ar^2,\ ar^3,\ \dots,$$

in which the first term is a, and the common ratio is r. It is easy to see that any term may be expressed in terms of a and r; thus, the 8th term is obtained by multiplying the first term by r seven times; the nth term is obtained by multiplying the first term by r $(n-1)$ times. These terms are, therefore, respectively ar^7, ar^{n-1}.

Example 11. *Find the* 10*th and nth terms of the series* $64, -32, 16, \dots$.

The series is a G.P. with first term 64 and C.R. $-\frac{1}{2}$;

$$\therefore \text{ the 10th term} = 64 \times (-\tfrac{1}{2})^9 = -\tfrac{1}{8};$$

$$\text{the } n\text{th term} = 64 \times (-\tfrac{1}{2})^{n-1}.$$

It is important to notice that the nth term may be further simplified. It equals $2^6 \times \dfrac{(-1)^{n-1}}{2^{n-1}} = (-1)^{n-1} \cdot 2^{7-n}$ or $\dfrac{(-1)^{n-1}}{2^{n-7}}$.

Note. If we do not know whether n is even or odd, we cannot simplify such expressions as $(-1)^{n-1}$, $(-1)^n$. But $(-1)^{2n} = 1$, for it is the product of an even number of factors each equal to -1; $(-1)^{2n-1} = -1$, for it is the product of an odd number of factors each equal to -1. Similarly $(-1)^{2n+1} = -1$.

Example 12. *The* 4*th and* 9*th terms of a G.P. are* $\frac{1}{3}$ *and* 81 *respectively; find the series.*

With the usual notation $ar^3 = \frac{1}{3}$, $ar^8 = 81$,

$$\therefore r^5 = 243, \quad \therefore r = 3 \quad \text{and} \quad a \times 27 = \tfrac{1}{3}, \text{ i.e. } a = \tfrac{1}{81};$$

$$\therefore \text{ the series is } \tfrac{1}{81},\ \tfrac{1}{27},\ \tfrac{1}{9},\ \dots.$$

It should be noted that a G.P. is completely determined when any two terms are known, for we can then write down two independent equations connecting a and r. By solving these equations one or more pairs of values of a and r can be found, and the possible series determined.

234. Geometric means. (1) When three numbers are in G.P. the middle term is called the **geometric mean** of the other two.

Thus, since 2, 4, 8 are in G.P., 4 is the geometric mean of 2 and 8. It should be noted that 2, -4, 8 are also in G.P., so that -4 may also be called the geometric mean, but it is customary to consider only positive values.

Example 13. *Find the geometric mean of x and y.*

Let G be the required mean; then since x, G, y are in G.P.,

the C.R. $=\frac{G}{x}=\frac{y}{G}$, $\therefore\ G^2=xy$, $\therefore\ G=\pm\sqrt{xy}$.

The geometric mean of x and y is therefore $+\sqrt{xy}$, adopting the usual convention.

This is an important result and should be committed to memory.

(2) When any number of numbers are in G.P., the terms intermediate between the first and the last are called the **geometric means** between these two terms. It is always possible to insert any required number of geometric means between two numbers.

Example 14. *Insert n geometric means between x and y.*

After insertion of the n numbers there will be $n+2$ numbers in G.P. It follows that y is the $(n+2)$th term of a G.P. of which x is the first, $\therefore\ y=xr^{n+1}$, if r is the common ratio; $\therefore\ r=\sqrt[n+1]{\frac{y}{x}}$, and the means are

$$x\left(\frac{y}{x}\right)^{\frac{1}{n+1}},\ x\left(\frac{y}{x}\right)^{\frac{2}{n+1}},\ \dots,\ x\left(\frac{y}{x}\right)^{\frac{n}{n+1}}.$$

This result should not be committed to memory. All cases which arise may be done from first principles as above.

EXERCISE 104

(*Many of these examples may be taken orally*)

1. Which of the following series are G.P.'s? What is the C.R.?

(i) 40, 20, 10, 5, ⋯ ; (ii) 81, -27, 9, -3, ⋯ ;

(iii) $5\frac{1}{2}$, $7\frac{1}{4}$, $9\frac{1}{8}$, $11\frac{1}{16}$, ⋯ ; (iv) 24, 20, 14, 10, ⋯ ;

(v) $ax, a^2x^2, a^3x^3, a^4x^4, \dots$; (vi) $y^{10}, -y^8, y^6, -y^4, \dots$;

(vii) $a+x, 2a+x^2, 3a+x^3, \dots$; (viii) $a-x, a^2-x^2, a^3-x^3, \dots$.

2. Find the 5th, 8th and nth terms of the following G.P.'s:

(i) $63+21+7+\dots$; (ii) $88-44+22-\dots$;

(iii) $\frac{4}{27}+\frac{2}{9}+\frac{1}{3}+\dots$; (iv) $\frac{9}{16}-\frac{3}{8}+\frac{1}{4}-\dots$;

(v) $4-2+1-\dots$; (vi) $a^4x^4+a^2x^6+x^8+\dots$.

3. Find the C.R., the 6th and nth terms of the following G.P.'s:

(i) $11-\frac{11}{3}+\frac{11}{9}-\dots$; (ii) $\frac{4}{5}+\frac{2}{5}+\frac{1}{5}+\dots$;

(iii) $x-\frac{1}{x}+\frac{1}{x^3}-\dots$; (iv) $x-xy^2+xy^4-\dots$.

4. Find the first 3 terms of the G.P.'s determined by the following data:

(i) $\mathbf{T}_5=32, \mathbf{T}_8=4$; (ii) $\mathbf{T}_4=\frac{1}{8}, \mathbf{T}_9=-4$; (iii) $\mathbf{T}_6=\frac{1}{27}, \mathbf{T}_8=\frac{1}{243}$.

5. Find the number of terms in the following G.P.'s:

(i) $96, 48, 24, \dots, \frac{3}{8}$; (ii) $324, -108, 36, \dots \frac{4}{729}$.

6. Find the geometric means of:

(i) 6 and 24, (ii) $5\sqrt{2}$ and $20\sqrt{18}$, (iii) x^3y^5 and x^5y.

7. Insert 5 geometric means between 18 and $\frac{9}{32}$.

8. Insert 4 geometric means between 21 and $-\frac{224}{81}$.

9. Insert 3 geometric means between 35 and 560.

10. In a G.P., $\mathbf{T}_2=6, \mathbf{T}_7=192$. Find $\mathbf{T}_n$.

11. In a G.P., $\mathbf{T}_3=8, \mathbf{T}_7=\frac{1}{2}$. Find $\mathbf{T}_n$.

12. What quantity must be added to each of the numbers 6, 22, and 63 to give three quantities in G.P.?

EXERCISE 104. c

1. In a G.P., $\mathbf{T}_2=6, \mathbf{T}_5=162$. Find $\mathbf{T}_1$ and the C.R.

2. In a G.P., $\mathbf{T}_5=27, \mathbf{T}_{10}=3\frac{5}{9}$. Find $\mathbf{T}_{15}$.

3. In a G.P., $\mathbf{T}_1=1, \mathbf{T}_4=-8\div(3\sqrt{3})$. Find $\mathbf{T}_7$.

4. Insert 2 geometric means between 5 and 135.

5. Find 5 geometric means between $81a^{-2}x^{15}$ and $\frac{1}{9}a^{10}x^{-9}$.

6. In a G.P., $\mathbf{T}_9+\mathbf{T}_{10}+\mathbf{T}_{11}=11{\cdot}7, \mathbf{T}_{10}=3$. Find the C.R.

7. Show that there are two G.P.'s in which $\mathbf{T}_1+\mathbf{T}_2=10$, $\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3=19$, and find them.

8. Find x and y when $5, x, y, 0{\cdot}04$ are in G.P.

9. The sum of three consecutive terms of a G.P. is $9\frac{1}{3}$, and 12 times the last term is equal to the square of the second term. Find the values of these terms.

10. The sum of the first five terms of a G.P. is $1\frac{3}{8}$, and the sum of the next five terms is -44. Find the first term and the C.R.

11. In a G.P., $\mathbf{T}_1=60$, $\mathbf{T}_5:\mathbf{T}_9=3:2$. Find $\mathbf{T}_{13}$.

12. In a G.P., $\mathbf{T}_4+\mathbf{T}_6=-5\frac{5}{9}$, $\mathbf{T}_5+\mathbf{T}_7=1\frac{23}{27}$. Find $\mathbf{T}_1$ and the C.R.

13. The sum of the first six terms of a G.P. is $15\frac{3}{4}$, and the sum of the terms from the fourth to the ninth, both inclusive, is $1\frac{31}{32}$. Find $\mathbf{T}_7$.

14. In a G.P., $\mathbf{T}_1$ is positive, $\mathbf{T}_1+\mathbf{T}_3+\mathbf{T}_5=52\frac{1}{2}$, $\mathbf{T}_1\times\mathbf{T}_5=100$. Find $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$, $\mathbf{T}_5$.

15. In a G.P., each term is double of the preceding term. Find $\mathbf{T}_1$ if the sum of the first nine terms is four times the sum of their reciprocals.

16. In a G.P., $\mathbf{T}_8-\mathbf{T}_9=11(\mathbf{T}_1-\mathbf{T}_2)$. Using logarithms, prove that the C.R. is approximately $1{\cdot}41$. If $\mathbf{T}_{15}=5{\cdot}5$, find $\mathbf{T}_1$.

17. The 4th, 8th and 24th terms of an A.P. are in G.P.; find the C.R. of the G.P.

18. Find three numbers in A.P. whose sum is 36, such that, when the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P.

19. If l, m, n are the 2nd, 9th and 11th terms respectively of a G.P., prove that $l^2n^7=m^9$.

20. The sum of the Arithmetic and Geometric means of two positive numbers is 96, and the ratio of the numbers is 9; find them.

Summation of a Geometrical Progression

235. To illustrate the general principle, we first work out in full a simple numerical case.

Example 15. *Find the sum,* **S**, *of* 10 *terms of the series*

$$\tfrac{1}{81}+\tfrac{1}{27}+\tfrac{1}{9}+\dots.$$

$$\mathbf{S}=\tfrac{1}{81}+\tfrac{1}{27}+\tfrac{1}{9}+\tfrac{1}{3}+1+3+9+27+81+243. \quad \dots\dots\dots(\text{i})$$

The series is a G.P. and if we multiply each side by the C.R. 3, each term is changed into the term which follows it;

$$\therefore\ 3\mathbf{S}=\tfrac{1}{27}+\tfrac{1}{9}+\tfrac{1}{3}+1+3+9+27+81+243+729. \quad \dots\dots\dots(\text{ii})$$

It is clear that, if we subtract, all terms cancel out except the first in (i) and the last in (ii);

$$\therefore\ 3\mathbf{S}-\mathbf{S}=729-\tfrac{1}{81}, \quad \therefore\ 2\mathbf{S}=728\tfrac{80}{81}, \quad \therefore\ \mathbf{S}=364\tfrac{40}{81}.$$

236. More generally, we proceed as in the following example.

Example 16. *Find the sum*, $\mathbf{S}$, *of n terms of the series $a, ar, ar^2, \ldots$*

We have $\mathbf{S} = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$. ....(i)

Multiply each term by the C.R. r; each term is then changed into the term which follows it;

$\therefore\ r\mathbf{S} = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$. ....(ii)

It is clear that, if we subtract, all terms cancel out except the first in (i) and the last in (ii);

$$\therefore\ r\mathbf{S} - \mathbf{S} = ar^n - a, \text{ or } \mathbf{S} - r\mathbf{S} = a - ar^n;$$

$$\text{i.e. } \mathbf{S} = \frac{a(r^n - 1)}{(r-1)}, \text{ or } \mathbf{S} = \frac{a(1 - r^n)}{(1-r)}.$$

237. The above formulae for S are both useful; the form $\dfrac{a(r^n - 1)}{(r-1)}$ is usually used if $r > 1$, the form $\dfrac{a(1-r^n)}{(1-r)}$ if $r < 1$, but either may be used on all occasions.

Note. Since $ar^n = ar^{n-1} \times r, = lr$, where l is the last term, these formulae may also be written in the forms

$$\frac{lr - a}{r - 1} \text{ and } \frac{a - lr}{1 - r}.$$

Example 17. *How many terms must be taken of the series* $1 - 3 + 9 - \ldots$ to make -14762?

The series is a G.P. with C.R. -3.

The sum of n terms is therefore $\dfrac{1(1-(-3)^n)}{1-(-3)}$,

$$\therefore\ \frac{1-(-3)^n}{4} = -14762, \quad \therefore\ 1 - (-3)^n = -59048,$$

$$\therefore\ 59049 = (-3)^n.$$

But $\qquad 59049 = 3^{10} = (-3)^{10}, \quad \therefore\ n = 10;$

$\therefore$ 10 terms of the series must be taken.

EXERCISE 105

Find the sum of the following G.P.'s :

1. $\frac{1}{6} + \frac{2}{3} + 2\frac{2}{3} + \ldots$ to 6 terms. **2.** $5 - 2\frac{1}{2} + 1\frac{1}{4} - \ldots$ to 12 terms

3. $5\frac{1}{16} - 3\frac{3}{8} + 2\frac{1}{4} - \ldots$ to 7 terms.

4. $\frac{4}{11} + \frac{3}{11} + \frac{9}{44} + \ldots$ to 7 terms.

5. $216 + 144 + 96 + \ldots$ to 8 terms.

6. $56 + 28 + 14 + \dots$ to n terms. **7.** $27 - 9 + 3 - \dots - \frac{1}{81}$.

8. $\frac{1}{125} + \frac{1}{25} + \frac{1}{5} + \dots + 625$. **9.** $\frac{1}{9} - \frac{1}{3} + 1 - 3 + \dots$ to n terms.

10. $a^2 + ab + b^2 + \dots$ to n terms.

How many terms must be taken of the series :

11. $11\frac{1}{4} - 7\frac{1}{2} + 5 - \dots$ to make $6\frac{17}{108}$?

12. $3\frac{1}{2} + 5\frac{1}{4} + 7\frac{7}{8} + \dots$ to make $46\frac{5}{32}$?

13. $27 - 18 + 12 - \dots$ to make $16\frac{151}{243}$?

14. $15 - 5 + 1\frac{2}{3} - \dots$ to make $11\frac{19}{81}$?

15. In a G.P., $T_3 = 3$, $T_8 = \frac{32}{81}$. Find the sum of the terms from the 3rd to the 8th inclusive.

16. In a G.P., $T_2 = 27$, $T_5 = 8$. Find S_8.

17. In a G.P., the first term is 3, the last term but three is -384 and the last term is 3072. Find the sum of the series.

18. In a G.P., whose C.R. is positive, $S_2 = 6$, $S_4 = 7\frac{1}{2}$. Find the C.R., T_1 and an expression for T_{15}.

19. In a G.P., whose C.R. is positive, $T_3 = \frac{4}{3}$, $T_5 = \frac{16}{27}$. Find T_6 and an expression for S_{20}.

20. In a G.P., $T_1 = a$, $T_2 = b$, $T_n = l$. Prove that $S_n = \dfrac{bl - a^2}{b - a}$.

238. Many questions involving G.P. are best solved with the aid of logarithms.

Example 18. *How many terms of the G.P.* 0·8, 1·2, 1·8, ..., *must be taken to give a sum greater than* 1600?

The sum of n terms of the series is $0{\cdot}8 \times \dfrac{(1{\cdot}5^n - 1)}{0{\cdot}5}$.

We need the smallest value of n satisfying the inequality

$$\tfrac{8}{5}(1{\cdot}5^n - 1) > 1600, \text{ i.e. } 1{\cdot}5^n - 1 > 1000,$$

$$\text{i.e. } 1{\cdot}5^n > 1001, \quad \therefore n \log 1{\cdot}5 > \log 1001,$$

$$\therefore n(0{\cdot}1761) > 3{\cdot}0005, \quad \therefore n > \frac{3{\cdot}0005}{0{\cdot}1761} > 17{\cdot}03\dots,$$

by ordinary division,

$\therefore$ the smallest value of n is 18, i.e. 18 terms must be taken.

Compound Interest

239. The most important practical application of series in G.P. is in connection with compound interest, annuities, repayments by equal instalments and insurances.

Compound Interest. If interest is credited annually, the amount of £P invested at r per cent. per annum compound interest for n years is £$P\left(1+\frac{r}{100}\right)^n$.

If interest is credited more frequently, at equal intervals, the amount is £$P\left(1+\frac{r}{100}\right)^n$, where £$r$ is the interest on £100 for the interest-period, and n is the number of periods. Thus, if interest is credited at the end of each half-year, £r is the interest on £100 for half a year, and n is the number of half-years.

Present Value. If a payment of £A is due in n years' time, and compound interest is credited annually as above, the present value of the payment may easily be calculated; for if £P is the present value, £P amounts to £A in n years at r per cent. per annum compound interest,

$$\therefore\ P\left(1+\frac{r}{100}\right)^n=A, \quad \therefore\ P=A\,.\left(1+\frac{r}{100}\right)^{-n}.$$

As before, the same formula may be used, if interest is credited at the end of equal periods; n is then the number of periods and £r is the interest on £100 for the interest-period.

Example 19. *A man borrows a sum of money from a Building Society, and agrees to repay £2 each month for the next 15 years. What sum should the Building Society advance, if interest is at the rate of 4·8 per cent. per annum, credited monthly?*

The required sum is the Present Value of £2 in one month's time, £2 in two months' time and so on, up to £2 in 180 months' time. In the formula £$A\left(1+\frac{r}{100}\right)^{-n}$, $A=2$, £r equals the interest on £100 for 1 month at 4·8 per cent. per annum, i.e. $r=0·4$, $n=$ the number of months. The required sum is therefore equal to

$$£2\,[(1·004)^{-1}+(1·004)^{-2}+\ldots+(1·004)^{-180}]$$
$$=£\frac{2(1·004)^{-1}[1-(1·004)^{-180}]}{1-(1·004)^{-1}}=£\frac{2[1-(1·004)^{-180}]}{0·004}$$
$$=£500\,[1-(1·004)^{-180}].$$

The expression $(1·004)^{-180}$ must be calculated by logarithms.

The 4-figure tables give $\log 1·004=0·0017$,

$\therefore$ if $x=(1·004)^{-180}$, $\log x=-180\times 0·0017=-0·306=\bar{1}·694$,

$$\therefore\ x=0·4943.$$

This is not reliable to more than three places of decimals; and in fact is only accurate to two; it follows that from 4-figure tables we can only obtain $(1{\cdot}004)^{-180}$ correct to two significant figures. The reason for this is that the error in taking $0{\cdot}0017$ as the value of $\log 1{\cdot}004$ has been multiplied by 180. In all such cases as this, i.e. whenever we have to calculate a high power of a number, 4-figure tables only give a very rough approximation. If we require a more accurate result, we must use tables giving 5 or more figures.

Using 7-figure tables we have,

$$\log x = -180 \times 0{\cdot}0017337 = -0{\cdot}312066 = \bar{1}{\cdot}687934,$$

$$\therefore\ x = 0{\cdot}48745.$$

Using this value we find that the required sum is equal to

$$£500(1 - 0{\cdot}48745) = £500(0{\cdot}51255) = £256 \text{ approx.}$$

240. For convenience the logarithms of a few important numbers are here given correct to seven figures:

$\log 1{\cdot}02 = 0{\cdot}0086002$,	$\log 1{\cdot}0225 = 0{\cdot}0096633$,
$\log 1{\cdot}025 = 0{\cdot}0107239$,	$\log 1{\cdot}0275 = 0{\cdot}0117818$,
$\log 1{\cdot}03 = 0{\cdot}0128372$,	$\log 1{\cdot}0325 = 0{\cdot}0138901$,
$\log 1{\cdot}035 = 0{\cdot}0149403$,	$\log 1{\cdot}0375 = 0{\cdot}0159881$,
$\log 1{\cdot}04 = 0{\cdot}0170333$,	$\log 1{\cdot}0425 = 0{\cdot}0180761$,
$\log 1{\cdot}045 = 0{\cdot}0191163$,	$\log 1{\cdot}0475 = 0{\cdot}0201540$,
$\log 1{\cdot}05 = 0{\cdot}0211893$,	$\log 1{\cdot}0525 = 0{\cdot}0222221$,
$\log 1{\cdot}055 = 0{\cdot}0232525$,	$\log 1{\cdot}0575 = 0{\cdot}0242804$,
$\log 1{\cdot}06 = 0{\cdot}0253059$.	

Infinite Series

241. Fig. 23 represents a straight line AB 2 in. long. Bisect it at D_1; then bisect D_1B at D_2, D_2B at D_3, and so on.

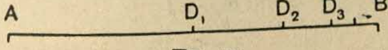


FIG. 23.

It is clear that each position of D is nearer to B than the preceding position, i.e. D_6 is nearer to B than D_5, D_{r+1} is nearer to B

than D_r. It is also clear that every position of D is to the left of B; at each successive step the distance between D and B is halved; this distance soon becomes so small that it is difficult to mark accurately the position of D on the paper, but D can never reach B.

This may also be shown by calculation; we have

$$AD_1 = 1 \text{ in.}, AD_2 = \left(1+\frac{1}{2}\right) \text{in.}, AD_3 = \left(1+\frac{1}{2}+\frac{1}{2^2}\right) \text{in., and so on.}$$

In general, $$AD_n = \left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^{n-1}}\right) \text{in.}$$
$$= \left(2-\frac{1}{2^{n-1}}\right) \text{in., summing the series.}$$

Thus, the distance $D_nB = \frac{1}{2^{n-1}}$ in., and by taking n sufficiently large, we can make this very small. In other words, the sum $\mathbf{S}_n$ of n terms of the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^{n-1}}$ is always less than 2, and the difference between S_n and 2 is $\frac{1}{2^{n-1}}$; this can be made as small as we please by taking n sufficiently large.

Also, if n is so large that D_nB is very small, then for all values of r greater than n, D_rB is still smaller. It follows that by taking a sufficient number of terms of the series, we can obtain a sum as near 2 as we please for this number of terms, *and for every greater number of terms.*

In other words, $\mathbf{S}_n$ can be made to approach as close as we please to the **limit** 2 by taking n large enough.

This is more concisely expressed by saying that $\mathbf{S}_n$ **tends** to 2 as n **tends to infinity.**

"Tends to infinity" means that n takes values greater than any stated number, however large.

The usual notation is:

When $n \to \infty$, $\frac{1}{2^{n-1}} \to 0$ and $\mathbf{S}_n \to 2$;

or
$$\operatorname*{Lt}_{n\to\infty} \mathbf{S}_n = 2.$$

The limit 2 is sometimes called the **sum to infinity** or **limiting sum** of the series, but the use of the word "sum" is misleading,

and the pupil must be careful only to use these phrases in the sense explained above.

242. Let us now consider the series 2, 6, 18, 54, $\cdots$.

The sum of n terms $=\dfrac{2(3^n-1)}{3-1}=3^n-1$.

By taking n large enough 3^n can be made as large as we please. This series has no sum to infinity.

More generally, the sum of n terms of the series

$$a+ar+ar^2+\dots$$

is (1) $\dfrac{a(1-r^n)}{1-r}=\dfrac{a}{1-r}-\dfrac{ar^n}{1-r}$, if r is positive and <1;

(2) $\dfrac{a(r^n-1)}{r-1}=\dfrac{ar^n}{r-1}-\dfrac{a}{r-1}$, if $r>1$;

(3) na, if $r=1$.

But (1) if $r<1$, r^n may be made as small as we please by taking n sufficiently large (see Example 20, below);

(2) if $r>1$, r^n may be made as large as we please by taking n sufficiently large (see Example 18, above);

(3) na may be made as large as we please by taking n sufficiently large.

In the above work we have considered positive values of r only. If r is negative, we have, writing $r=-R$, the sum of n terms is

(1) $\dfrac{a}{1+R}-\dfrac{aR^n(-1)^n}{1+R}$, where R is positive and <1;

(2) $\dfrac{a}{1+R}-\dfrac{aR^n(-1)^n}{1+R}$, where $R>1$;

(3) 0, if n is even; a, if n is odd, where $R=1$.

As before, we have a definite sum to infinity only in (1).

We conclude therefore that the G.P.

$$a+ar+ar^2+\cdots$$

has a sum to infinity, **if and only if** r is a fraction between 0 and 1 or between 0 and -1, and that the sum to infinity is then $\dfrac{\mathbf{a}}{\mathbf{1-r}}$.

Example 20. *Find the sum to n terms and to infinity of the G.P. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$. How many terms must be taken that the sum may differ from the sum to infinity by less than* $0{\cdot}00001$?

$$\mathbf{S}_n=\frac{\frac{1}{2}\{1-(\frac{1}{2})^n\}}{\frac{1}{2}}=1-\frac{1}{2^n}.$$

The sum to infinity is therefore 1.

The difference between $\mathbf{S}_n$ and the sum to infinity $=\frac{1}{2^n}$. If this is less than $0{\cdot}00001$, we have

$$\frac{1}{2^n}<0{\cdot}00001, \quad \therefore \ -n\log 2<\log(0{\cdot}00001),$$

$$\therefore \ -n(0{\cdot}3010)<-5, \quad \therefore \ 5<n(0{\cdot}3010),$$

$$\therefore \ n(0{\cdot}3010)>5,$$

$\therefore$ $n>16{\cdot}6$, by ordinary division,

$\therefore$ at least 17 terms must be taken.

243. Application to recurring decimals.

Example 21. *Find the value of* $4{\cdot}2\dot{8}\dot{3}$.

Either, $4{\cdot}2\dot{8}\dot{3}=4+\frac{2}{10}+\frac{8}{10^2}+\frac{3}{10^3}+\frac{8}{10^4}+\frac{3}{10^5}+\frac{8}{10^6}+\frac{3}{10^7}+\dots$.

But $\frac{8}{10^2}+\frac{8}{10^4}+\frac{8}{10^6}+\dots$ is a G.P. with C.R. $\frac{1}{10^2}$.

It therefore has a sum to infinity and this sum is

$$\frac{8}{10^2}\div\left(1-\frac{1}{10^2}\right)=\frac{8}{99}.$$

Similarly, $\frac{3}{10^3}+\frac{3}{10^5}+\frac{3}{10^7}+\dots$ has a sum to infinity equal to

$$\frac{3}{10^3}\div\left(1-\frac{1}{10^2}\right)=\frac{3}{990}.$$

Thus, $4{\cdot}2\dot{8}\dot{3}=4+\frac{2}{10}+\frac{8}{99}+\frac{3}{990}=4+\frac{198+80+3}{990}=4\frac{281}{990}$.

Or, we may make use of the general method of summation.

Let $\qquad \mathbf{S}_n=\ \ 4{\cdot}2838383\dots$, where 83 occurs n times;

then $\qquad 100\mathbf{S}_n=428{\cdot}38383\dots$, " " " .

[We multiply by 10^r, r being the number of figures that recur.]

By subtraction, $\quad 99\mathbf{S}_n=424{\cdot}1-\frac{83}{10^{2n+1}},$

$$\therefore\ \mathbf{S}_n = \frac{424{\cdot}1}{99} - \frac{83}{99\,.\,10^{2n+1}}.$$

As n tends to infinity, $\frac{83}{99\,.\,10^{2n+1}}$ tends to zero,

$$\therefore\ \mathbf{S}_n \text{ tends to } \frac{424{\cdot}1}{99}.$$

We may therefore take $\frac{424{\cdot}1}{99}$ as the value of $4{\cdot}2\dot{8}\dot{3}$ for all practical purposes.

Hence $4{\cdot}2\dot{8}\dot{3} = \frac{424{\cdot}1}{99} = \frac{4241}{990} = 4\frac{281}{990}$, as above.

Once the method is thoroughly understood, there is no need to write down the term $\frac{83}{10^{2n+1}}$, which is small enough to be neglected.

244. Miscellaneous series.

Example 22. *Sum to n terms the series whose rth term is* $4\,.\,5^r - 3r$.

We have
$$\mathbf{T}_1 = 4\,.\,5 - 3,$$
$$\mathbf{T}_2 = 4\,.\,5^2 - 6,$$
$$\mathbf{T}_3 = 4\,.\,5^3 - 9,$$
$$\cdots\cdots\cdots\cdots$$

$\therefore$ adding $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \ldots = 4(5 + 5^2 + 5^3 + \ldots) - (3 + 6 + 9 + \ldots)$;

$\therefore\ \mathbf{S}_n = 5(5^n - 1) - \frac{3n(n+1)}{2}$, summing the series in brackets.

EXERCISE 106

In this Exercise logarithms may be used to shorten the working.

1. What is the sum of n terms of the series
$$1 + 1{\cdot}05 + 1{\cdot}05^2 + 1{\cdot}05^3 + \ldots?$$
Find how many terms of this series must be taken, so that the sum shall exceed $170{\cdot}3$.

2. Obtain the 10th term of the series $3 + 2 + 1\frac{1}{3} + \frac{8}{9} + \ldots$, and evaluate it correct to 3 sig. figs.

3. In a G.P., $\mathbf{T}_1 = 2$, $\mathbf{T}_2 = 2{\cdot}4$. Find $\mathbf{S}_n$. Find also the least value of n for which this sum exceeds 2000.

4. Show that the sum of the first 20 terms of the G.P. $7 + 2{\cdot}1 + 0{\cdot}63 + \ldots$ differs from 10 by an amount which is less than half the twentieth term. Which term is first less than one millionth?

5. Find the sum of 20 terms of the series $1 + 1{\cdot}04 + 1{\cdot}04^2 + \dots$, and also the sum of 20 terms of the series $1 + \dfrac{1}{1{\cdot}04} + \dfrac{1}{1{\cdot}04^2} + \dots$, being given that $1{\cdot}04^{20} = 2{\cdot}19112$. Give results to 3 dec. places.

6. A chess-board has 64 squares. Show that ten thousand million men each prepared to bring a million pounds could not bring sufficient money to put 1d. on the first square, 2d. on the second, 4d. on the third, 8d. on the fourth and so on for the 64 squares.

7. Find the amount at compound interest of £432 in 20 years at $4\frac{1}{2}$ per cent.

8. What sum will amount to £2140 in 25 years at $3\frac{1}{4}$ per cent. compound interest?

9. A man borrows £5000 to be repaid with interest at 3 per cent. per annum in 10 equal annual instalments, the first payment being due at the end of 1 year. What sum (to the nearest pound) must be repaid each year?

10. A man pays a premium of £100 at the beginning of every year to an Insurance Company on the understanding that at the end of 15 years he can receive back the premiums which he has paid with $2\frac{1}{2}$ per cent. compound interest. What should he receive?

11. A ball is dropped from a height of 50 ft., rebounds to a height of 30 ft., and continues to fall and rebound, rising after each rebound to three-fifths of the height it previously fell from. Through what distance will it move before it comes to rest on the ground?

12. The population of a town has increased from 160,000 to 212,000 in 10 years. What will be its population five years afterwards, if it goes on increasing at the same rate?

Sum to infinity the series :

13. $3 + 2 + 1\frac{1}{3} + \dots$.

14. $18 - 6 + 2 - \dots$

15. $9{\cdot}6 - 4{\cdot}8 + 2{\cdot}4 - \dots$.

16. $10 + 3 + 0{\cdot}9 + \dots$.

17. How many terms of the G.P., whose first term is $1\frac{1}{2}$ and common ratio $\frac{1}{2}$, must be taken in order that the sum of the terms may differ from the sum to infinity by less than $0{\cdot}0001$?

18. Find the first term and C.R. of the G.P. whose second term is -21 and whose sum to infinity is 16.

19. Find the sum of n terms of the G.P. in which $\mathbf{T}_3 = -24$, $\mathbf{T}_6 = 3$. How many terms must be taken so that this sum may differ from the sum to infinity by less than $0{\cdot}001$?

20. In a G.P. continued to infinity, the sum of all the odd terms exceeds the sum of all the even terms by half the sum of the whole series. What is the common ratio?

21. A point moves in a straight line in such a way that in each second it moves half as far as it moved in the previous second. During the first second it moves 12″. How long does it take to move $23\frac{1}{4}$″? How long would it take to move the next $\frac{3}{4}$″?

22. The yearly output of a gold mine decreases every year 13 per cent. of its amount during the previous year. Given that the first year's output is £260,000, and that $(0{\cdot}87)^{10}=0{\cdot}24842$ approx., find the total output, (*a*) for the first 10 years, (*b*) for all time.

Find the value of :

23. $0{\cdot}\dot{2}9\dot{6}$. **24.** $0{\cdot}5\dot{7}\dot{3}$. **25.** $5{\cdot}1\dot{6}$.

26. $3{\cdot}3\dot{8}\dot{2}$. **27.** $0{\cdot}\dot{5}$. **28.** $2{\cdot}3\dot{2}0\dot{7}$.

Sum to n terms the series whose rth term is :

29. $3^{r-1}-2^{r-1}$. **30.** $2\,.\,7^r-5r+2$

31. $4^{2r}-3(r+2)$. **32.** $(1-c^r)^2$.

EXERCISE 106. c

Miscellaneous Series

(*These examples have not been arranged in order of difficulty*)

1. Prove that the series $\frac{8}{13}, \frac{9}{15}, \frac{10}{17}, \frac{11}{19}, \ldots$ is neither an Arithmetic nor a Geometric Series ; and find a formula for its nth term.

2. If a man saves £4 more each year than he did the year before, and if he saves £20 in the first year, after how many complete years will his savings first come to more than £1000 altogether, and what will then be the exact sum?

3. In a G.P., $\mathbf{T}_1=a^5x^{-8}$, $\mathbf{T}_4=a^{-1}x$. Find $\mathbf{S}_{21}$.

4. If $\mathbf{S}_n=n(n+8)$, find $\mathbf{T}_1$ and $\mathbf{T}_n$. What sort of series is it?

5. If the natural numbers are grouped as follows :

$$(1),\ (2,\ 3),\ (4,\ 5,\ 6),\ (7,\ 8,\ 9,\ 10),\ \cdots,$$

so that there are n numbers in the nth group, find (i) the last number in the nth group, (ii) the first number of that group.

6. Each term of the series 57, 4, 49, 20·5, etc. is formed by adding together corresponding terms of an A.P. and a G.P., and the 1st term of the A.P. is 7 less than the 1st term of the G.P. Find the A.P. and the G.P. and the 7th term of the series.

7. If $2a$, $3b$, $4c$ are in A.P., and $2a$, $3b-2a$, $4c-2a$ are in G.P., prove that $2c=5a$.

8. The 5th term of an A.P. is 33, and the 17th term is 60. Find the 1st term and the C.D. What term in this series has a value nearest to, but less than 100? What is the sum of this series up to and including this term?

9. The first term of a certain G.P. is 0·6. A new series is formed by taking the square of each term. Prove that the new series is a G.P. and, if its sum to infinity is nine-tenths of the sum to infinity of the first G.P., find the C.R. of the first G.P.

10. The 1st, 5th and 11th terms of an A.P., whose 1st term is a, form the first three terms of a G.P. Find the C.D. of the A.P. and the C.R. of the G.P., and show that only one more term of the latter series is also a term of the former.

11. If $\mathbf{S}_n = 2 - \frac{2^{n+1}}{3^n}$, find $\mathbf{T}_1$ and $\mathbf{T}_n$. What type of series is it?

12. If $\mathbf{T}_n = 3n - 1$, prove that the series is an A.P., and find $\mathbf{S}_n$. Check the result by putting $n = 8$.

13. If 1, a, A are in A.P., 1, a, G are in G.P., and 1, a, H are in H.P., show that $4G = (A+1)^2$, and find H in terms of A.

14. Prove that $\log a + \log ax + \log ax^2 + \dots$ to n terms

$$= n \log a + \frac{n(n-1)}{2} \log x.$$

15. If x and y are positive, $x + y = 1$, and

$a = 1 + x + x^2 + x^3 + \dots,$ $\qquad b = 1 + y + y^2 + y^3 + \dots,$

$c = 1 + xy + x^2y^2 + x^3y^3 + \dots$; prove that

$$ab = a + b, \quad \text{and} \quad abc = a + b + c.$$

16. If A, G, H are the arithmetic, geometric and harmonic means respectively between two positive quantities a and b, prove that $G^2 = AH$.

17. What number must be added to each of the numbers 1, 3, 9 in order that the three numbers so obtained may be in H.P.?

18. In a G.P., $\mathbf{S}_7 = 158\frac{3}{4}$, and $\mathbf{S}_{16} - \mathbf{S}_9 = \frac{635}{2048}$. Find $\mathbf{T}_1$ and the C.R.

19. Find x and y when $\log 3$, x, y, $\log 9$ are in A.P.

20. If b is the arithmetic mean between a and c, prove that $b^2(a+c)$ is the arithmetic mean between $a^2(b+c)$ and $c^2(a+b)$.

21. If y is the arithmetic mean between x and z, and z is the geometric mean between x and y, prove that x is the harmonic mean between y and z.

22. If n is an odd integer, prove that

$$(1+x+x^2+\dots+x^{n-1})(1-x+x^2-x^3\dots+x^{n-1})$$
$$=1+x^2+x^4+\dots+x^{2n-2}.$$

23. Show that the sum of the first n odd numbers is a perfect square. Show also that 57^2-13^2 is the sum of certain consecutive odd numbers, and find them.

24. A man starts in business with a capital of £8000. He expects to lose £600 during each of the first six years; after that he reckons that during each year his capital will increase by one-sixth of what it was at the beginning of the year. Write down an expression for his expected capital at the end of n years, when (i) $n<6$, (ii) $n>6$. Also find during what year he may expect his capital to amount to £12,000.

25. Find all the G.P.'s in each of which $\mathbf{T}_2+\mathbf{T}_3-\mathbf{T}_1=1$ and $\mathbf{T}_1\times\mathbf{T}_5=16$.

26. If $\mathbf{A}_n=1+\frac{3}{2}+(\frac{3}{2})^2+\dots+(\frac{3}{2})^{n-1}$, and
$\mathbf{B}_n=1+\frac{2}{3}+(\frac{2}{3})^2+\dots+(\frac{2}{3})^{n-1}$, prove that $\mathbf{A}_n=(\frac{3}{2})^{n-1}\mathbf{B}_n$.

27. If $\mathbf{S}_n=\frac{1}{3}n(4n^2+6n-1)$, find $\mathbf{T}_r$.

28. If $\mathbf{S}_n=5n^2-2n+7$, find $\mathbf{T}_{12}$. What sort of series is it?

29. Find the sum of the series

$$1-2+3-4+5-6+\dots+(2n+1).$$

30. The first term of an A.P. and a G.P. are each $\frac{2}{3}$; the C.D. of the former and the C.R. of the latter are each equal to x. The sums of the first three terms of the series are also equal. Show that there are two values of x, and find them. Find the sum of 20 terms of each of the two possible A.P.'s.

31. Find the sum to n terms of the series

$$(a+bc)^2+(a+bc^2)^2+(a+bc^3)^2+\cdots.$$

32. Find the least number of terms of the G.P. $2+2\frac{1}{2}+\cdots$ which must be taken that their sum may exceed the sum of 100 terms of the A.P. $2+2\frac{1}{2}+\cdots$.

33. A man deposits £100 annually to accumulate at compound interest at 4 per cent. per annum. How much will he have standing to his credit just after he has made the tenth deposit, if the interest is credited (i) each year, (ii) each half-year? Give the results to the nearest pound.

34. Find a G.P. whose second term is 6 and sum to infinity 49.

35. Find the sum to infinity of the G.P. whose first term is 4, and whose third term is 3, all the terms being positive.

36. If $\mathbf{T}_r=4(r+5)-3a^r$, find $\mathbf{S}_n$.

37. The salary of a clerk begins at £150 a year, and rises by £12 10s. each year. Find the total amount he has been paid at the end of 25 years' service.

38. In an A.P., $T_1=35$, $T_2=39$, $T_3=43$. Find the sum of all the terms from the 15th to the 41st (both inclusive).

39. From a sufficiently long piece of cord, 10 portions are cut off successively, their lengths forming a G.P. The first and second portions are respectively 10 yd. and $9\frac{1}{2}$ yd. in length. What is the total length cut off, correct to the nearest foot? Also, if the process is continued as long as the portion cut off is not less than a foot, how many portions are obtained?

40. Evaluate $1000\{1+1\cdot03+1\cdot03^2+1\cdot03^3+\ldots+1\cdot03^{19}\}$, being given that $1\cdot03^{20}=1\cdot8061$.

41. Find the sum of n terms of the series $1+\frac{2}{3}+\frac{4}{9}+\ldots$, and the least number of terms that must be summed in order to give a result greater than 2·9.

42. In a motor-car endurance contest lasting several days, 112 cars started. A number of cars fell out each day, 16 the first day, 14 the second, 12 the third, and so on in A.P. Find an expression for the number of cars starting on the nth day, and find n when this number is equal to 46.

43. One side of a staircase is to be closed in by rectangular planks from the floor to each step. The width of each plank is 9 in., and their heights are successively 6 in., 12 in., 18 in., ... There are 24 planks required. Find their total area in sq. ft.

44. A man saves £100 each year and invests it at the *end* of the year at 5 per cent. compound interest. How much will the combined savings and interest amount to at the end of 15 years?

45. Find the amount at compound interest of £9200 in 18 years at $3\frac{3}{4}$ per cent.

46. What sum will amount to £7000 in 40 years at $2\frac{1}{2}$ per cent. compound interest?

47. At the beginning of each year a man puts by £50 to accumulate at compound interest, interest at the rate of 5 per cent. per annum being added at the end of each year. Find, to the nearest pound, the total amount of his accumulated savings at the end of the 10th year.

48. Find the sum to infinity of the series $1+\frac{1}{2}+\frac{1}{4}+\ldots$, and find the least number of terms of the progression which must be taken for the sum to exceed 1·9999.

CHAPTER XXXII

THEORY OF QUADRATIC EQUATIONS AND FUNCTIONS. FURTHER GRAPHS MISCELLANEOUS EQUATIONS

245. In Chapter XXI it was shown that the roots of the equation

$$ax^2+bx+c=0$$

are $$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}. \quad \ldots\ldots\ldots\ldots(1)$$

In Chapter XVII we explained the use of the terms real, rational, irrational, unreal (imaginary), as applied to the roots of an equation. This work should be carefully revised.

246. Character of the roots of a quadratic equation. From (1) above it is clear that the character of the roots of the equation $ax^2+bx+c=0$ depends upon the value of b^2-4ac, the quantity under the root sign :

(i) If b^2-4ac is a perfect square, the roots are rational and unequal.

(ii) If b^2-4ac is zero, each root is $-\frac{b}{2a}$, i.e. the roots are rational and equal.

(iii) If b^2-4ac is positive, but not a perfect square, the roots are real, irrational and unequal.

(iv) If b^2-4ac is negative the roots are unreal (imaginary) and unequal.

The expression b^2-4ac is called the **discriminant** ; it is usually denoted by the symbol Δ. It is important that the pupil should be able to write down at once the discriminant of any quadratic equation.

It should also be noted that, if $b=0$, the roots of the equation are equal in magnitude but opposite in sign.

Example 1. *For what value of a will the equation $5x^2 - 7x + a = 0$ have equal roots?*

The roots will be equal if $\Delta = 0$, i.e. if $(-7)^2 - 4 \cdot 5 \cdot a = 0$,

i.e. $49 - 20a = 0$, whence $a = 2{\cdot}45$.

Example 2. *What is the nature of the roots of $5x^2 - 4x + 3 = 0$?*

$$\Delta = (-4)^2 - 4 \cdot 5 \cdot 3 = 16 - 60 = -44;$$

$\therefore$ the roots are imaginary.

Example 3. *What is the nature of the roots of*

$$x^2 - 2ax + a^2 - b^2 - c^2 = 0?$$

$$\Delta = (-2a)^2 - 4(a^2 - b^2 - c^2) = 4(b^2 + c^2).$$

This is positive for all real values of b and c,

$\therefore$ the roots are real; if $b^2 + c^2$ is a perfect square they are rational, otherwise they are irrational.

247. Sum and product of the roots of a quadratic equation. If we denote by p and q respectively the roots of the equation $ax^2 + bx + c = 0$, we have

$$p + q = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a},$$

$$pq = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

These results are very important and may be expressed:

$$\textbf{Sum of roots} = -\frac{\textbf{coefficient of x}}{\textbf{coefficient of x}^2}.$$

$$\textbf{Product of roots} = \frac{\textbf{constant term}}{\textbf{coefficient of x}^2}.$$

248. These results may also be obtained by a more general method.

Let p, q be the roots of the equation. Then the equation may be written in the form $(x - p)(x - q) = 0$.

To compare this with $ax^2+bx+c=0$, we must write the latter equation in the form $x^2+\frac{b}{a}x+\frac{c}{a}=0$, i.e. with the coefficient of x^2 unity. Hence

$$x^2-(p+q)x+pq\equiv x^2+\frac{b}{a}x+\frac{c}{a}.$$

But if two polynomials in x are identically equal, the coefficients of corresponding powers of x must be equal;

$$\therefore\ p+q=-\frac{b}{a},\quad pq=\frac{c}{a}.$$

249. This method may be extended to equations of higher degree. Thus, if p, q, r are the roots of $ax^3+bx^2+cx+d=0$, we have similarly

$$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}\equiv(x-p)(x-q)(x-r)$$
$$\equiv x^3-(p+q+r)x^2+(qr+rp+pq)x-pqr;$$
$$\therefore\ p+q+r=-\frac{b}{a},\quad pq+qr+rp=\frac{c}{a},\quad pqr=-\frac{d}{a}.$$

The first result is of great importance in connection with graphical work, as we shall show in the next chapter.

It is true for rational integral equations of any degree n and may be expressed in words thus:

$$\textbf{Sum of roots}=-\frac{\textbf{coefficient of x}^{n-1}}{\textbf{coefficient of x}^{n}}.$$

250. When one root of a quadratic is obvious by inspection, the other root may be obtained by using the properties of the roots proved above.

Example 4. *Solve the equation* $\dfrac{5x^2-7x+2}{3x^2-5x+1}=\dfrac{5a^2-7a+2}{3a^2-5a+1}.$

This is a quadratic, and it is clearly satisfied by $x=a$.

Also, the equation may be written

$$(5x^2-7x+2)(3a^2-5a+1)=(3x^2-5x+1)(5a^2-7a+2),$$

i.e. $x^2(4a+1)-x(4a^2+3)-a(a-3)=0$, on reduction.

The product of the roots is therefore $-\dfrac{a(a-3)}{4a+1}$, and since one root is a, the other must be $-\dfrac{(a-3)}{4a+1}$.

251. In questions involving the roots of a quadratic equation the roots should not be considered separately. Most questions can be more easily solved by making use of the expressions for the sum and product of the roots.

Example 5. *If p, q are the roots of $ax^2+bx+c=0$, find* (i) *the value of p^3+q^3,* (ii) *the equation whose roots are $\frac{p}{q}$ and $\frac{q}{p}$.*

From the data, $p+q=-\frac{b}{a}$ and $pq=\frac{c}{a}$.

(i) $p^3+q^3=(p+q)(p^2-pq+q^2)=(p+q)[(p+q)^2-3pq]$

$$=\left(-\frac{b}{a}\right)\left[\frac{b^2}{a^2}-\frac{3c}{a}\right]=-\frac{b(b^2-3ac)}{a^3}.$$

(ii) The required equation is $\left(x-\frac{p}{q}\right)\left(x-\frac{q}{p}\right)=0$, i.e.

$$x^2-x\left(\frac{p}{q}+\frac{q}{p}\right)+1=0.$$

But $$\frac{p}{q}+\frac{q}{p}=\frac{p^2+q^2}{pq}=\frac{(p+q)^2-2pq}{pq}=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c}{a}}=\frac{b^2-2ac}{ac};$$

$\therefore$ the required equation is $x^2-x\left(\frac{b^2-2ac}{ac}\right)+1=0$, or

$$acx^2-(b^2-2ac)x+ac=0.$$

Example 6. *Find the relation connecting the coefficients of the equation $ax^2+bx+c=0$, when one root is five times the other.*

Let p, $5p$ represent the roots;

then sum of roots $=6p=-\frac{b}{a}$; product of roots $=5p^2=\frac{c}{a}$.

From the first result $p^2=\frac{b^2}{36a^2}$; from the second $p^2=\frac{c}{5a}$;

$$\therefore \frac{b^2}{36a^2}=\frac{c}{5a}, \text{ or } 5b^2=36ac.$$

Example 7. *Find, without solving the second equation, the values of p and q, so that the roots of $4x^2+px+q=0$ may be the squares of the roots of $5x^2-3x-4=0$.*

Let α, β represent the roots of $5x^2-3x-4=0$.

Then $$\alpha+\beta=\frac{3}{5},\quad \alpha\beta=-\frac{4}{5}.$$

Also $$-\frac{p}{4}=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{9}{25}+\frac{8}{5}=\frac{49}{25};$$

$$\frac{q}{4}=\alpha^2\beta^2=\frac{16}{25};$$

$$\therefore\ p=-\frac{196}{25},\quad q=\frac{64}{25}.$$

EXERCISE 107

Find the nature of the roots of the following equations :

1. $2x^2+x-120=0$.

2. $3x^2+32=28x$.

3. $4x^2-24x+9=0$.

4. $9x^2+30x+25=0$.

5. $8x^2+63=18x$.

6. $(3x+1)^2=6x+7$.

7. In each of the following quadratic functions, find whether the factors are rational or irrational, real or unreal :

(i) $x^2-7x-14$; (ii) $x^2+8x+20$.

8. For what values of a will the following equations have equal roots?

(i) $6x^2+5x+a=0$; (ii) $3x^2+ax+27=0$;

(iii) $x^2-x(a-5)+a^2-5a+7=0$; (iv) $4a^2x^2+2(a+3)x+9=0$.

9. For the following equations one solution is given. Obtain the other *without solving the equation* :

(i) $81x^2+85x+4=0$, $x=-1$; (ii) $94x^2+15x=31$, $x=\frac{1}{2}$;

(iii) $\dfrac{2x^2+1}{2a^2+1}=\dfrac{3x-1}{3a-1}$, $x=a$; (iv) $x^2+3lm=x(3l+m)$, $x=m$.

10. Find the condition that the roots of $a^2x^2+(2ac-b^2)x+c^2=0$ shall be real.

11. If one of the roots of $x^2+px+32=0$ is -8, whilst $x^2+px+q=0$ has equal roots, find the value of q.

12. Prove that the roots of $x^2-4ax+4a^2-b^2-6bc-9c^2=0$ are rational.

13. If p, q are the roots of $3x^2-8x+2=0$, find (i) p^2+q^2, (ii) the quadratic equation whose roots are $\dfrac{p}{q}$ and $\dfrac{q}{p}$.

14. If p, q are the roots of $5x^2-20x+12=0$, find p^3+q^3.

15. If α, β are the roots of $x^2-4x+1=0$, find the equation whose roots are $\alpha^2+\frac{1}{\alpha}$, $\beta^2+\frac{1}{\beta}$.

16. If α, β are the roots of $7x^2-8x+2=0$, find the equation whose roots are $2\alpha+1$ and $2\beta+1$.

17. If α, β are the roots of $x^2-4x+1=0$, find (i) $\alpha^3+\beta^3$, (ii) $\alpha^6+\beta^6$.

18. If α, β are the roots of $x^2-x+5=0$, find the equation whose roots are $\alpha^2+\frac{1}{\beta^2}$, $\beta^2+\frac{1}{\alpha^2}$.

19. If p, q are the roots of $ax^2+bx+c=0$, find the values of:

(i) p^2+q^2; (ii) p^3q+pq^3; (iii) $5p^2-3pq+5q^2$;

(iv) $p-q$; (v) p^4+q^4; (vi) $\frac{p^2}{q^2}+\frac{q^2}{p^2}$.

20 If α, β are the roots of $x^2+px+q=0$, show that the roots of $qx^2+(2q-p^2)x+q=0$ are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

21 If $\alpha\ \beta$ are the roots of $x^2-px+q=0$, find the equation whose roots are $\alpha+\frac{1}{\beta}$, $\beta+\frac{1}{\alpha}$.

22. If α, β are the roots of $ax^2+2bx+c=0$, find the equation whose roots are $2\alpha+\frac{1}{\beta}$, $2\beta+\frac{1}{\alpha}$; and prove that, when $a+c=0$, this equation is the same as the original equation.

23. If α, β are the roots of $ax^2+bx+c=0$, find the value of $(1-\alpha^3)(1-\beta^3)$.

24. If p and q are the roots of $ax^2+ax+c=0$, and if (p^2+q^2) and $(p+1)(q+1)$ are the roots of $a^2y^2+ky+l=0$, obtain k and l in terms of a and c.

25. One root of $3x^2-2kx+(k^2-3)=0$ is three times the other. Find two possible values of k.

26. The squares of the roots of $4x^2-17x+c=0$ differ by $3\frac{3}{16}$. Find c.

27. If $a^2=3a+7$ and $b^2=3b+7$ (a and b being unequal), find (i) a^2+b^2, (ii) $\frac{a^2}{b}+\frac{b^2}{a}$.

28. Find the two values of p for which the roots of $x^2-19x+25=0$ are the squares of the roots of $x^2+px=5$.

29. If α, β are the roots of $x^2 - 6x + k = 0$, and if $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ are the roots of $8x^2 + 10x + l = 0$, find k and l.

30. If a, p are the roots of $x^2 - 100x + 2491 = 0$, and a, q are the roots of $x^2 + 50x - 4559 = 0$, find the values of $p - q$ and pq.

31. Find the equation whose roots are the cubes of the roots of $x^2 + 4x + 1 = 0$.

32. If α, β are the roots of $x^2 + px + q = 0$, obtain the equation whose roots are $\alpha + 2\beta$, $\beta + 2\alpha$.

33. If α, β are the roots of $2x^2 + 15x = 24$, while $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are the roots of $px^2 + qx + 1 = 0$, find the values of p and q.

34. The arithmetic mean between the roots of a certain quadratic is $\frac{7}{3}$ and the harmonic mean is $\frac{5}{7}$; find the equation in its simplest form.

35. If p, q are the roots of $ax^2 + bx + c = 0$, find the values of:

(i) $(ap + b)(aq + b)$, (ii) $(bp + c)(bq + c)$,

(iii) $(ap + b)^{-2} + (aq + b)^{-2}$.

36. If the equations $ax^2 + bx + c = 0$, $px^2 + qx + r = 0$ have one root in common, prove that

$$(br - cq)(aq - bp) = (cp - ar)^2.$$

Quadratic Functions. Variations in Sign

252. Consider the function $ax^2 + bx + c$, and let p, q be the roots of $ax^2 + bx + c = 0$; then $ax^2 + bx + c \equiv a(x - p)(x - q)$.

(1) Suppose the roots are real and different, and let p be the greater root.

Then, for *all* values of x greater than p, the factors $x - p$, $x - q$ are both positive; for *all* values of x less than q, the factors $x - p$, $x - q$ are both negative. In each case the product $(x - p)(x - q)$ is positive and the function $ax^2 + bx + c$ has the same sign as a.

But if x lies between p and q, $x - p$ is negative and $x - q$ is positive, and the product $(x - p)(x - q)$ is negative; the sign of the function $ax^2 + bx + c$ is opposite to that of a.

(2) Suppose the roots are real and equal. Then since $p = q$,

$$ax^2 + bx + c = a(x - q)^2.$$

But $(x - q)^2$ is positive for real values of x;

$$\therefore\ ax^2 + bx + c \text{ has the same sign as } a.$$

(3) Suppose the roots are imaginary.

Then $ax^2+bx+c=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right\},$$

where Δ is the discriminant.

But since Δ is negative, $-\frac{\Delta}{4a^2}$ is positive; also $\left(x+\frac{b}{2a}\right)^2$ is positive for real values of x; $\therefore\ ax^2+bx+c$ has the same sign as a.

To sum up: For real values of x the function ax^2+bx+c always has the same sign as a, except when the roots of the equation $ax^2+bx+c=0$ are real and unequal, and x lies between them.

253. The pupil will more easily remember this result, if he thinks graphically. It is easily seen that, if the function changes sign, the graph of the function must *cross* the axis of x; also, since it is a quadratic function, if it *crosses* the axis once, it must cross it a second time, and the value of the function changes sign whenever the graph crosses the axis. Also, if we know that the function never changes sign, we know that the graph never *crosses* the axis; in this case the discriminant must be negative or zero.

Example 8. *Find the signs of the following functions for real values of x:*

$$\text{(i) } 2x^2+3x+8\,; \quad \text{(ii) } 5-x-6x^2.$$

(i) $\Delta=9-64=-55$, $\therefore$ the roots of $2x^2+3x+8=0$ are unreal, $\therefore\ 2x^2+3x+8$ always has the same sign as $+2$.

The function is therefore always positive.

(ii) $\Delta=1+120=121$, $\therefore$ the roots of $5-x-6x^2=0$ are rational. They are $\frac{5}{6}$ and -1. Hence the expression has the same sign as -6, except when x lies between $\frac{5}{6}$ and -1; $\therefore$ the expression is positive when x lies between $\frac{5}{6}$ and -1, zero when x equals $\frac{5}{6}$ and -1, and negative for all other values of x.

254. Quadratic functions. Variations in magnitude.

Example 9. *If x is real, find whether $5+4x-3x^2$ is capable of all values.*

Put $5+4x-3x^2=y$; then $3x^2-4x+(y-5)=0$. If this equation gives real values of x for any particular value of y,

$$\Delta=(-4)^2-4\cdot 3\cdot(y-5)$$

must be positive or zero for that particular value of y, i.e. the function can only have a particular value of y, if that value of y is such that $16-12y+60$, i.e. $76-12y$ is positive or zero. It follows that y must be less than or equal to $\frac{76}{12}$, i.e. $6\frac{1}{3}$.

Thus the function $5+4x-3x^2$ can never be greater than $6\frac{1}{3}$ it may be equal to $6\frac{1}{3}$ or have any value less than $6\frac{1}{3}$. The maximum value of the function is therefore $6\frac{1}{3}$.

Example 10. *If x is real, prove that the expression* $\dfrac{x^2+4x+10}{2x+5}$ *can have all numerical values except such as lie between 2 and -3.*

Let $\quad \dfrac{x^2+4x+10}{2x+5}=y$; then $x^2+4x+10=y(2x+5)$;

$$\text{i.e. } x^2+2x(2-y)+5(2-y)=0.$$

If this equation gives real values of x for any particular value of y, $\Delta=4(2-y)^2-20(2-y)$ must be positive or zero for that particular value of y.

We have $\Delta=4(2-y)[2-y-5]=4(2-y)(-3-y)$.

For all values of $y>2$, both factors are negative and the product is positive. If $y=2$, the product is zero.

For all values of $y<-3$, both factors are positive and the product is positive. If $y=-3$, the product is zero.

For all values of y between 2 and -3, the first factor is positive and the second factor is negative; the product is then negative.

Hence y may have all values except such as lie between 2 and -3.

Example 11. *Prove that* $\dfrac{x^2-3x+1}{x^2-x}$ *can have all numerical values, if x is real.*

Let $\dfrac{x^2-3x+1}{x^2-x}=y$; then $x^2-3x+1=y(x^2-x)$;

$$\therefore\ x^2(1-y)+x(y-3)+1=0.$$

Proceeding as above, we have

$$\Delta=(y-3)^2-4(1-y)\cdot 1=y^2-2y+5=(y-1)^2+4,$$

which is always positive for real values of y. Hence the result.

255. The use of the discriminant is of great value when the graphs of functions, such as $\frac{x^2+4x+10}{2x+5}$, have to be drawn.

Before discussing such graphs, we must consider the value of a function of x for values of x which make the denominator of the function zero.

Let us consider the value of $\frac{1}{x}(=y)$, as x approaches 0.

If $x=\frac{1}{2}$, $y=2$; if $x=\frac{1}{10}$, $y=10$; if $x=\frac{1}{1000}$, $y=1000$; if $x=\frac{1}{1,000,000}$, $y=1,000,000$; and so on.

It is clear that, as x approaches 0, the value of y becomes too big to be plotted on the graph. If $x=0$, y has no meaning; the number $\frac{1}{0}$ has not been defined, and we cannot plot any point corresponding to $x=0$.

The line $x=0$ is therefore for all practical purposes a barrier which cannot be crossed by the graph; points on one side of the barrier cannot be joined to points on the other side of the barrier.

Similarly, for the graph of $\frac{x^2+4x+10}{2x+5}$, the line $2x+5=0$ is a barrier which cannot be crossed by the graph.

256. We may now consider the graphs of

$$(1)\ y=\frac{x^2+4x+10}{2x+5} \quad \text{and} \quad (2)\ \frac{x^2-3x+1}{x^2-x}.$$

(1) As shown above (Ex. 10), y may have all values except such as lie between 2 and -3.

If $y=2$, $x^2=0$, i.e. $x=0$; if $y=-3$, $x^2+10x+25=0$, i.e. $(x+5)^2=0$, i.e. $x=-5$.

(It should be noted that, after substituting $y=2$, $y=-3$, we must obtain equal roots for x. This is a very valuable check of the accuracy of the working.)

Since y cannot have values between 2 and -3, the curve must turn when $y=2$ and when $y=-3$. The points (0, 2), (-5, -3) are therefore turning points on the curve.

When $2x+5=0$, i.e. when $x=-2\frac{1}{2}$, y has no meaning. The line $x=-2\frac{1}{2}$ is therefore a barrier dividing one branch of the curve from the other. Other points are obtained as usual by plotting.

x	-8	-7	-6	-5	-4	$-3\frac{1}{2}$	-3	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	0	1	2	3
$x^2+4x+10$	42	31	22	15	10	$8\frac{1}{4}$	7	$6\frac{1}{4}$	6	$6\frac{1}{4}$	7	10	15	22	31
$2x+5$	-11	-9	-7	-5	-3	-2	-1	0	1	2	3	5	7	9	11
y	$-3\frac{9}{11}$	$-3\frac{4}{9}$	$-3\frac{1}{7}$	-3	$-3\frac{1}{3}$	$-4\frac{1}{8}$	-7	No Value	6	$3\frac{1}{8}$	$2\frac{1}{3}$	2	$2\frac{1}{7}$	$2\frac{4}{9}$	$2\frac{9}{11}$

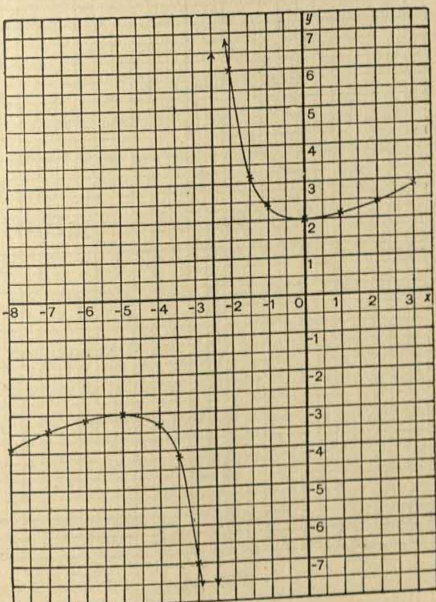


FIG. 24.

It is useful to notice the points (if any) where the curve crosses the axis of x. In this case, $y=0$ if $x^2+4x+10=0$, i.e. $x=-2\pm\sqrt{-6}$. The values of x are unreal, so that the curve does not cross the axis. It is convenient to put arrows, as in the figure, to denote that the curve approaches the barrier as x approaches $-2\frac{1}{2}$.

(2) As shown above (Ex. 11), y may have all numerical values, if x is real. If the discriminant is positive for all real values of y, the curve has no turning points.

When $x^2-x=0$, i.e. when $x=0$ or $x=1$, y has no meaning. The lines $x=0$ and $x=1$ are therefore barriers, and the curve has three branches.

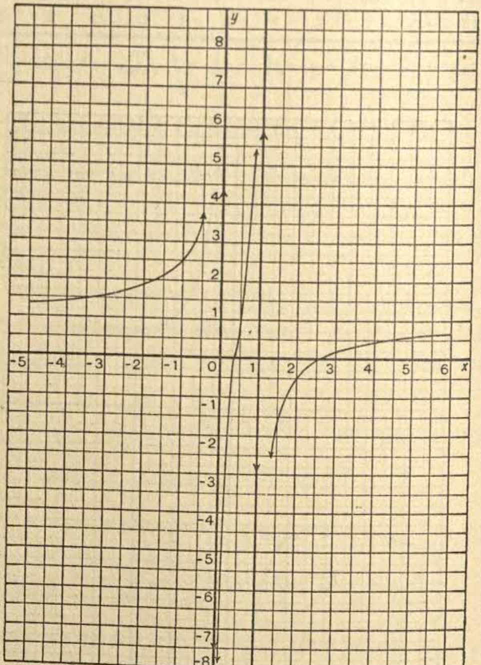


FIG. 25.

The curve crosses the axis of x when $x^2-3x+1=0$,

$$\text{i.e. when } x=\frac{3\pm\sqrt{5}}{2}=2\cdot62 \text{ or } 0\cdot38 \text{ approx.}$$

Other points are obtained as usual by plotting.

x	−5	−4	−3	−2	−1	−0·5	0	0·1	0·2	0·3	0·4	0·5	0·6
x^2-3x+1	41	29	19	11	5	2·75	1	0·71	0·44	0·19	−0·04	−0·25	−0·44
x^2-x	30	20	12	6	2	0·75	0	−0·09	−0·16	−0·21	−0·24	−0·25	−0·24
y	1·37	1·45	1·58	1·83	2·5	3·67	No Value	−7·89	−2·75	−0·90	0·17	1	1·83

x	0·7	0·8	0·9	1	1·5	2	3	4	5	6
x^2-3x+1	−0·61	−0·76	−0·89	−1	−1·25	−1	1	5	11	19
x^2-x	−0·21	−0·16	−0·09	0	0·75	2	6	12	20	30
y	2·90	4·75	9·89	No Value	−1·67	−0·5	0·17	0·42	0·55	0·63

EXERCISE 108

(*In this exercise x takes only real values*)

For what values of x are the following functions negative?

1. $x^2+x-20.$ **2.** $2x^2-17x+30.$ **3.** $3x^2+2x+8.$

For what values of x are the following functions positive?

4. $10+3x-x^2.$ **5.** $-2+x-7x^2.$ **6.** $21-32x-5x^2.$

7. Find the greatest value of k for which (i) $x^2-5x+2k=0$, (ii) $3x^2+7x+k=0$ has real roots.

8. Find the greatest value of (i) $3-5x-2x^2$, (ii) $(2x-1)(3-2x)$.

9. Find the least value of (i) $5x^2-7x-3$, (ii) $(4x-3)(5x-2)$.

10. What is the least value of k, if $3x^2-11x+k$ is never negative?

11. What is the greatest value of k, if $k-5x-2x^2$ is never positive?

12. What is the greatest value of k, if $k-4x-3x^2$ is never greater than 5?

13. If $y=\dfrac{(2x+7)(2x-1)}{x-1}$, show that y cannot lie between 8 and 32. Sketch the curve.

14. If $y=\dfrac{(x-1)(x-4)}{x-3}$, show that y can have all real values. Sketch the curve.

15. If $y=\dfrac{x^2+x+1}{x^2+1}$, show that y must lie between $1\frac{1}{2}$ and $\frac{1}{2}$. Sketch the curve.

16. If $y=\dfrac{x+1}{x^2+3}$, show that y must lie between two numbers. Find the numbers.

17. If $y=\dfrac{x^2-x+1}{x-1}$, show that there are two numbers between which y cannot lie. Find the numbers.

18. Find the turning points of the curve $y=\dfrac{3x^2-9x+7}{2x^2-7x+6}$. Sketch the curve.

19. Sketch the curve $y^2=(2-x)(3+x)$. Between what values must x lie?

20. If $y=\dfrac{2-x^2}{(1-x)^2}$, prove that the minimum value of y is -2. Sketch the curve.

257. Miscellaneous equations. We now consider some miscellaneous equations; it will be seen that many equations are reducible to quadratics.

Example 12. *Solve* $x^{\frac{1}{3}}-x^{-\frac{1}{3}}=1\frac{1}{2}$.

Put $y=x^{\frac{1}{3}}$, then $y-\dfrac{1}{y}=1\dfrac{1}{2}$, i.e. $2y^2-3y-2=0$,

$$\therefore (2y+1)(y-2)=0, \text{ whence } y=-\tfrac{1}{2} \text{ or } 2;$$

$$\therefore x^{\frac{1}{3}}=-\tfrac{1}{2} \text{ or } 2, \quad \therefore x=-\tfrac{1}{8} \text{ or } 8.$$

Note. All equations of the type

$$ax^{2n}+bx^n+c=0, \text{ or } ax^n+b+cx^{-n}=0$$

may be solved in a similar manner.

Example 13. *Solve* $(x-3)(x-5)(x+6)(x+8)=504$.

We have, rearranging, $(x-3)(x+6)(x-5)(x+8)=504$,

$$\therefore (x^2+3x-18)(x^2+3x-40)=504.$$

Put $y=x^2+3x$, then $(y-18)(y-40)=504$,

$$\therefore y^2-58y+720-504=0, \quad \therefore y^2-58y+216=0,$$

$$\therefore (y-54)(y-4)=0, \text{ whence } y=54 \text{ or } 4;$$

$$\therefore x^2+3x-54=0, \text{ or } x^2+3x-4=0,$$

whence

$$x=6, \ -9, \ 1, \ -4.$$

Note. All equations of the type $(x+a)(x+b)(x+c)(x+d)=e$ may be solved in a similar manner, if the sum of any two of the quantities a, b, c, d is equal to the sum of the other two.

Example 14. *Solve* $3x^4 - 20x^3 - 94x^2 - 20x + 3 = 0$.

Since $x = 0$ is not a solution, the equation may be written in the form

$$3\left(x^2 + \frac{1}{x^2}\right) - 20\left(x + \frac{1}{x}\right) - 94 = 0.$$

Put $y = x + \frac{1}{x}$, then $y^2 = x^2 + \frac{1}{x^2} + 2$, so that the equation becomes

$$3(y^2 - 2) - 20y - 94 = 0, \text{ i.e. } 3y^2 - 20y - 100 = 0,$$

$$\therefore\ (3y + 10)(y - 10) = 0, \text{ whence } y = -\frac{10}{3} \text{ or } 10;$$

$$\therefore\ x + \frac{1}{x} = -\frac{10}{3} \text{ or } x + \frac{1}{x} = 10,$$

whence $$x = -3,\ -\frac{1}{3},\ 5 \pm 2\sqrt{6}.$$

Note 1. All equations of the type $ax^4 + bx^3 + cx^2 + bx + a = 0$ may be solved in a similar manner.

Note 2. Equations in which the coefficients of terms equidistant from the beginning and end are equal are called **reciprocal equations.** Reciprocal equations of even degree can be reduced to an equation in y of half the degree, by putting $y = x + \frac{1}{x}$. Reciprocal equations of odd degree have a factor $x + 1$ and therefore a root -1. When this root has been taken out, the resulting equation is a reciprocal equation of even degree. Thus, to solve

$$3x^5 - 17x^4 - 114x^3 - 114x^2 - 17x + 3 = 0,$$

we first write $(x + 1)(3x^4 - 20x^3 - 94x^2 - 20x + 3) = 0$; whence

$$x + 1 = 0 \text{ or } 3x^4 - 20x^3 - 94x^2 - 20x + 3 = 0, \text{ etc., as above.}$$

Note 3. The equation $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$, i.e. a similar equation in which the coefficients of corresponding *odd* powers of x are equal in magnitude but opposite in sign, although not a reciprocal equation, may be solved by putting $y = x - \frac{1}{x}$. For (since $x \neq 0$) the equation may be written

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0,$$

$$\text{i.e. } 6(y^2 + 2) - 25y + 12 = 0, \text{ etc.}$$

Example 15. *Solve* $(x+8)^4+(x+6)^4=2$.

Put $$y=\frac{x+8+x+6}{2}=x+7.$$

Then $(y+1)^4+(y-1)^4=2$, $\therefore\ 2(y^4+6y^2+1)=2$;

$$\therefore\ y^2(y^2+6)=0,\quad \therefore\ y=0,\ 0,\ \pm\sqrt{-6};$$

$$\therefore\ x=-7,\ -7,\ -7\pm\sqrt{-6}.$$

Note. All equations of the type $(x+a)^4+(x+b)^4=c$ may be solved by putting $y=\dfrac{x+a+x+b}{2}$.

EXERCISE 109

Solve the equations:

1. $x^6+26x^3=27$. **2.** $x^4+400=41x^2$. **3.** $144=25x^2-x^4$.

4. $36=13x^2-x^4$. **5.** $4x^4-17x^2+4=0$. **6.** $x^6+27=28x^3$.

7. $3x^{\frac{1}{2}}+3x^{-\frac{1}{2}}=10$. **8.** $3\sqrt{\dfrac{2}{x}}+5\sqrt{\dfrac{x}{2}}=16$. **9.** $9x^{\frac{1}{2n}}-x^{\frac{1}{n}}-20=0$.

10. $x^{-2}-2x^{-1}=24$. **11.** $12x^{\frac{3}{5}}=17x^{\frac{1}{5}}-6x^{-\frac{1}{5}}$. **12.** $x^{\frac{2}{n}}+12=7x^{\frac{1}{n}}$.

13. $(x+7)(x-5)(x-9)(x+3)=385$.

14. $(2x+1)(2x+3)(2x+5)(2x+7)=9$.

15. $(x+4)(x+6)(x-1)(x-3)=120$.

16. $(2x-5)(x-2)(x+4)(2x+7)=91$.

17. $x(x+2)(2x+3)(2x-1)=63$. **18.** $x(x-4)(x+8)(x+4)=1680$.

19. $x^4+x^3-4x^2+x+1=0$. **20.** $9x^4+8x^2+9=27x(1+x^2)$.

21. $2x^4-5x^3+6x^2-5x+2=0$.

22. $12x^4-56x^3+89x^2-56x+12=0$.

23. $9x^4-2x^2+9=24x(x^2+1)$. **24.** $(x^2-1)^2=3x(x^2+1)$.

25. $x^5+x^4+x+1=2x^2(x+1)$. **26.** $8x^4+29x^2+8=42x(x^2-1)$.

27. $10(x^4+1)+52x^2=63x(x^2-1)$.

28. $x^5+2x^4-3x^3-3x^2+2x+1=0$.

29. $(2x-1)^4+(2x-5)^4=256$. **30.** $(x+1)^4+(x+3)^4=82$.

31. $(x-2)^4+(x-3)^4=1$. **32.** $(2x+1)^4+(2x-1)^4=16$.

33. $2^{2x+6}+1=8\cdot 2^{x+1}$. **34.** $3^{2x+5}-28(3^{x+1}-2)=55$.

35. $(x+2a)(x-6a)(x+3a)(x-5a)=180a^4$.

36. $(2b+x)^{\frac{2}{3}}+4(2b-x)^{\frac{2}{3}}=5(4b^2-x^2)^{\frac{1}{3}}$.

CHAPTER XXXIII

GRADIENT OF A CURVE. MAXIMA AND MINIMA. GRAPHICAL SOLUTION OF EQUATIONS (CONTINUED)

258. In Chapter XIV we discussed the gradient of a straight line. We now proceed to consider the gradient of a curve.

At present we cannot find the gradient of a curve, for the term "gradient" has only been defined for a straight line. But we can find the gradient of any chord AB of the curve, and this may be regarded as the **average gradient** of the arc AB of the curve.

Example 1. *Find the gradient of the chord PQ and of the chord PR, of the curve $y=x^3+5$, P, Q, R being the points on the curve where $x=1, 3, 1+h$ respectively.*

P is $(1, 6)$, Q is $(3, 32)$, R is $(1+h, 6+3h+3h^2+h^3)$.

The gradient of $PQ=\dfrac{y_Q-y_P}{x_Q-x_P}=\dfrac{26}{2}=13$;

the gradient of $PR=\dfrac{y_R-y_P}{x_R-x_P}=\dfrac{3h+3h^2+h^3}{h}=3+3h+h^2$, since $h\neq 0$.

The latter result is a general formula giving the gradient of all the chords of $y=x^3+5$ which pass through P. It includes the former result, which may be obtained from it by putting $h=2$.

259. Let us now consider what happens as h gets very small, i.e. as the point R gets nearer and nearer to P on the curve.

If $h=1, \frac{1}{2}, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$, the gradient of $PR=7, 4\cdot75, 3\cdot31, 3\cdot0301, 3\cdot003001, \ldots$.

It is clear that, as R moves along the curve towards P, the gradient of PR gets closer and closer to 3 and approaches the **limit 3.** In other words, as $R\to P$ along the curve, $h\to 0$, and the gradient of $PR\to 3$. This is written more concisely

$$\underset{R\to P}{\mathrm{Lt}}\ (\text{Gradient of } PR)=3.$$

This limit is called the **gradient of the curve at P.** Its value was

found by taking the gradient of the chord PR and finding the limit as R approached P along the curve. But the limiting position of PR as R approaches P along the curve is, by definition, the tangent at P to the curve. The gradient of the curve at P is therefore the same as the gradient of the tangent to the curve at P.

Example 2. *Find the gradient of the curve* $y=x^2$ (i) *at the point* (2, 4), (ii) *at the point* (a, a^2).

(i) With the notation used above let P be (2, 4) and R be the point where $x=2+h$, i.e. the point $(2+h, 4+4h+h^2)$.

Then the gradient of $PR=\dfrac{4h+h^2}{h}=4+h$, since $h\neq 0$.

The limit of this as $h\to 0$ is 4,

$\therefore$ the gradient of the curve at (2, 4) is 4.

(ii) Let P, R be (a, a^2), $(a+h, a^2+2ah+h^2)$ respectively.

Then the gradient of $PR=\dfrac{2ah+h^2}{h}=2a+h$, since $h\neq 0$.

The limit of this as $h\to 0$ is $2a$,

$\therefore$ the gradient of the curve at (a, a^2) is $2a$.

The result of (i) may be deduced from this by putting $a=2$. We have shown that the gradient of $y=x^2$ at the point where $x=a$ is $2a$; in other words, the gradient at any point on the curve is twice the value of x at that point, i.e. $2x$ is the formula for the gradient at any point.

EXERCISE 110

Find the gradient of:

1. $y=2x^2-x-1$ at (5, 44).

2. $y=x^3-x^2-7x-1$ at (2, −11).

3. $y=7x^2-5x+3$ at (−3, 81).

4. $y=2x^3+7x-5$ at (−1, −14).

5. $y=5x^2-4x-2$ at (4, 62).

6. $y=2x^3+3x^2+5x-3$ at (−2, −17).

7. $y=x^3$ at (a, a^3).

8. $y=kx^3$ at (a, ka^3), k being a constant.

Find the gradient at the point where $x=a$ of:

9. (i) $y=4x^3$, (ii) $y=3x^2$, (iii) $y=5x$, (iv) $y=4x^3+3x^2+5x$.

What is the connection between the last result and the first three?

10. (i) $y=2x^2$, (ii) $y=-5x$, (iii) $y=7$, (iv) $y=2x^2-5x+7$.

What is the connection between the last result and the first three?

260. Examples Nos. 7–10 of Ex. 110 suggest the truth of the following rules. The formal rigorous proofs of these rules are beyond the scope of this book and we shall assume the results.

(1) If n is rational, the gradient of $y=x^n$ at the point (a, a^n) is na^{n-1}.

(2) If k is a constant, the gradient of $y=kx^n$ at the point (a, ka^n) is k times the gradient of $y=x^n$ at the point (a, a^n). It is therefore kna^{n-1}.

(3) The gradient of $y=\Sigma A_r x^r$ at the point $(a, \Sigma A_r a^r)$ is $\Sigma A_r \,.\, r \,.\, a^{r-1}$. It should be noted that a constant term, e.g. A_0, contributes **0** to the gradient, for the constant term disappears when we subtract y_P from y_R in forming the gradient of the chord PR.

Example 3. *Write down the gradients at the points where* $x=a$ of (i) $y=3x^2-7x+5$, (ii) $y=4x^4-5x^3+2x-3$, (iii) $y=\dfrac{x^2+1}{x}$, (iv) $y=x\sqrt{x}+5x-3\sqrt{x}+2$.

(i) Gradient $=3\times 2a-7\times 1\,.\,a^0+\mathbf{0}=6a-7$;

(ii) Gradient $=4\times 4a^3-5\times 3a^2+2\times 1\,.\,a^0-\mathbf{0}=16a^3-15a^2+2$;

(iii) This is not written in the form $y=\Sigma A_r x^r$, but it may be so written, for $\dfrac{x^2+1}{x}=x+\dfrac{1}{x}=x+x^{-1}$;

$$\therefore \text{ the gradient}=1\,.\,a^0+(-1)a^{-2}=1-\frac{1}{a^2};$$

(iv) This must first be written in the form $y=x^{\frac{3}{2}}+5x-3x^{\frac{1}{2}}+2$, and the gradient $=\dfrac{3}{2}a^{\frac{1}{2}}+5\times 1\,.\,a^0-3\times\dfrac{1}{2}\,.\,a^{-\frac{1}{2}}+\mathbf{0}=\dfrac{3\sqrt{a}}{2}+5-\dfrac{3}{2\sqrt{a}}$.

It is essential that these rules should be mastered by the pupil.

EXERCISE 111

In Nos. 1–20, *write down* the gradient at the point where $x=a$ of:

1. $y=x^5$. **2.** $y=x^8$. **3.** $y=5x^2$.

4. $y=8x^6$. **5.** $y=3x^3-2x$. **6.** $y=4x^2-x-7$.

7. $y=(2x^2)^2$. **8.** $y=(2x-1)^2$. **9.** $y=4\sqrt{x}$.

10. $y=x^2-\dfrac{3}{x}$. **11.** $y=\dfrac{5x^2-x}{x^2}$. **12.** $y=\dfrac{5x+2}{x^3}$.

13. $y=3x^3-5x+1$. **14.** $y=2x^4+3x^2+1$.

15. $y=(x+1)(2x^2+9)$. **16.** $y=\left(2x-\frac{3}{x}\right)\left(3x+\frac{2}{x}\right)$.

17. $y=3x^3-8x^2+7x-2+\frac{5}{x}$. **18.** $y=(3x+1)^3$.

19. $(4x^5)^2$. **20.** $y=\frac{2x-7}{x^4}$.

21. Show that the gradient at the point (2, 4) of the curve $y=-1+3x-\frac{x^2}{4}$ is double that at the point (4, 7); find the point at which the gradient is -1.

22. Find the points at which the gradient of the curve $y=2x^3-15x^2+36x-20$ is (i) 0, (ii) 12.

23. If $y=x+x^{-1}$, show that the gradient of the graph of y is positive or negative according as x is numerically greater than 1 or numerically less than 1. Sketch the graph of y, and show that y is never numerically less than 2.

24. Find the gradient of the curve $y=3x^3-4x^2+x$ at each of the points where it cuts the axis of x, and draw a rough sketch of the curve.

261. Increasing and decreasing functions. Turning points. A function $F(x)(=y)$ which increases as x increases from a to b is called an **increasing function** for the range of values a to b, and a function which decreases as x increases is called a **decreasing function.** Figs. 26 and 27 represent graphs of increasing functions, and Figs. 28 and 29 graphs of decreasing functions for the range of values shown.

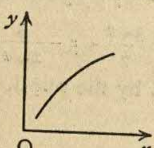


FIG. 26.

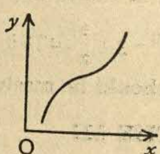


FIG. 27.

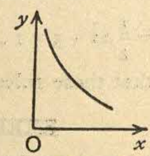


FIG. 28.

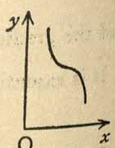


FIG. 29.

If the function $F(x)$ is increasing for a given range of values, the gradient of $y=F(x)$ is positive or zero at each point within the given range of values. Similarly, if the function is decreasing, the gradient is negative or zero.

Turning points. Fig. 30 represents the graph of a function

$F(x)$ which is an increasing function for some ranges of values of x and a decreasing function for other ranges.

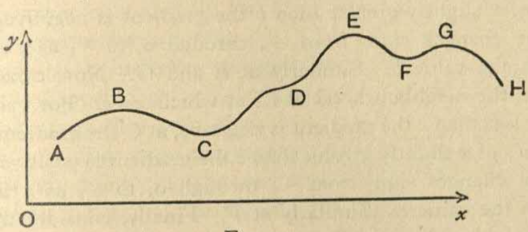


FIG. 30.

$F(x)$ is an increasing function from A to B, from C to E and from F to G. It is a decreasing function from B to C, from E to F and from G to H. The separating points B, C, E, F, G are called **turning points.**

B, E, G correspond to values of x for which $F(x)$ is greater than at any neighbouring points ; at these turning points $F(x)$, or y is said to have a **maximum** value ; C, F correspond to values of x for which $F(x)$ is less than at any neighbouring points ; at these turning points $F(x)$ is said to have a **minimum** value.

It should be particularly noted that " maximum " and " minimum " do not necessarily mean greatest or least in the arithmetical sense ; they only mean greatest or least in the immediate neighbourhood. Thus, in Fig. 30 the value of y at F is greater than the value of y at B. But y has at B a maximum value and at F a minimum value. The pupil must distinguish carefully between the technical use of the terms " maximum " and " minimum " and the arithmetical use of " greatest " and " least ".

At a turning point the tangent to the curve is parallel to Ox and the gradient of the curve is therefore zero ; we find the turning points of a curve by examining the points where the gradient is zero. But it should be noted that the gradient may be zero for a value of x for which $F(x)$ is neither a maximum nor a minimum. This is the case at the point D (Fig. 30). Such a point is called a **point of inflexion.**

262. Let us now consider how to discriminate between maxima, minima and points of inflexion. Consider the curve (Fig. 30) in

the neighbourhood of B, at which $x=b$. For values of x slightly less than b the gradient is positive, at B the gradient is zero, for values of x slightly greater than b the gradient is negative, i.e. the gradient changes sign, from $+$, through 0, to $-$, as x increases through the value b. Similarly at E and G. Now consider the curve in the neighbourhood of C, at which $x=c$. For values of x slightly less than c the gradient is negative, at C the gradient is zero, for values of x slightly greater than c the gradient is positive, i.e. the gradient changes sign, from $-$, through 0, to $+$, as x increases through the value c. Similarly at F. Finally, consider the curve in the neighbourhood of D, at which $x=d$. For values of x slightly less than d the gradient is positive, at D the gradient is zero, for values of x slightly greater than d the gradient is positive, i.e. the gradient does not change sign as x increases through the value d. It is clear that to obtain maximum and minimum values of a function $F(x)$ we must:

(1) **Find the values of x at which the gradient of $y=F(x)$ is zero.**

(2) **Find whether, as x increases through these values, the gradient changes (a) from + to − (a Maximum)**

or (b) from − to + (a Minimum).

(Both criteria are necessary. If the gradient does not change sign, the point is neither a maximum nor a minimum.)

(3) **Find the values of F(x) for the values of x which give a maximum or a minimum.**

Example 4. *Find the maximum or minimum values (if any) of* $2+5x-3x^2$.

The gradient, G, of $y=2+5x-3x^2$ at $x=a$ is $5-6a$. This is zero if $a=\frac{5}{6}$. For values of a slightly less than $\frac{5}{6}$, G is $+$; for values of a slightly greater than $\frac{5}{6}$, G is $-$; $\therefore$ as a increases through the value $\frac{5}{6}$, G changes sign from $+$, through 0, to $-$. Hence, when $x=\frac{5}{6}$, $2+5x-3x^2$ has a maximum value; this value is

$$2+5\cdot\tfrac{5}{6}-3\cdot\tfrac{5}{6}\cdot\tfrac{5}{6}=4\tfrac{1}{12}.$$

Example 5. *Find the greatest rectangular area that can be enclosed by a wire* 160 *yards long.*

The perimeter of the rectangle is 160 yards. Suppose that one side is x yards, then the other side is $(80-x)$ yards; and the area is $x(80-x)$ sq. yd.

Let $y = x(80 - x) = 80x - x^2$. The gradient, G, at $x = a$ is $80 - 2a$. This is zero, if $a = 40$.

For values of a slightly less than 40, G is $+$; for values of c slightly greater than 40, G is $-$.

Hence, as a increases through the value 40, G changes sign from $+$, through 0, to $-$; $\therefore$ when $x = 40$, y has a maximum value; this value is $40(80 - 40) = 1600$.

Also it is clear from the graph of $y = x(80 - x)$, which has only one turning point, that the maximum value is the greatest value; $\therefore$ the greatest rectangular area that can be enclosed is 1600 sq. yd.

Note. The pupil should make a habit of drawing a *rough* sketch of any graph needed.

Example 6. *A man orders a plumber to make a cistern with a square base and closed at the top to hold* 3375 *cu. ft. of water. It has to be lined inside with sheet lead at* 4*d. per square foot. Find the least that the cost of lining can be.*

Let V cu. ft. be the volume, S sq. ft. the area to be lined, x ft. the side of the square base, h ft. the height.

Then $$V = x^2h = 3375, \quad S = 4xh + 2x^2.$$

We require the least value of S, when x is positive. We cannot write down the gradient of $S = 4xh + 2x^2$ as it stands, for x and h are both variable quantities. But we may replace h by $\frac{3375}{x^2}$, and write $S = 13500x^{-1} + 2x^2$. The gradient of this at $x = a$ is $-13500a^{-2} + 4a$. This is zero, if $a^3 = 3375 = 15^3$, i.e. if $a = 15$.

For values of a slightly less than 15, $G = 4a\left(-\frac{3375}{a^3} + 1\right)$ is $-$; for values of a slightly greater than 15, G is $+$. Hence, as a increases through the value 15, G changes sign from $-$, through 0, to $+$; $\therefore$ when $x = 15$, S has a minimum value. This value is $\frac{13500}{15} + 2 \cdot 15^2 = 900 + 450 = 1350$. Also it is clear from the graph of $S = 13500x^{-1} + 2x^2$, which has only one turning point, that the minimum value is the least value; $\therefore$ the least area which must be lined with sheet lead is 1350 sq. ft. The cost of lining this at 4d. per sq. ft. is £$1350 \times \frac{1}{60}$ = £22 10s.

Example 7. *Find the greatest value of* $4 - 3x + x^2$ *for values of* x *from* 0 *to* 5 *inclusive.*

If $y = 4 - 3x + x^2$, the gradient, G, at $x = a$ is $-3 + 2a$.

This is zero if $a = 1\frac{1}{2}$, and applying the usual test we find that when $a = 1\frac{1}{2}$, y has a minimum value, which is $1\frac{3}{4}$.

When $x = 0$, $y = 4$; when $x = 5$, $y = 14$.

For values of x between 0 and $1\frac{1}{2}$, G is $-$; for values of x between $1\frac{1}{2}$ and 5, G is $+$; $\therefore$ y decreases from the value 4 when $x = 0$ to the value $1\frac{3}{4}$ when $x = 1\frac{1}{2}$; it then increases to the value 14 when $x = 5$. It is clear that the greatest value of y, for the stated range of values of x, is 14.

EXERCISE 112

Find the maximum and minimum values (if any) of:

1. $7 + 6x + x^2$. **2.** $7 + 5x - 3x^2$. **3.** $5 - 4x - 2x^2$.

4. $12x^3 - 18x^2 + 9x$. **5.** $2x^3 - 9x^2 + 12x - 5$. **6.** $x^4 - 2x^2 + 10$.

7. $2x^3 - 3x^2 - 12x + 7$. **8.** $4x^3 - 6x^2 - 9x + 1$. **9.** $x(x-1)^2 + 2$.

10. $2x^3 + 3x^2 - 36x - 28$.

11. $x^2(a - x)$ (a being a positive constant).

12. x^3y^2, x and y being connected by the relation $x + y = 10$.

13. A rectangular block with a square base has a total surface area of 150 sq. in. Find the greatest volume of the block.

14. A line AB, 8 in. long, is divided at P. Find P, so that $AP^2 + PB^2$ is a minimum.

15. What number exceeds its cube by the greatest number possible?

16. A rectangular field is bounded on one side by a straight river, and on the other three sides by a fence whose total length is 160 yd. Show that the area of the field cannot exceed 3200 sq. yd.

17. If 10 solid cubes of side x in. and 40 of side y in. are to be made, where $x + y = 12$, find the values of x and y that will make the total volume a minimum.

18. If a thin rod 12 in. long swings like a pendulum, the tendency to break at a point x ft. from the point of suspension varies as $x(1-x)^2$. Find where the rod is most likely to break.

19. A circular tin canister closed at both ends has a surface area of 400 sq. cm. Find the greatest volume it can contain.

20. An isosceles $\triangle$ has a base of length 2 in. and its height is 5 in. Find the area of the largest rectangle which can be inscribed in the $\triangle$

21. A closed rectangular box is made of sheet metal of negligible thickness, the length of the box being twice its width. Find the dimensions of the box of least surface that has a capacity of 243 cu. in.

22. Find the height of the right circular cone of greatest volume, the sum of the height and radius of the base being 12 in.

23. A closed rectangular box is to be made from 1600 sq. in. of thin metal, the perimeter of the base being always 80 in. Find the dimensions of the box that has the greatest volume.

24. Find the value of x for which the sum of the corresponding ordinates of the curves

$$y=2x^3-15x^2+36x+5 \quad \text{and} \quad y=x^2-4x+3$$

is a maximum, and show that, for this value, the corresponding ordinate of one curve is a maximum, while that of the other is a minimum.

25. From each corner of a thin rectangular sheet of metal 8 in. long and 6 in. wide a square of side x in. is cut away and the projecting portions of the remainder are turned up so as to form the sides of a rectangular box. Determine x to the nearest tenth of an inch, so that the box may contain the greatest volume.

26. A hat-box with a square base has a slip-on-lid of depth 1 in., which fits tightly round the sides of the box. The box and lid are made out of a sheet of thin cardboard of area 680 sq. in. Find the dimensions of the box when the volume is greatest.

27. Post Office regulations prescribe that the combined length and girth of a parcel must not exceed 6 ft. Find the greatest volume of a parcel whose shape is a right prism with a square base. Find also the greatest volume of a parcel whose shape is a right circular cylinder.

28. A waste-paper basket made of thin material is in the form of a right cylinder on a circular base, open at the top. If the volume is to be 2 cu. ft., find the radius of the base, if the amount of material used is to be as small as possible.

29. The side AB of the rectangle $ABCD$ is $6x$ ft. On AB, outside the rectangle, is drawn a $\triangle OAB$, such that OA and OB each equal $\frac{5AB}{6}$. Find the greatest area of the figure $OADCB$, if its perimeter is always 50 ft.

30. A piece of wire is cut into two pieces and each piece is bent into the form of a circle. Show that the sum of the areas of the two circles so formed is least when the wire is cut into two equal pieces.

31. A hat-box with a circular base has a slip-on-lid of depth 0·5″, which fits tightly round the box. The box and lid are made out of a sheet of thin metal of area 60π sq. in. Find the height and radius of the base when the volume is greatest.

32. The running cost, C, of a ship, in pounds per hour, is given by the formula $C = 4 + \dfrac{s^3}{1000}$, where s is the speed in knots. Find the speed which causes the least cost for a *given* voyage.

EXERCISE 112. c

(*In this exercise all cylinders are right circular cylinders*)

1. A match-box of the usual type (i.e. an open rectangular box with a sliding cover open at each end) is made of material of negligible thickness. Its length is 5 cm., and its girth is 10 cm. Find the breadth and height when the area of the material used is greatest.

2. CAB is an isosceles $\triangle$, right-angled at C and having $CA = CB$. P lies on AB; PM is perpendicular to CA, and PN to CB. Find the position of P in which the area of the rectangle $PMCN$ is greatest.

3. A and B are fixed points whose coordinates are $(0, a)$ and (b, c) respectively. $P(x, 0)$ is a variable point on the x-axis. Express $AP^2 + BP^2$ in terms of x and the constants, and find the position of P which makes this function a minimum.

4. An open cylindrical vessel is to be constructed from a given amount of uniform thin material. Show that it contains the greatest possible volume when its height is equal to the radius of its base.

5. A cylindrical tin canister of height h and radius r has a slip-on-lid of depth a (a constant). If the capacity of the tin is a maximum for a given expenditure of metal, including the lid, prove that $h = a + 2r$. Neglect the thickness of the metal.

6. The expenses each day in running a ship consist of a fixed amount a together with a variable amount bx^3, where x miles is the distance run per day and b is another constant. Write down the total cost of a voyage of length s miles. Prove that the cost is least when the fixed part of the cost per day is twice the variable part.

7. An open cylindrical tub of height h and diameter $2r$ holds a definite quantity of water when full. Show that the surface of the tub itself will be least when $h = r$.

8. Show that $x^n(a - x)$, a being a positive constant and n a positive integer, has a maximum value where $x = \dfrac{na}{n+1}$.

9. A five-sided window is to be made in the form of a rectangle surmounted by an equilateral triangle. If the perimeter of the window must be equal to P, prove that the greatest possible area is $P^2(6+\sqrt{3})/132$.

10. If a and b are positive quantities, determine whether x^5-5a^4x+b has any maximum or minimum values and find any such values, distinguishing between them. Hence, or otherwise, prove that, if $b>4a^5$, the given expression cannot be negative for any positive value of x.

11. A beam AB, 4 ft. long and of weight W lb., is supported at its ends A and B, and a weight W' lb. is attached to a point C of the beam, 1 ft. from A. Assuming that the bending moment at a point of the beam in AC, distant x ft. from A, is proportional to $(4W+6W')x-Wx^2$, and the bending moment at a point in CB, distant y ft. from B, is proportional to $(4W+2W')y-Wy^2$, find the point of the beam at which the bending moment is greatest, (i) when $W'=2W$, (ii) when $2W'=W$. Illustrate your results by drawing graphs of the bending moments in these two cases.

12. The brightness of a small object at P due to a source of light at A may be measured by $\dfrac{k}{PA^2}$, where k is the candle-power of the source. A source of candle-power 10 is 20 ft. from a second source of candle-power 640. Find the point on the line between the sources where the brightness is least.

Graphical solution of equations (continued)

263. In Chapter XVIII, the following result was proved:

If we have a pair of simultaneous equations in x and y, and if the graphs corresponding to the equations are drawn with the same axes and with the same scales, then, at the points of intersection of the graphs,

(1) the coordinates are the roots of the simultaneous equations;

(2) the x-coordinates are roots of the equation in x obtained by eliminating y from the two equations;

(3) the y-coordinates are the roots of the equation in y obtained by eliminating x from the two equations.

We shall now show how to obtain graphically the solution of the standard equation of the third degree, usually called a cubic equation.

264. By applying the general theorem quoted above, we may obtain the real solutions of the equations $ax^3+bx^2+cx+d=0$:

(1) by drawing the graph of $y=ax^3+bx^2+cx+d$ and finding the values of x at its intersections with the graph of $y=0$;

(2) by drawing the graph of $y=ax^3+bx^2+cx$ and finding the values of x at its intersections with the graph of $y=-d$;

(3) by drawing the graph of $y=ax^3+bx^2$ and finding the values of x at its intersections with the graph of $y=-cx-d$;

(4) by drawing the graph of $y=ax^3$ and finding the values of x at its intersections with the graph of $y=-bx^2-cx-d$; etc.

In general, (3) is the best method, for the graph of $y=ax^3+bx^2$ has two turning points, unless $b=0$. (The values of x at the turning points are given by $3ax^2+2bx=0$, i.e. $x=0$ or $-\dfrac{2b}{3a}$. Thus one turning point is always at the origin.) If $b=0$, we have the standard graph $y=ax^3$.

Also, the graph of $y=-cx-d$ is a straight line, so that, whatever the values of the constants, a, b, c, d, the work involved is reasonably short and simple. It may happen, however, that the turning points are inconveniently close together; in such a case it may be desirable to use one of the other methods.

Example 8. *Solve graphically the equation* $x^3-5x^2-12x+34=0$.

The required solutions are the values of x at the intersections of the graphs of $y=x^3-5x^2$ and $y=12x-34$.

(1) Consider $y=x^3-5x^2$. The turning points occur when $3x^2-10x=0$, i.e. when $x=0$ or $3\frac{1}{3}$. Plotting points as usual, we have

x	-3	-2	-1	0	1	2	3	$3\frac{1}{3}$	$3\frac{2}{3}$	4	5
x^3	-27	-8	-1	0	1	8	27	$37\frac{1}{27}$	$49\frac{8}{27}$	64	125
$-5x^2$	-45	-20	-5	0	-5	-20	-45	$-55\frac{5}{9}$	$-67\frac{2}{9}$	-80	-125
y	-72	-28	-6	0	-4	-12	-18	$-18\frac{14}{27}$	$-17\frac{25}{27}$	-16	0

(2) Consider $y = 12x - 34$:

x	2	3	4
y	-10	2	14

The graphs are as shown. It is clear that the roots are approximately $-2{\cdot}95$ and $1{\cdot}9$.

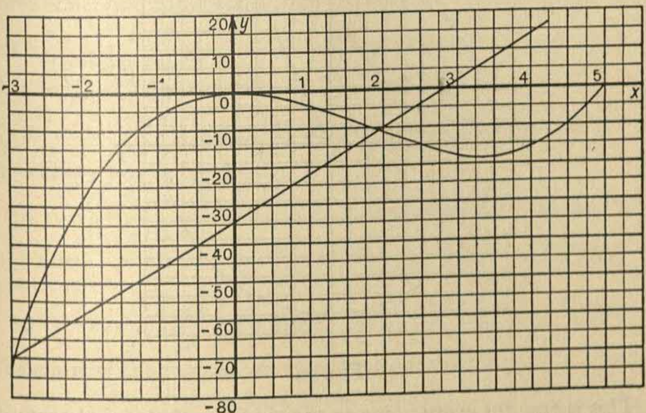


FIG. 31.

To get the third root we might draw the graphs on a smaller scale, to get an approximate value of x at the third intersection, **but it is not necessary to do this.**

Instead, we may use the fact that the sum of the roots of $x^3 - 5x^2 - 12x + 34 = 0$ is 5 (see Ch. XXXII, Art. 249),

$\therefore$ the third root is approximately $6{\cdot}05$.

If more accurate results are desired, we may draw a portion of the graphs on a larger scale in the neighbourhood of

$$x = -2{\cdot}95 \quad \text{and} \quad x = 1{\cdot}9 \text{ (or } x = 6{\cdot}05).$$

It should be particularly noted that the third root may be deduced from the other two by using the formula for the sum of the roots.

265. The circle. If P is a point (x, y) on a circle, of radius r, whose centre is at the origin, we have (see Fig. 32),

$$ON^2 + NP^2 = OP^2, \text{ by the theorem of Pythagoras ;}$$

$$\text{i.e. } x^2 + y^2 = r^2.$$

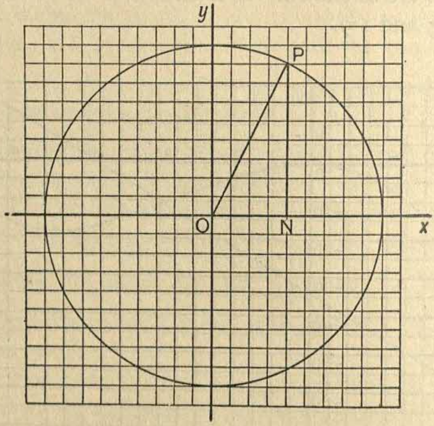


FIG. 32.

This is true for every point on the circle, so that a circle with its centre at the origin, and with radius r, is the graph corresponding to the equation $x^2 + y^2 = r^2$. This is of great importance in solving graphically equations of the type $x^2 + y^2 = r^2$, $ax + by + c = 0$; for we have only to find the intersections of a circle and a straight line.

EXERCISE 113

Solve graphically Nos. 1–6 :

1. $x^3 - 3x^2 - 6x + 4 = 0$. **2.** $x^3 - 12x + 12 = 0$.

3. $4x^3 - x^2 - 8x = 2$. **4.** $2x^3 + 3x^2 + 3x + 1 = 0$.

5. $2x^3 - 5x = 4$. **6.** $x^3 - 3x^2 - 4x + 3 = 0$.

Draw, for the values of x stated, the graphs of (Nos. 7–15) :

7. $(x - 1)(x - 2)^2$ [0 to 3·4]. Hence find to 2 sig. figs. a root of $2(x - 1)(x - 2)^2 = 1$.

8. $y = (x - 1)(x - 2)(x - 3)$ [0 to 4]. Find the value of x where the graph meets that of $2y = x$. Of what cubic equation is this value of x a solution?

9. $y=x^3-12x+6$ $[-4$ to $4]$. Find from the graph, as nearly as you can, for what values of y there is more than one positive value of x.

10. $y=15+4x-x^3$ $[-2$ to $3]$. Find from the graph for what range of positive values of x the value of y exceeds that of x.

11. $4y=x(x^2-4)$ $[-3$ to $3]$. From your graph determine (i) the value of x for which $y=3$, (ii) the values of x at the points on the curve which are equidistant from the points $(-1, 1)$ and $(1, -1)$.

12. $\dfrac{(x-2)(x-4)}{(x-1)(x-3)}$ $[-1$ to $5]$. Use your graph to find a root of $2x(x-1)(x-3)=5(x-2)(x-4)$.

13. $y=x+\dfrac{1}{x}$ $[-3$ to $3]$. From your graph find the values of x for which $y-2x$ is positive.

14. $y=\dfrac{(x-1)(x-2)}{(x-3)(x+1)}$ $[-2$ to $5]$. Determine the limiting value of y when x tends to infinity, and find whether y is ever equal to this value.

15. $y=3x-4+\dfrac{200}{3x^2}$ $\left[1\frac{1}{2} \text{ to } 8\right]$. Find the minimum value of y. Explain how the roots of an equation of the form

$$3x^2(a+4-3x)=200,$$

where a is a constant, may be found from the graph. Apply the method to find the roots of $9x^3-60x^2+200=0$.

16. On the same diagram, draw the graphs of $y=\dfrac{(x+1)^2}{4}$ and $y=\dfrac{x(x-4)^2}{4}$ for values of x from 0 to 4. For what range of values of x between 0 and 4 is $x(x-4)^2$ greater than $(x+1)^2$?

17. On the same diagram, draw the graphs of (a) x^2 from -1 to 4, (b) $\dfrac{15}{x}$ from 0·5 to 4. Hence find $\sqrt[3]{15}$, correct to one decimal place.

18. On the same diagram, draw the graphs of $\log_{10} x$ and $\dfrac{x}{10}$ for values of x from 0·5 to 10. Hence find two solutions of $\log_{10} x=\dfrac{x}{10}$, correct to 2 sig. figs.

19. On the same diagram, draw the graphs of $y=\dfrac{1}{5}(8-x^3)$ and $y=-\dfrac{4x}{5}$. Write down a formula for the length parallel to the

y-axis intercepted between the two graphs for any value of x. Find a value for x which satisfies the equation $x^3 = 4x + 8$.

20. (1) If $y = \frac{1}{2}x^2 - 8\frac{1}{2} + \frac{8}{x^2}$, find the values of x which make y a maximum or a minimum, and the corresponding values of y. (2) What symmetry has the curve $y = \frac{1}{2}x^2 - 8\frac{1}{2} + \frac{8}{x^2}$? (3) Where does it cut the x-axis? (4) What is the sign of y when x is (a) very great, (b) very small? (5) From these indications sketch the graph.

21. If $y = (x^2 - 2)(x^2 - 3)$, (1) calculate the values of x for which y is a maximum or a minimum, and the corresponding values of y; (2) prove that y is positive when $x^2 < 2$ or when $x^2 > 3$, and that y is negative when $2 < x^2 < 3$; (3) prove that the graph of

$$y = (x^2 - 2)(x^2 - 3)$$

is symmetrical with regard to the axis of y; (4) sketch the graph.

Solve graphically Nos. 22–24:

22. $x^2 + y^2 = 25$, $2x + y = 2$.

23. $x^2 + y^2 = 10$, $7x + 2y = 1$.

24. $x^2 + y^2 = 9$, $3x + y = 3$.

TEST PAPERS IX

A

1. If p be a root of $x^2 + ax + b = 0$, show that it is also a root of $x^3 - ax^2 - (2a^2 - b)x - 2ab = 0$.

2. Simplify (i) $\dfrac{16a^4 - 81b^4}{8a^3 - 12a^2b + 18ab^2 - 27b^3}$,

(ii) $1 - \dfrac{3}{(2a-1)} + \dfrac{3}{(2a-1)^2} - \dfrac{1}{(2a-1)^3}$.

3. Evaluate (i) $\dfrac{(0\cdot5933)^{\frac{2}{5}} \times 25\cdot13}{(5\cdot526)^2 - (0\cdot526)^2}$,

(ii) $(1\cdot73)^{-0\cdot4} + (0\cdot08392)^{-1\cdot1}$.

4. The sum of the first five terms of an A.P. is 40. Eight times the 7th term is equal to three times the 19th term. Find the 1st term and the common difference.

5. The first and last terms of a G.P. are x and $2x$. If there are 16 terms, find the square root of the product of all the terms.

6. The volume of a solid right prism of square section is 64 cu. in., the side of the square being x in. Express the length of the prism in terms of x, and find its *total* surface area. Show that this area is least when the prism is a cube.

B

1. Find the minimum value of $2x^3 - 3x^2 - 36x + 1$.

2. Find the sum of 40 terms of the A.P. whose 7th term is 2 and whose 19th term is -4.

3. The weight of a body varies jointly as its height and the square of the diameter of its base. If the weight is 125 lb. when the height is 5 cm. and the diameter of the base is 4 cm., find (i) the weight when the height is 8 cm. and the diameter of the base is 1·2 cm., (ii) the diameter of the base when the height is 4 cm. and the weight is 36 lb.

4. Given $\log_{10}2 = p$ and $\log_{10}3 = q$, find in terms of p and q and without the use of tables the values of $\log_{10}6$, $\log_{10}5$ and $\log_{10}24$. Find also $\log_e 27 \div \log_e 9$, without using tables.

5. A number of squares are described whose sides are in G.P. Prove that the areas of the squares are also in G.P. The side of the $(2m)$th square is a ft. and the side of the $(2n)$th square is b ft.; find the area of the $(m+n)$th square.

6. Solve the equations (i) $21x - 3x^2 + 4\sqrt{x^2 - 7x + 15} = 41$,
(ii) $3250 \times 0{\cdot}5 = x[1 - (1{\cdot}05)^{-15}]$.

C

1. If $x = \dfrac{a+1}{a-1}$ and $y = \dfrac{2a+1}{2a-1}$, find $\dfrac{x-y}{x+y}$ in terms of a, in its simplest form.

2. Find the x-coordinates of the points on the curve

$$y = x^3 - x^2 - 5x + 1$$

at which the gradient is (i) 0, (ii) -4.

3. From the relation $p_1 v_1{}^x = p_2 v_2{}^x$, find the value of x to two decimal places, if $p_1 = 12{\cdot}43$, $p_2 = 27{\cdot}89$, $v_1 = 102{\cdot}5$, $v_2 = 58{\cdot}02$.

4. (i) Find the sum of n terms of the series $9 + 7 + 5 + 3 + \cdots$.

(ii) Show that the square of the sum of the first n natural numbers exceeds the square of the sum of the first $(n-1)$ natural numbers by n^3.

5. A man receives a pension starting with £100 the first year, but each year he receives 90 per cent. of what he received the previous year. Find the total amount he receives in the first 6 years; find also the greatest amount he could possibly receive, even if he were to live for ever.

6. The weight of a right circular cylinder of given material varies as the square of the radius and also as the height. If the radius is increased by 20 per cent. and the height by 10 per cent., find the percentage increase in the weight.

D

1. Find the gradient at any point on the curve $6y = 4x^3 + 3x - 7$. Find a point on this curve at which the tangent to the curve is equally inclined to the axes. How many such points are there? Prove that there is no point on the curve at which the tangent is parallel to the axis of x.

2. Find what conditions must be satisfied by the real numbers a, b, c, in order that the roots of the equation $ax^2 + bx + c = 0$ may be real and positive. Show also that, if both roots of the equation are greater than unity, $a + b + c$ has the same sign as a, but $2a + b$ has the opposite sign.

3. Evaluate $\sqrt{\dfrac{3{\cdot}7226}{0{\cdot}5819} - \dfrac{0{\cdot}5819}{3{\cdot}7226}}$.

4. (i) In the series $210 + 192 + 174 + \dots$, the sum of n terms is 1320. Find n.

(ii) In a certain week the expenses of a shop exceeded the takings by £8. In the next week the loss is £5, and in the following week £2. If this improvement is maintained regularly, how much profit is made in 20 weeks from the start?

5. Using logarithms, sum the G.P. $3 + 2^{\frac{1}{4}}3^{\frac{3}{4}} + 2^{\frac{1}{2}}3^{\frac{1}{2}} + \dots$ to 10 terms.

6. Find c, so that the roots of the equation $(x-2)(4-x) = x - c$ may be equal. Taking this value of c, draw the graphs of $(x-2)(4-x)$ and $x - c$ and state the geometrical meaning of the result.

E

1. Reduce the fractions :

$$\text{(i)}\ \frac{p^4 - p^2 + 2p - 1}{p^4 + 2p^3 + p^2 - 1}, \quad \text{(ii)}\ \frac{3a^5 - 5a^3 + 2}{2a^5 - 5a^2 + 3}$$

to their lowest terms.

2. A square sheet of thin metal has sides of length a. Equal square pieces are cut out of each corner of the sheet, so that the piece that remains can be bent to form an open box. Find the greatest volume of the box that can be so formed.

3. If H is the horse-power required to drive a given type of ship whose weight is W tons at V knots, the relation between H, W and V is $H^3 = A \,.\, W^2 \,.\, V^9$, where A is a constant. When W is 1520 and V is 12, then H is 550. Find H, by using logarithms, when W is 2100 and V is 5.

4. Solve the equations (i) $(x+2)^2 = 5\left(\sqrt{x^2 + 4x + 5} - 1\right)$,

(ii) $7.(1{\cdot}03)^x = 10{\cdot}38$.

5. The 6th term of an A.P. is four times the 2nd term, and the sum of the first 24 terms is 1704; find the sum of the first 48 terms.

6. Some machinery belonging to a company was originally valued at £5400. At the end of each subsequent year it was re-valued as being x per cent. of the estimated value at the beginning of the year. Find the value of x (to the nearest whole number), if at the end of 10 years the estimated value was £100.

F

1. Find three quadratics which are such that, in each, the sum of the squares of the roots is greater by 40 than the sum of the roots, and that the sum of the cubes of the roots is greater by 20 than the sum of the squares of the roots.

2. (i) Find the sum of all multiples of 11 between 300 and 3000.

(ii) Prove that in any G.P. the sum of the 4th, 5th and 6th terms is the geometric mean between the sum of the 1st 2nd and 3rd terms and the sum of the 7th, 8th and 9th terms.

3. The cost of providing a school dinner is partly constant and partly varies as the number of pupils who take dinner. When 306 pupils took dinner the total daily cost was £8 9s. 6d., but when the number dropped to 270 the total daily cost was £7 14s. 6d. Assuming that other conditions do not vary, what will be the total daily cost when 324 pupils take the dinner?

4. Evaluate $\sqrt[3]{\dfrac{0{\cdot}6234 \times 17{\cdot}02}{(4{\cdot}176)^3 + 1{\cdot}61}}$,

5. Without using tables, find x, if

$$\log x = \log 0{\cdot}3 + 3 \log 2 - \tfrac{2}{5} \log 32.$$

6. Find the gradient of $(x-1)^2(x-2)$ and the values of x for which the gradient vanishes. Draw the graph of the function for values of x between 0 and 4. Deduce that $x^3 - 4x^2 + 5x = 4$ has only one real root, and that $x^3 - 4x^2 + 5x = \frac{17}{9}$ has three real roots.

G

1. If $y = \dfrac{4+3x}{3+2x}$, express the fraction $\dfrac{y-x}{2-y^2}$ in terms of x, and simplify the expression as far as possible.

2. (i) Find the sum of the first 84 terms of the A.P. whose 7th term is 22 and whose 15th term is 78.

(ii) A man's salary started at £234 a year, and was raised £15 at the end of each year; he received in all £3798. For how long did he hold the post?

3. Calculate $\dfrac{a}{b}(1 - 10^x)$, when $a=4$, $b=2$, $x = -2{\cdot}065$;

4. What is the error made in taking the sum of the infinite series 1, 0·2, 0·04, 0·008, ... as being 1·248? Find, to 3 dec. places, the sum of the square roots of the terms of this series, (i) taken as all +, (ii) taken as alternatively + and −.

5. Prove that the roots of the equation $2x^2 - 2x(a+2) + 3a + 1 = 0$ are real for all real values of a and that the difference of the roots is greater than the numerical value of $a - 1$.

6. A right pyramid stands on a square base. If the distance from the vertex to the middle point of each edge of the base is a fixed length a, show that the volume of the pyramid is a maximum when the base has an area $8a^2/3$.

H

1. If an A.P. and a G.P. are added together, the sum of their first terms is 6, of their second terms is −1, and of their third terms is 1. If the first term of the G.P. is twice the first term of the A.P., find the common difference and the common ratio.

2. Given that $x^{\frac{2}{3}}y^{\frac{3}{5}}z^{-\frac{1}{2}} = 2$, express x in terms of y and z in the form $x = ay^bz^c$, and similarly express y in terms of x and z. Calculate z, to 2 decimal places, when $x = y = 2$.

3. Prove that the sum of the integers from 1 to 100 inclusive less twice the sum of the integers from 1 to 50 inclusive is equal to the sum of all the odd numbers from 1 to 99 inclusive.

4. If α, β are the roots of $x^2 + 5x + 1 = 0$, find the equation whose roots are $\alpha(\alpha + 2\beta)$, $\beta(\beta + 2\alpha)$.

5. Solve the equations (i) $10 - 4\sqrt{(1-x)(2-x)} = x^2 - 3x$,

(ii) $3^{x+2} - 3^x = 216$.

6. Find the dimensions of the right circular cone of greatest volume when the sum of the height of the cone and the radius of the base is 3 ft.

I

1. Express the value of $\dfrac{ab^2}{b+x} - \dfrac{a^2b}{a+x}$, when $x = \dfrac{ab}{a+b}$, as a single fraction in terms of a and b only, and in its lowest terms.

2. The population of a county increases by the same number of persons each year throughout n years: the rate of increase per annum during the last year is k times that in the first year; find an expression for the total percentage of increase in n years.

3. Find a point on the curve $y = x^3 - 2x^2 - 2x + 8$ at which the tangent is parallel to the tangent to the curve at (2, 4).

4. Evaluate (i) $3{\cdot}765^{-0{\cdot}8234}$,

(ii) $x^{\frac{3}{2}} \div \sqrt{y^2 + z^2}$, when $x = 1{\cdot}026$, $y = 0{\cdot}137$, $z = 0{\cdot}462$.

5. If A is the sum of n terms of the series $1+\frac{1}{4}+\frac{1}{16}+\ldots$ and B is the sum of $2n$ terms of the series $1+\frac{1}{2}+\frac{1}{4}+\ldots$, find the value of $\frac{A}{B}$.

6. A number n is the algebraic sum of two numbers a and b; $a \propto x$ when y is constant, and $a \propto y^2$ when x is constant; $b \propto x^3$ when y is constant and b varies inversely as y^2 when x is constant. If $n=213$ when $x=3$, $y=1$, and $n=-8$ when $x=-2$, $y=2$, find the equation connecting n, x, y.

J

1. (i) Prove that $(1+3 \cdot 2^{\frac{1}{3}}-3 \cdot 2^{\frac{2}{3}})(19+15 \cdot 2^{\frac{1}{3}}+12 \cdot 2^{\frac{2}{3}})=1$.

(ii) Find the rational solutions of the equations

$$2x-yz=16, \quad 2xyz=-15, \quad y^2+z^2=34.$$

2. (i) If $\log_e 2 = 0{\cdot}6931$, find e.

(ii) If $n=\log_{10}\left(\frac{Ar}{100P}+1\right)\div\log_{10}\left(1+\frac{r}{100}\right)$, find n to the nearest integer when $P=150$, $A=1720$, $r=6$.

3. What are the 1000th term and the sum of 1000 terms of the series 3, 4, 6, 7, 9, ... in which the terms increase alternately by 1 and 2?

4. If α, β are the roots of $ax^2+2bx+c=0$, find the equation whose roots are $\alpha+\frac{1}{\beta}$, $\beta+\frac{1}{\alpha}$.

5. A man wishes to found an annual prize of value £5 to be given in 20 successive years; find the sum that he should pay now in order to cover the annual payments, the first prize being awarded at the end of the first year, and interest being reckoned at 5 per cent. per annum. If he were to pay £80 now, for how many years could the prize be given under the same conditions?

6. An open vessel of thin material has a square horizontal base and four vertical rectangular sides. Show that, if the volume V is kept constant, and a, the length of the edge of the base, varies, the total surface is least when $a^3=2V$.

K

1. If $y^2=ax^7$ and $y^7z^3=a^3$, express x in its simplest form in terms of a and z. Find y and z when $a=2{\cdot}475$, $x=0{\cdot}72$.

2. A farmer has 240 hens and sufficient corn to feed them all for 15 weeks. He sends the same number of birds to market at the end of each week, and thus makes the corn last for 25 weeks. How

many does he send away each week, and how many has he during the last week for which the corn lasts?

3. Prove that the sum of the squares of four consecutive odd numbers always exceeds four times the square of the average of the numbers by 20. Use this fact to calculate $37^2+39^2+41^2+43^2$.

4. Write down the nth term of the series $2^{\frac{2}{1}}+2^{\frac{3}{2}}+2^{\frac{4}{3}}+\dots$, and find the first term whose value is less than $2{\cdot}1$.

5. Find the turning points of $y=\dfrac{5x-13}{x^2-1}$. Draw a rough graph of the equation.

6. A point P whose x-coordinate is a is taken on the line $y=3x-7$. If Q is the point $(4, 1)$, show that $PQ^2=10a^2-56a+80$. Find the value of a which will make this expression a minimum. Hence show that the coordinates of N, the foot of the perpendicular from Q on to the line, are $(2{\cdot}8, 1{\cdot}4)$.

L

1. Prove that the equation $x=k(x-1)(x+2)$ has real roots for all real values of k; but that the equation $x=k(x-1)(x-2)$ has real roots only if k has a value *not* lying between $-3-2\sqrt{2}$ and $-3+2\sqrt{2}$. What conclusion can be drawn about the graphs of the functions $\dfrac{x}{(x-1)(x+2)}$ and $\dfrac{x}{(x-1)(x-2)}$?

2. If $2^x \cdot 3^y=3^x \cdot 4^y=6$, show that $x^2-2y^2=2x-3y$.

3. (i) Construct an A.P. whose first term is 1 and which is such that the sum of the first 20 terms is half the sum of the next 20 terms.

(ii) Find the sum of $2n$ terms of the series

$$\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{8}+\tfrac{1}{27}+\tfrac{1}{16}+\dots.$$

4. If the receipts on a railway vary as the excess of speed over 30 m.p.h., while the expenses vary as the square of that excess, find the speed at which the profits will be greatest, if at 60 m.p.h. the expenses are just covered.

5. Solve the equations

$$\text{(i)}\quad 30-16x=50x^2-3\sqrt{125x^2+40x-21},$$

$$\text{(ii)}\quad 6=3(2{\cdot}718)^x+2(2{\cdot}718)^{-x}.$$

6. (i) Given that $10^{0{\cdot}3010}=2$ and that $y=2^{4{\cdot}136}$, find $\log_{10}y$.

(ii) On the graph of 2^x three points P, Q, R are taken for which the values of x are $p-1$, p, $p+1$ respectively. Show that the difference of the ordinates of R and P bears to the ordinate of Q a ratio which is independent of p, and find this ratio.

TABLES

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	5 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	16 15	20 18	23 22	27 26	31 29	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	6 7	10 10	13 13	16 16	19 19	23 22	26 25	29 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 14	19 17	22 20	25 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 6	9 8	11 11	14 14	17 17	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	6 5	8 8	11 10	14 13	16 16	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 3	5 5	8 8	10 10	13 12	15 15	18 17	20 20	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 4	7 7	9 9	12 11	14 14	17 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
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·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

SQUARE ROOTS. FROM 1 TO 10.

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
1·0	1·000	1·005	1·010	1·015	1·020	1·025	1·030	1·034	1·039	1·044	0	1	1	2	2	3	3	4	4
1·1	1·049	1·054	1·058	1·063	1·068	1·072	1·077	1·082	1·086	1·091	0	1	1	2	2	3	3	4	4
1·2	1·095	1·100	1·105	1·109	1·114	1·118	1·122	1·127	1·131	1·136	0	1	1	2	2	3	3	4	4
1·3	1·140	1·145	1·149	1·153	1·158	1·162	1·166	1·170	1·175	1·179	0	1	1	2	2	3	3	3	4
1·4	1·183	1·187	1·192	1·196	1·200	1·204	1·208	1·212	1·217	1·221	0	1	1	2	2	2	3	3	4
1·5	1·225	1·229	1·233	1·237	1·241	1·245	1·249	1·253	1·257	1·261	0	1	1	2	2	2	3	3	4
1·6	1·265	1·269	1·273	1·277	1·281	1·285	1·288	1·292	1·296	1·300	0	1	1	2	2	2	3	3	3
1·7	1·304	1·308	1·311	1·315	1·319	1·323	1·327	1·330	1·334	1·338	0	1	1	2	2	2	3	3	3
1·8	1·342	1·345	1·349	1·353	1·356	1·360	1·364	1·367	1·371	1·375	0	1	1	1	2	2	3	3	3
1·9	1·378	1·382	1·386	1·389	1·393	1·396	1·400	1·404	1·407	1·411	0	1	1	1	2	2	3	3	3
2·0	1·414	1·418	1·421	1·425	1·428	1·432	1·435	1·439	1·442	1·446	0	1	1	1	2	2	2	3	3
2·1	1·449	1·453	1·456	1·459	1·463	1·466	1·470	1·473	1·476	1·480	0	1	1	1	2	2	2	3	3
2·2	1·483	1·487	1·490	1·493	1·497	1·500	1·503	1·507	1·510	1·513	0	1	1	1	2	2	2	3	3
2·3	1·517	1·520	1·523	1·526	1·530	1·533	1·536	1·539	1·543	1·546	0	1	1	1	2	2	2	3	3
2·4	1·549	1·552	1·556	1·559	1·562	1·565	1·568	1·572	1·575	1·578	0	1	1	1	2	2	2	3	3
2·5	1·581	1·584	1·587	1·591	1·594	1·597	1·600	1·603	1·606	1·609	0	1	1	1	2	2	2	3	3
2·6	1·612	1·616	1·619	1·622	1·625	1·628	1·631	1·634	1·637	1·640	0	1	1	1	2	2	2	2	3
2·7	1·643	1·646	1·649	1·652	1·655	1·658	1·661	1·664	1·667	1·670	0	1	1	1	2	2	2	2	3
2·8	1·673	1·676	1·679	1·682	1·685	1·688	1·691	1·694	1·697	1·700	0	1	1	1	1	2	2	2	3
2·9	1·703	1·706	1·709	1·712	1·715	1·718	1·720	1·723	1·726	1·729	0	1	1	1	1	2	2	2	3
3·0	1·732	1·735	1·738	1·741	1·744	1·746	1·749	1·752	1·755	1·758	0	1	1	1	1	2	2	2	3
3·1	1·761	1·764	1·766	1·769	1·772	1·775	1·778	1·780	1·783	1·786	0	1	1	1	1	2	2	2	3
3·2	1·789	1·792	1·794	1·797	1·800	1·803	1·806	1·808	1·811	1·814	0	1	1	1	1	2	2	2	2
3·3	1·817	1·819	1·822	1·825	1·828	1·830	1·833	1·836	1·838	1·841	0	1	1	1	1	2	2	2	2
3·4	1·844	1·847	1·849	1·852	1·855	1·857	1·860	1·863	1·865	1·868	0	1	1	1	1	2	2	2	2
3·5	1·871	1·873	1·876	1·879	1·881	1·884	1·887	1·889	1·892	1·895	0	1	1	1	1	2	2	2	2
3·6	1·897	1·900	1·903	1·905	1·908	1·910	1·913	1·916	1·918	1·921	0	1	1	1	1	2	2	2	2
3·7	1·924	1·926	1·929	1·931	1·934	1·936	1·939	1·942	1·944	1·947	0	1	1	1	1	2	2	2	2
3·8	1·949	1·952	1·954	1·957	1·960	1·962	1·965	1·967	1·970	1·972	0	1	1	1	1	2	2	2	2
3·9	1·975	1·977	1·980	1·982	1·985	1·987	1·990	1·992	1·995	1·997	0	1	1	1	1	2	2	2	2
4·0	2·000	2·002	2·005	2·007	2·010	2·012	2·015	2·017	2·020	2·022	0	0	1	1	1	1	2	2	2
4·1	2·025	2·027	2·030	2·032	2·035	2·037	2·040	2·042	2·045	2·047	0	0	1	1	1	1	2	2	2
4·2	2·049	2·052	2·054	2·057	2·059	2·062	2·064	2·066	2·069	2·071	0	0	1	1	1	1	2	2	2
4·3	2·074	2·076	2·078	2·081	2·083	2·086	2·088	2·090	2·093	2·095	0	0	1	1	1	1	2	2	2
4·4	2·098	2·100	2·102	2·105	2·107	2·110	2·112	2·114	2·117	2·119	0	0	1	1	1	1	2	2	2
4·5	2·121	2·124	2·126	2·128	2·131	2·133	2·135	2·138	2·140	2·142	0	0	1	1	1	1	2	2	2
4·6	2·145	2·147	2·149	2·152	2·154	2·156	2·159	2·161	2·163	2·166	0	0	1	1	1	1	2	2	2
4·7	2·168	2·170	2·173	2·175	2·177	2·179	2·182	2·184	2·186	2·189	0	0	1	1	1	1	2	2	2
4·8	2·191	2·193	2·195	2·198	2·200	2·202	2·205	2·207	2·209	2·211	0	0	1	1	1	1	2	2	2
4·9	2·214	2·216	2·218	2·220	2·223	2·225	2·227	2·229	2·232	2·234	0	0	1	1	1	1	2	2	2
5·0	2·236	2·238	2·241	2·243	2·245	2·247	2·249	2·252	2·254	2·256	0	0	1	1	1	1	2	2	2
5·1	2·258	2·261	2·263	2·265	2·267	2·269	2·272	2·274	2·276	2·278	0	0	1	1	1	1	2	2	2
5·2	2·280	2·283	2·285	2·287	2·289	2·291	2·293	2·296	2·298	2·300	0	0	1	1	1	1	2	2	2
5·3	2·302	2·304	2·307	2·309	2·311	2·313	2·315	2·317	2·319	2·322	0	0	1	1	1	1	2	2	2
5·4	2·324	2·326	2·328	2·330	2·332	2·335	2·337	2·339	2·341	2·343	0	0	1	1	1	1	1	2	2

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
5·5	2·345	2·347	2·349	2·352	2·354	2·356	2·358	2·360	2·362	2·364	0	0	1	1	1	1	1	2	2
5·6	2·366	2·369	2·371	2·373	2·375	2·377	2·379	2·381	2·383	2·385	0	0	1	1	1	1	1	2	2
5·7	2·387	2·390	2·392	2·394	2·396	2·398	2·400	2·402	2·404	2·406	0	0	1	1	1	1	1	2	2
5·8	2·408	2·410	2·412	2·415	2·417	2·419	2·421	2·423	2·425	2·427	0	0	1	1	1	1	1	2	2
5·9	2·429	2·431	2·433	2·435	2·437	2·439	2·441	2·443	2·445	2·447	0	0	1	1	1	1	1	2	2
6·0	2·449	2·452	2·454	2·456	2·458	2·460	2·462	2·464	2·466	2·468	0	0	1	1	1	1	1	2	2
6·1	2·470	2·472	2·474	2·476	2·478	2·480	2·482	2·484	2·486	2·488	0	0	1	1	1	1	1	2	2
6·2	2·490	2·492	2·494	2·496	2·498	2·500	2·502	2·504	2·506	2·508	0	0	1	1	1	1	1	2	2
6·3	2·510	2·512	2·514	2·516	2·518	2·520	2·522	2·524	2·526	2·528	0	0	1	1	1	1	1	2	2
6·4	2·530	2·532	2·534	2·536	2·538	2·540	2·542	2·544	2·546	2·548	0	0	1	1	1	1	1	2	2
6·5	2·550	2·551	2·553	2·555	2·557	2·559	2·561	2·563	2·565	2·567	0	0	1	1	1	1	1	2	2
6·6	2·569	2·571	2·573	2·575	2·577	2·579	2·581	2·583	2·585	2·587	0	0	1	1	1	1	1	2	2
6·7	2·588	2·590	2·592	2·594	2·596	2·598	2·600	2·602	2·604	2·606	0	0	1	1	1	1	1	2	2
6·8	2·608	2·610	2·612	2·613	2·615	2·617	2·619	2·621	2·623	2·625	0	0	1	1	1	1	1	2	2
6·9	2·627	2·629	2·631	2·632	2·634	2·636	2·638	2·640	2·642	2·644	0	0	1	1	1	1	1	2	2
7·0	2·646	2·648	2·650	2·651	2·653	2·655	2·657	2·659	2·661	2·663	0	0	1	1	1	1	1	2	2
7·1	2·665	2·666	2·668	2·670	2·672	2·674	2·676	2·678	2·680	2·681	0	0	1	1	1	1	1	1	2
7·2	2·683	2·685	2·687	2·689	2·691	2·693	2·694	2·696	2·698	2·700	0	0	1	1	1	1	1	1	2
7·3	2·702	2·704	2·706	2·707	2·709	2·711	2·713	2·715	2·717	2·718	0	0	1	1	1	1	1	1	2
7·4	2·720	2·722	2·724	2·726	2·728	2·729	2·731	2·733	2·735	2·737	0	0	1	1	1	1	1	1	2
7·5	2·739	2·740	2·742	2·744	2·746	2·748	2·750	2·751	2·753	2·755	0	0	1	1	1	1	1	1	2
7·6	2·757	2·759	2·760	2·762	2·764	2·766	2·768	2·769	2·771	2·773	0	0	1	1	1	1	1	1	2
7·7	2·775	2·777	2·778	2·780	2·782	2·784	2·786	2·787	2·789	2·791	0	0	1	1	1	1	1	1	2
7·8	2·793	2·795	2·796	2·798	2·800	2·802	2·804	2·805	2·807	2·809	0	0	1	1	1	1	1	1	2
7·9	2·811	2·812	2·814	2·816	2·818	2·820	2·821	2·823	2·825	2·827	0	0	1	1	1	1	1	1	2
8·0	2·828	2·830	2·832	2·834	2·835	2·837	2·839	2·841	2·843	2·844	0	0	1	1	1	1	1	1	2
8·1	2·846	2·848	2·850	2·851	2·853	2·855	2·857	2·858	2·860	2·862	0	0	1	1	1	1	1	1	2
8·2	2·864	2·865	2·867	2·869	2·871	2·872	2·874	2·876	2·877	2·879	0	0	1	1	1	1	1	1	2
8·3	2·881	2·883	2·884	2·886	2·888	2·890	2·891	2·893	2·895	2·897	0	0	1	1	1	1	1	1	2
8·4	2·898	2·900	2·902	2·903	2·905	2·907	2·909	2·910	2·912	2·914	0	0	1	1	1	1	1	1	2
8·5	2·915	2·917	2·919	2·921	2·922	2·924	2·926	2·927	2·929	2·931	0	0	1	1	1	1	1	1	2
8·6	2·933	2·934	2·936	2·938	2·939	2·941	2·943	2·944	2·946	2·948	0	0	1	1	1	1	1	1	2
8·7	2·950	2·951	2·953	2·955	2·956	2·958	2·960	2·961	2·963	2·965	0	0	1	1	1	1	1	1	2
8·8	2·966	2·968	2·970	2·972	2·973	2·975	2·977	2·978	2·980	2·982	0	0	1	1	1	1	1	1	2
8·9	2·983	2·985	2·987	2·988	2·990	2·992	2·993	2·995	2·997	2·998	0	0	1	1	1	1	1	1	2
9·0	3·000	3·002	3·003	3·005	3·007	3·008	3·010	3·012	3·013	3·015	0	0	0	1	1	1	1	1	1
9·1	3·017	3·018	3·020	3·022	3·023	3·025	3·027	3·028	3·030	3·032	0	0	0	1	1	1	1	1	1
9·2	3·033	3·035	3·036	3·038	3·040	3·041	3·043	3·045	3·046	3·048	0	0	0	1	1	1	1	1	1
9·3	3·050	3·051	3·053	3·055	3·056	3·058	3·059	3·061	3·063	3·064	0	0	0	1	1	1	1	1	1
9·4	3·066	3·068	3·069	3·071	3·072	3·074	3·076	3·077	3·079	3·081	0	0	0	1	1	1	1	1	1
9·5	3·082	3·084	3·085	3·087	3·089	3·090	3·092	3·094	3·095	3·097	0	0	0	1	1	1	1	1	1
9·6	3·098	3·100	3·102	3·103	3·105	3·106	3·108	3·110	3·111	3·113	0	0	0	1	1	1	1	1	1
9·7	3·114	3·116	3·118	3·119	3·121	3·122	3·124	3·126	3·127	3·129	0	0	0	1	1	1	1	1	1
9·8	3·130	3·132	3·134	3·135	3·137	3·138	3·140	3·142	3·143	3·145	0	0	0	1	1	1	1	1	1
9·9	3·146	3·148	3·150	3·151	3·153	3·154	3·156	3·158	3·159	3·161	0	0	0	1	1	1	1	1	1

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
1·0	1·000	1·005	1·010	1·015	1·020	1·025	1·030	1·034	1·039	1·044	0	1	1	2	2	3	3	4	4
1·1	1·049	1·054	1·058	1·063	1·068	1·072	1·077	1·082	1·086	1·091	0	1	1	2	2	3	3	4	4
1·2	1·095	1·100	1·105	1·109	1·114	1·118	1·122	1·127	1·131	1·136	0	1	1	2	2	3	3	4	4
1·3	1·140	1·145	1·149	1·153	1·158	1·162	1·166	1·170	1·175	1·179	0	1	1	2	2	3	3	3	4
1·4	1·183	1·187	1·192	1·196	1·200	1·204	1·208	1·212	1·217	1·221	0	1	1	2	2	2	3	3	4
1·5	1·225	1·229	1·233	1·237	1·241	1·245	1·249	1·253	1·257	1·261	0	1	1	2	2	2	3	3	4
1·6	1·265	1·269	1·273	1·277	1·281	1·285	1·288	1·292	1·296	1·300	0	1	1	2	2	2	3	3	3
1·7	1·304	1·308	1·311	1·315	1·319	1·323	1·327	1·330	1·334	1·338	0	1	1	2	2	2	3	3	3
1·8	1·342	1·345	1·349	1·353	1·356	1·360	1·364	1·367	1·371	1·375	0	1	1	1	2	2	3	3	3
1·9	1·378	1·382	1·386	1·389	1·393	1·396	1·400	1·404	1·407	1·411	0	1	1	1	2	2	3	3	3
2·0	1·414	1·418	1·421	1·425	1·428	1·432	1·435	1·439	1·442	1·446	0	1	1	1	2	2	2	3	3
2·1	1·449	1·453	1·456	1·459	1·463	1·466	1·470	1·473	1·476	1·480	0	1	1	1	2	2	2	3	3
2·2	1·483	1·487	1·490	1·493	1·497	1·500	1·503	1·507	1·510	1·513	0	1	1	1	2	2	2	3	3
2·3	1·517	1·520	1·523	1·526	1·530	1·533	1·536	1·539	1·543	1·546	0	1	1	1	2	2	2	3	3
2·4	1·549	1·552	1·556	1·559	1·562	1·565	1·568	1·572	1·575	1·578	0	1	1	1	2	2	2	3	3
2·5	1·581	1·584	1·587	1·591	1·594	1·597	1·600	1·603	1·606	1·609	0	1	1	1	2	2	2	3	3
2·6	1·612	1·616	1·619	1·622	1·625	1·628	1·631	1·634	1·637	1·640	0	1	1	1	2	2	2	2	3
2·7	1·643	1·646	1·649	1·652	1·655	1·658	1·661	1·664	1·667	1·670	0	1	1	1	2	2	2	2	3
2·8	1·673	1·676	1·679	1·682	1·685	1·688	1·691	1·694	1·697	1·700	0	1	1	1	1	2	2	2	3
2·9	1·703	1·706	1·709	1·712	1·715	1·718	1·720	1·723	1·726	1·729	0	1	1	1	1	2	2	2	3
3·0	1·732	1·735	1·738	1·741	1·744	1·746	1·749	1·752	1·755	1·758	0	1	1	1	1	2	2	2	3
3·1	1·761	1·764	1·766	1·769	1·772	1·775	1·778	1·780	1·783	1·786	0	1	1	1	1	2	2	2	3
3·2	1·789	1·792	1·794	1·797	1·800	1·803	1·806	1·808	1·811	1·814	0	1	1	1	1	2	2	2	2
3·3	1·817	1·819	1·822	1·825	1·828	1·830	1·833	1·836	1·838	1·841	0	1	1	1	1	2	2	2	2
3·4	1·844	1·847	1·849	1·852	1·855	1·857	1·860	1·863	1·865	1·868	0	1	1	1	1	2	2	2	2
3·5	1·871	1·873	1·876	1·879	1·881	1·884	1·887	1·889	1·892	1·895	0	1	1	1	1	2	2	2	2
3·6	1·897	1·900	1·903	1·905	1·908	1·910	1·913	1·916	1·918	1·921	0	1	1	1	1	2	2	2	2
3·7	1·924	1·926	1·929	1·931	1·934	1·936	1·939	1·942	1·944	1·947	0	1	1	1	1	2	2	2	2
3·8	1·949	1·952	1·954	1·957	1·960	1·962	1·965	1·967	1·970	1·972	0	1	1	1	1	2	2	2	2
3·9	1·975	1·977	1·980	1·982	1·985	1·987	1·990	1·992	1·995	1·997	0	1	1	1	1	2	2	2	2
4·0	2·000	2·002	2·005	2·007	2·010	2·012	2·015	2·017	2·020	2·022	0	0	1	1	1	1	2	2	2
4·1	2·025	2·027	2·030	2·032	2·035	2·037	2·040	2·042	2·045	2·047	0	0	1	1	1	1	2	2	2
4·2	2·049	2·052	2·054	2·057	2·059	2·062	2·064	2·066	2·069	2·071	0	0	1	1	1	1	2	2	2
4·3	2·074	2·076	2·078	2·081	2·083	2·086	2·088	2·090	2·093	2·095	0	0	1	1	1	1	2	2	2
4·4	2·098	2·100	2·102	2·105	2·107	2·110	2·112	2·114	2·117	2·119	0	0	1	1	1	1	2	2	2
4·5	2·121	2·124	2·126	2·128	2·131	2·133	2·135	2·138	2·140	2·142	0	0	1	1	1	1	2	2	2
4·6	2·145	2·147	2·149	2·152	2·154	2·156	2·159	2·161	2·163	2·166	0	0	1	1	1	1	2	2	2
4·7	2·168	2·170	2·173	2·175	2·177	2·179	2·182	2·184	2·186	2·189	0	0	1	1	1	1	2	2	2
4·8	2·191	2·193	2·195	2·198	2·200	2·202	2·205	2·207	2·209	2·211	0	0	1	1	1	1	2	2	2
4·9	2·214	2·216	2·218	2·220	2·223	2·225	2·227	2·229	2·232	2·234	0	0	1	1	1	1	2	2	2
5·0	2·236	2·238	2·241	2·243	2·245	2·247	2·249	2·252	2·254	2·256	0	0	1	1	1	1	2	2	2
5·1	2·258	2·261	2·263	2·265	2·267	2·269	2·272	2·274	2·276	2·278	0	0	1	1	1	1	2	2	2
5·2	2·280	2·283	2·285	2·287	2·289	2·291	2·293	2·296	2·298	2·300	0	0	1	1	1	1	2	2	2
5·3	2·302	2·304	2·307	2·309	2·311	2·313	2·315	2·317	2·319	2·322	0	0	1	1	1	1	2	2	2
5·4	2·324	2·326	2·328	2·330	2·332	2·335	2·337	2·339	2·341	2·343	0	0	1	1	1	1	1	2	2

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
5·5	2·345	2·347	2·349	2·352	2·354	2·356	2·358	2·360	2·362	2·364	0	0	1	1	1	1	1	2	2
5·6	2·366	2·369	2·371	2·373	2·375	2·377	2·379	2·381	2·383	2·385	0	0	1	1	1	1	1	2	2
5·7	2·387	2·390	2·392	2·394	2·396	2·398	2·400	2·402	2·404	2·406	0	0	1	1	1	1	1	2	2
5·8	2·408	2·410	2·412	2·415	2·417	2·419	2·421	2·423	2·425	2·427	0	0	1	1	1	1	1	2	2
5·9	2·429	2·431	2·433	2·435	2·437	2·439	2·441	2·443	2·445	2·447	0	0	1	1	1	1	1	2	2
6·0	2·449	2·452	2·454	2·456	2·458	2·460	2·462	2·464	2·466	2·468	0	0	1	1	1	1	1	2	2
6·1	2·470	2·472	2·474	2·476	2·478	2·480	2·482	2·484	2·486	2·488	0	0	1	1	1	1	1	2	2
6·2	2·490	2·492	2·494	2·496	2·498	2·500	2·502	2·504	2·506	2·508	0	0	1	1	1	1	1	2	2
6·3	2·510	2·512	2·514	2·516	2·518	2·520	2·522	2·524	2·526	2·528	0	0	1	1	1	1	1	2	2
6·4	2·530	2·532	2·534	2·536	2·538	2·540	2·542	2·544	2·546	2·548	0	0	1	1	1	1	1	2	2
6·5	2·550	2·551	2·553	2·555	2·557	2·559	2·561	2·563	2·565	2·567	0	0	1	1	1	1	1	2	2
6·6	2·569	2·571	2·573	2·575	2·577	2·579	2·581	2·583	2·585	2·587	0	0	1	1	1	1	1	2	2
6·7	2·588	2·590	2·592	2·594	2·596	2·598	2·600	2·602	2·604	2·606	0	0	1	1	1	1	1	2	2
6·8	2·608	2·610	2·612	2·613	2·615	2·617	2·619	2·621	2·623	2·625	0	0	1	1	1	1	1	2	2
6·9	2·627	2·629	2·631	2·632	2·634	2·636	2·638	2·640	2·642	2·644	0	0	1	1	1	1	1	2	2
7·0	2·646	2·648	2·650	2·651	2·653	2·655	2·657	2·659	2·661	2·663	0	0	1	1	1	1	1	2	2
7·1	2·665	2·666	2·668	2·670	2·672	2·674	2·676	2·678	2·680	2·681	0	0	1	1	1	1	1	1	2
7·2	2·683	2·685	2·687	2·689	2·691	2·693	2·694	2·696	2·698	2·700	0	0	1	1	1	1	1	1	2
7·3	2·702	2·704	2·706	2·707	2·709	2·711	2·713	2·715	2·717	2·718	0	0	1	1	1	1	1	1	2
7·4	2·720	2·722	2·724	2·726	2·728	2·729	2·731	2·733	2·735	2·737	0	0	1	1	1	1	1	1	2
7·5	2·739	2·740	2·742	2·744	2·746	2·748	2·750	2·751	2·753	2·755	0	0	1	1	1	1	1	1	2
7·6	2·757	2·759	2·760	2·762	2·764	2·766	2·768	2·769	2·771	2·773	0	0	1	1	1	1	1	1	2
7·7	2·775	2·777	2·778	2·780	2·782	2·784	2·786	2·787	2·789	2·791	0	0	1	1	1	1	1	1	2
7·8	2·793	2·795	2·796	2·798	2·800	2·802	2·804	2·805	2·807	2·809	0	0	1	1	1	1	1	1	2
7·9	2·811	2·812	2·814	2·816	2·818	2·820	2·821	2·823	2·825	2·827	0	0	1	1	1	1	1	1	2
8·0	2·828	2·830	2·832	2·834	2·835	2·837	2·839	2·841	2·843	2·844	0	0	1	1	1	1	1	1	2
8·1	2·846	2·848	2·850	2·851	2·853	2·855	2·857	2·858	2·860	2·862	0	0	1	1	1	1	1	1	2
8·2	2·864	2·865	2·867	2·869	2·871	2·872	2·874	2·876	2·877	2·879	0	0	1	1	1	1	1	1	2
8·3	2·881	2·883	2·884	2·886	2·888	2·890	2·891	2·893	2·895	2·897	0	0	1	1	1	1	1	1	2
8·4	2·898	2·900	2·902	2·903	2·905	2·907	2·909	2·910	2·912	2·914	0	0	1	1	1	1	1	1	2
8·5	2·915	2·917	2·919	2·921	2·922	2·924	2·926	2·927	2·929	2·931	0	0	1	1	1	1	1	1	2
8·6	2·933	2·934	2·936	2·938	2·939	2·941	2·943	2·944	2·946	2·948	0	0	1	1	1	1	1	1	2
8·7	2·950	2·951	2·953	2·955	2·956	2·958	2·960	2·961	2·963	2·965	0	0	1	1	1	1	1	1	2
8·8	2·966	2·968	2·970	2·972	2·973	2·975	2·977	2·978	2·980	2·982	0	0	1	1	1	1	1	1	2
8·9	2·983	2·985	2·987	2·988	2·990	2·992	2·993	2·995	2·997	2·998	0	0	1	1	1	1	1	1	2
9·0	3·000	3·002	3·003	3·005	3·007	3·008	3·010	3·012	3·013	3·015	0	0	0	1	1	1	1	1	1
9·1	3·017	3·018	3·020	3·022	3·023	3·025	3·027	3·028	3·030	3·032	0	0	0	1	1	1	1	1	1
9·2	3·033	3·035	3·036	3·038	3·040	3·041	3·043	3·045	3·046	3·048	0	0	0	1	1	1	1	1	1
9·3	3·050	3·051	3·053	3·055	3·056	3·058	3·059	3·061	3·063	3·064	0	0	0	1	1	1	1	1	1
9·4	3·066	3·068	3·069	3·071	3·072	3·074	3·076	3·077	3·079	3·081	0	0	0	1	1	1	1	1	1
9·5	3·082	3·084	3·085	3·087	3·089	3·090	3·092	3·094	3·095	3·097	0	0	0	1	1	1	1	1	1
9·6	3·098	3·100	3·102	3·103	3·105	3·106	3·108	3·110	3·111	3·113	0	0	0	1	1	1	1	1	1
9·7	3·114	3·116	3·118	3·119	3·121	3·122	3·124	3·126	3·127	3·129	0	0	0	1	1	1	1	1	1
9·8	3·130	3·132	3·134	3·135	3·137	3·138	3·140	3·142	3·143	3·145	0	0	0	1	1	1	1	1	1
9·9	3·146	3·148	3·150	3·151	3·153	3·154	3·156	3·158	3·159	3·161	0	0	0	1	1	1	1	1	1

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
10	3·162	3·178	3·194	3·209	3·225	3·240	3·256	3·271	3·286	3·302	2	3	5	6	8	9	11	12	14
11	3·317	3·332	3·347	3·362	3·376	3·391	3·406	3·421	3·435	3·450	1	3	4	6	7	9	10	12	13
12	3·464	3·479	3·493	3·507	3·521	3·536	3·550	3·564	3·578	3·592	1	3	4	6	7	8	10	11	13
13	3·606	3·619	3·633	3·647	3·661	3·674	3·688	3·701	3·715	3·728	1	3	4	5	7	8	10	11	12
14	3·742	3·755	3·768	3·782	3·795	3·808	3·821	3·834	3·847	3·860	1	3	4	5	7	8	9	11	12
15	3·873	3·886	3·899	3·912	3·924	3·937	3·950	3·962	3·975	3·987	1	3	4	5	6	8	9	10	11
16	4·000	4·012	4·025	4·037	4·050	4·062	4·074	4·087	4·099	4·111	1	2	4	5	6	7	9	10	11
17	4·123	4·135	4·147	4·159	4·171	4·183	4·195	4·207	4·219	4·231	1	2	4	5	6	7	8	10	11
18	4·243	4·254	4·266	4·278	4·290	4·301	4·313	4·324	4·336	4·347	1	2	3	5	6	7	8	9	10
19	4·359	4·370	4·382	4·393	4·405	4·416	4·427	4·438	4·450	4·461	1	2	3	5	6	7	8	9	10
20	4·472	4·483	4·494	4·506	4·517	4·528	4·539	4·550	4·561	4·572	1	2	3	4	6	7	8	9	10
21	4·583	4·593	4·604	4·615	4·626	4·637	4·648	4·658	4·669	4·680	1	2	3	4	5	6	8	9	10
22	4·690	4·701	4·712	4·722	4·733	4·743	4·754	4·764	4·775	4·785	1	2	3	4	5	6	7	8	9
23	4·796	4·806	4·817	4·827	4·837	4·848	4·858	4·868	4·879	4·889	1	2	3	4	5	6	7	8	9
24	4·899	4·909	4·919	4·930	4·940	4·950	4·960	4·970	4·980	4·990	1	2	3	4	5	6	7	8	9
25	5·000	5·010	5·020	5·030	5·040	5·050	5·060	5·070	5·079	5·089	1	2	3	4	5	6	7	8	9
26	5·099	5·109	5·119	5·128	5·138	5·148	5·158	5·167	5·177	5·187	1	2	3	4	5	6	7	8	9
27	5·196	5·206	5·215	5·225	5·235	5·244	5·254	5·263	5·273	5·282	1	2	3	4	5	6	7	8	9
28	5·292	5·301	5·310	5·320	5·329	5·339	5·348	5·357	5·367	5·376	1	2	3	4	5	6	7	7	8
29	5·385	5·394	5·404	5·413	5·422	5·431	5·441	5·450	5·459	5·468	1	2	3	4	5	5	6	7	8
30	5·477	5·486	5·495	5·505	5·514	5·523	5·532	5·541	5·550	5·559	1	2	3	4	4	5	6	7	8
31	5·568	5·577	5·586	5·595	5·604	5·612	5·621	5·630	5·639	5·648	1	2	3	3	4	5	6	7	8
32	5·657	5·666	5·675	5·683	5·692	5·701	5·710	5·718	5·727	5·736	1	2	3	3	4	5	6	7	8
33	5·745	5·753	5·762	5·771	5·779	5·788	5·797	5·805	5·814	5·822	1	2	3	3	4	5	6	7	8
34	5·831	5·840	5·848	5·857	5·865	5·874	5·882	5·891	5·899	5·908	1	2	3	3	4	5	6	7	8
35	5·916	5·925	5·933	5·941	5·950	5·958	5·967	5·975	5·983	5·992	1	2	2	3	4	5	6	7	8
36	6·000	6·008	6·017	6·025	6·033	6·042	6·050	6·058	6·066	6·075	1	2	2	3	4	5	6	7	7
37	6·083	6·091	6·099	6·107	6·116	6·124	6·132	6·140	6·148	6·156	1	2	2	3	4	5	6	7	7
38	6·164	6·173	6·181	6·189	6·197	6·205	6·213	6·221	6·229	6·237	1	2	2	3	4	5	6	6	7
39	6·245	6·253	6·261	6·269	6·277	6·285	6·293	6·301	6·309	6·317	1	2	2	3	4	5	6	6	7
40	6·325	6·332	6·340	6·348	6·356	6·364	6·372	6·380	6·387	6·395	1	2	2	3	4	5	6	6	7
41	6·403	6·411	6·419	6·427	6·434	6·442	6·450	6·458	6·465	6·473	1	2	2	3	4	5	5	6	7
42	6·481	6·488	6·496	6·504	6·512	6·519	6·527	6·535	6·542	6·550	1	2	2	3	4	5	5	6	7
43	6·557	6·565	6·573	6·580	6·588	6·595	6·603	6·611	6·618	6·626	1	2	2	3	4	5	5	6	7
44	6·633	6·641	6·648	6·656	6·663	6·671	6·678	6·686	6·693	6·701	1	2	2	3	4	5	5	6	7
45	6·708	6·716	6·723	6·731	6·738	6·745	6·753	6·760	6·768	6·775	1	1	2	3	4	4	5	6	7
46	6·782	6·790	6·797	6·804	6·812	6·819	6·826	6·834	6·841	6·848	1	1	2	3	4	4	5	6	7
47	6·856	6·863	6·870	6·877	6·885	6·892	6·899	6·907	6·914	6·921	1	1	2	3	4	4	5	6	7
48	6·928	6·935	6·943	6·950	6·957	6·964	6·971	6·979	6·986	6·993	1	1	2	3	4	4	5	6	6
49	7·000	7·007	7·014	7·021	7·029	7·036	7·043	7·050	7·057	7·064	1	1	2	3	4	4	5	6	6
50	7·071	7·078	7·085	7·092	7·099	7·106	7·113	7·120	7·127	7·134	1	1	2	3	4	4	5	6	6
51	7·141	7·148	7·155	7·162	7·169	7·176	7·183	7·190	7·197	7·204	1	1	2	3	4	4	5	6	6
52	7·211	7·218	7·225	7·232	7·239	7·246	7·253	7·259	7·266	7·273	1	1	2	3	3	4	5	6	6
53	7·280	7·287	7·294	7·301	7·308	7·314	7·321	7·328	7·335	7·342	1	1	2	3	3	4	5	5	6
54	7·348	7·355	7·362	7·369	7·376	7·382	7·389	7·396	7·403	7·409	1	1	2	3	3	4	5	5	6

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
55	7·416	7·423	7·430	7·436	7·443	7·450	7·457	7·463	7·470	7·477	1	1	2	3	3	4	5	5	6
56	7·483	7·490	7·497	7·503	7·510	7·517	7·523	7·530	7·537	7·543	1	1	2	3	3	4	5	5	6
57	7·550	7·556	7·563	7·570	7·576	7·583	7·589	7·596	7·603	7·609	1	1	2	3	3	4	5	5	6
58	7·616	7·622	7·629	7·635	7·642	7·649	7·655	7·662	7·668	7·675	1	1	2	3	3	4	5	5	6
59	7·681	7·688	7·694	7·701	7·707	7·714	7·720	7·727	7·733	7·740	1	1	2	3	3	4	4	5	6
60	7·746	7·752	7·759	7·765	7·772	7·778	7·785	7·791	7·797	7·804	1	1	2	3	3	4	4	5	6
61	7·810	7·817	7·823	7·829	7·836	7·842	7·849	7·855	7·861	7·868	1	1	2	3	3	4	4	5	6
62	7·874	7·880	7·887	7·893	7·899	7·906	7·912	7·918	7·925	7·931	1	1	2	3	3	4	4	5	6
63	7·937	7·944	7·950	7·956	7·962	7·969	7·975	7·981	7·987	7·994	1	1	2	3	3	4	4	5	6
64	8·000	8·006	8·012	8·019	8·025	8·031	8·037	8·044	8·050	8·056	1	1	2	2	3	4	4	5	6
65	8·062	8·068	8·075	8·081	8·087	8·093	8·099	8·106	8·112	8·118	1	1	2	2	3	4	4	5	6
66	8·124	8·130	8·136	8·142	8·149	8·155	8·161	8·167	8·173	8·179	1	1	2	2	3	4	4	5	5
67	8·185	8·191	8·198	8·204	8·210	8·216	8·222	8·228	8·234	8·240	1	1	2	2	3	4	4	5	5
68	8·246	8·252	8·258	8·264	8·270	8·276	8·283	8·289	8·295	8·301	1	1	2	2	3	4	4	5	5
69	8·307	8·313	8·319	8·325	8·331	8·337	8·343	8·349	8·355	8·361	1	1	2	2	3	4	4	5	5
70	8·367	8·373	8·379	8·385	8·390	8·396	8·402	8·408	8·414	8·420	1	1	2	2	3	4	4	5	5
71	8·426	8·432	8·438	8·444	8·450	8·456	8·462	8·468	8·473	8·479	1	1	2	2	3	4	4	5	5
72	8·485	8·491	8·497	8·503	8·509	8·515	8·521	8·526	8·532	8·538	1	1	2	2	3	3	4	5	5
73	8·544	8·550	8·556	8·562	8·567	8·573	8·579	8·585	8·591	8·597	1	1	2	2	3	3	4	5	5
74	8·602	8·608	8·614	8·620	8·626	8·631	8·637	8·643	8·649	8·654	1	1	2	2	3	3	4	5	5
75	8·660	8·666	8·672	8·678	8·683	8·689	8·695	8·701	8·706	8·712	1	1	2	2	3	3	4	5	5
76	8·718	8·724	8·729	8·735	8·742	8·746	8·752	8·758	8·764	8·769	1	1	2	2	3	3	4	5	5
77	8·775	8·781	8·786	8·792	8·798	8·803	8·809	8·815	8·820	8·826	1	1	2	2	3	3	4	4	5
78	8·832	8·837	8·843	8·849	8·854	8·860	8·866	8·871	8·877	8·883	1	1	2	2	3	3	4	4	5
79	8·888	8·894	8·899	8·905	8·911	8·916	8·922	8·927	8·933	8·939	1	1	2	2	3	3	4	4	5
80	8·944	8·950	8·955	8·961	8·967	8·972	8·978	8·983	8·989	8·994	1	1	2	2	3	3	4	4	5
81	9·000	9·006	9·011	9·017	9·022	9·028	9·033	9·039	9·044	9·050	1	1	2	2	3	3	4	4	5
82	9·055	9·061	9·066	9·072	9·077	9·083	9·088	9·094	9·099	9·105	1	1	2	2	3	3	4	4	5
83	9·110	9·116	9·121	9·127	9·132	9·138	9·143	9·149	9·154	9·160	1	1	2	2	3	3	4	4	5
84	9·165	9·171	9·176	9·182	9·187	9·192	9·198	9·203	9·209	9·214	1	1	2	2	3	3	4	4	5
85	9·220	9·225	9·230	9·236	9·241	9·247	9·252	9·257	9·263	9·268	1	1	2	2	3	3	4	4	5
86	9·274	9·279	9·284	9·290	9·295	9·301	9·306	9·311	9·317	9·322	1	1	2	2	3	3	4	4	5
87	9·327	9·333	9·338	9·343	9·349	9·354	9·359	9·365	9·370	9·375	1	1	2	2	3	3	4	4	5
88	9·381	9·386	9·391	9·397	9·402	9·407	9·413	9·418	9·423	9·429	1	1	2	2	3	3	4	4	5
89	9·434	9·439	9·445	9·450	9·455	9·460	9·466	9·471	9·476	9·482	1	1	2	2	3	3	4	4	5
90	9·487	9·492	9·497	9·503	9·508	9·513	9·518	9·524	9·529	9·534	1	1	2	2	3	3	4	4	5
91	9·539	9·545	9·550	9·555	9·560	9·566	9·571	9·576	9·581	9·586	1	1	2	2	3	3	4	4	5
92	9·592	9·597	9·602	9·607	9·612	9·618	9·623	9·628	9·633	9·638	1	1	2	2	3	3	4	4	5
93	9·644	9·649	9·654	9·659	9·664	9·670	9·675	9·680	9·685	9·690	1	1	2	2	3	3	4	4	5
94	9·695	9·701	9·706	9·711	9·716	9·721	9·726	9·731	9·737	9·742	1	1	2	2	3	3	4	4	5
95	9·747	9·752	9·757	9·762	9·767	9·772	9·778	9·783	9·788	9·793	1	1	2	2	3	3	4	4	5
96	9·798	9·803	9·808	9·813	9·818	9·823	9·829	9·834	9·839	9·844	1	1	2	2	3	3	4	4	5
97	9·849	9·854	9·859	9·864	9·869	9·874	9·879	9·884	9·889	9·894	1	1	1	2	3	3	4	4	5
98	9·899	9·905	9·910	9·915	9·920	9·925	9·930	9·935	9·940	9·945	0	1	1	2	2	3	3	4	4
99	9·950	9·955	9·960	9·965	9·970	9·975	9·980	9·985	9·990	9·995	0	1	1	2	2	3	3	4	4

INDEX

The numbers refer to the pages

(*See also Table of Contents*)

ANSWERS

PART I

EXERCISE I A (Pp. 4, 5)

1. 3 times 5 equals 15. **2.** 9 is greater than 7. **3.** $2\frac{1}{5}$ equals 2·2.
4. 1·3 is less than $1\frac{1}{3}$. **5.** x is greater than 5. **6.** y is less than 13.
7. s equals 7. **8.** z is not greater than 1·8.
9. 15·99 is approximately equal to 16.
10. x is identically equal to y. **11.** c is not equal to d.
12. The difference between 15 and 11 is 4.
13. l is not less than m. **14.** g is approximately equal to h.
15. b times a equals c.
16. r divided by s is identically equal to t. **17.** $9>5$.
18. $Q=8$. **19.** $Y \not< 7$. **20.** $r>3$. **21.** $t \neq 3$.
22. $p \equiv q$. **23.** $a \not> b$. **24.** $3y=12$, $\therefore\ y=4$.
25. $Z+4=30$, $\therefore\ Z=26$. **26.** $X-20=16$, $\therefore\ X=36$.
27. $3q$, $3q$, $25c$, $25c$, 125, xy, xy. **28.** $\frac{q}{3}, \frac{3}{q}, \frac{25}{c}, \frac{c}{25}, \frac{x}{y}, \frac{y}{x}$.
29. $30 \times 4 = 120$. **30.** Correct. **31.** Correct. **32.** $113 \times 7 = 791$.
33. x times 5, 20. **34.** x plus 7, 11. **35.** One half of x, 2.
36. x minus 1, 3. **37.** One half of x, 2. **38.** 8 divided by x, 2.
39. One quarter of three times x, 3. **40.** 10 divided by x, $2\frac{1}{2}$.
41. One third of twice x, $2\frac{2}{3}$. **42.** z times x times 2, 16.
43. Twice z plus x, 8. **44.** Twice x minus three times z, 2.
45. z times y times x, 24. **46.** z divided by y, $\frac{2}{3}$.
47. The difference between y and z, 1.
48. x plus the product of y and z, 10.

EXERCISE I B (Pp. 5, 6)

1. 11 is greater than 9. **2.** 5 is less than 6.
3. 4 times 6 equals 24. **4.** $3\frac{2}{5}$ equals 3·4. **5.** 2·6 is less than $2\frac{2}{3}$.
6. v is not less than 1·9. **7.** y is less than 11. **8.** s equals 14.
9. q is not equal to 5. **10.** 8 divided by 2 equals 4.

11. m is approximately equal to n. **12.** c is not less than d.
13. The difference between 8 and 5 is 3. **14.** e is not equal to f.
15. p is identically equal to q. **16.** d times c equals f.
17. $z \not< 4$. **18.** $X = 7$. **19.** $2 < 7$. **20.** $e \not> f$.
21. $q \doteqdot r$. **22.** $s \neq 0$. **23.** $c \not> d$. **24.** $\frac{Z}{3} = 4, \therefore Z = 12$.
25. $2X = 18, \therefore X = 9$. **26.** $Y - 12 = 8, \therefore Y = 20$.
27. $5r, 5r, \frac{23}{d}, \frac{d}{23}, 63, ps, ps$. **28.** $\frac{r}{5}, \frac{5}{r}, 23d, 23d, \frac{p}{s}, \frac{s}{p}$.
29. $112 \times 3 = 336$. **30.** Correct. **31.** Correct. **32.** $20 \times 3 = 60$.
33. p times 4, 24. **34.** p plus 5, 11.
35. One third of twice p, 4. **36.** One half of p, 3.
37. p minus 4, 2. **38.** 1 plus the product of p and 3, 19.
39. One-third of p, 2. **40.** One-fifth of three times p, 3·6.
41. Three times q plus twice r, 23. **42.** 15 divided by p, $2\frac{1}{2}$.
43. r times p, 24. **44.** r times q times 3, 60.
45. r times q times p times 2, 240. **46.** Three times p minus twice q, 8.
47. q plus the product of 2, p and r, 53.
48. Twice q minus r, all divided by p, 1.

EXERCISE 2 A (Pp. 6, 7, 8, 9)

1. (*a*) 36, (*b*) 84, (*c*) $12a$, (*d*) $12x$. **2.** (*a*) 36, (*b*) 84, (*c*) $12a$, (*d*) $12x$.
3. (*a*) 2, (*b*) 8, (*c*) $\frac{a}{12}$, (*d*) $\frac{h}{12}$. **4.** (*a*) 4, (*b*) 5, (*c*) $\frac{x}{16}$, (*d*) $\frac{f}{16}$.
5. (*a*) 240, (*b*) 420, (*c*) $60b$, (*d*) $420m$.
6. (*a*) 8, (*b*) 12, (*c*) $\frac{a}{60}$, (*d*) $\frac{n}{60}$. **7.** (*a*) 2, (*b*) 7, (*c*) $\frac{x}{8}$, (*d*) $\frac{5q}{8}$.
8. (*a*) 700, (*b*) 2300, (*c*) $100q$, (*d*) $100r$.
9. (*a*) 1 ft., (*b*) $2\frac{1}{4}$ ft., (*c*) $(3 - a)$ ft.
10. (*a*) $(x - 2)$ ft., (*b*) $(x - \frac{3}{4})$ ft., (*c*) $(x - a)$ ft.
11. (i) 4 lb., (ii) 4 lb., (iii) $(10 - x)$ lb., (iv) $(y - 4)$ lb., (v) $(a - b)$ lb., (vi) $(2P - 3Q)$ lb.
12. $(p + q)$ miles. **13.** (i) 13, (ii) $p + q$.
14. (i) 7, (ii) $s - r$. **15.** (i) 11, (ii) $x - y$.
16. (*a*) $3s$, (*b*) $s + 7$, (*c*) $s - 8$, (*d*) $s - 12$, (*e*) $\frac{s}{2}$, (*f*) $\frac{2s}{3}$.
17. (i) 6, (ii) $4C$, (iii) $\frac{60}{D}$, (iv) $\frac{12x}{z}$.
18. (i) 72 miles, (ii) $5k$ miles, (iii) $30N$ miles, (iv) ab miles.
19. (i) $(24 - y)$ years, (ii) 23 years, (iii) $(24 - x)$ years; $(z - y)$ years, $(z - 1)$ years, $(z - x)$ years.

20. $r-2s$. **21.** $x-1, x+1$. **22.** 5. **23.** 34.
24. 11, 8. **25.** 4, 11. **26.** 11. **27.** 7.
28. 16. **29.** 15. **30.** 6. **31.** 40.
32. 6. **33.** 1. **34.** 5. **35.** 9.
36. 7. **37.** 24. **38.** 40. **39.** 42.

40. (i) 240 shillings, (ii) 9000 shillings, (iii) $3p$ shillings, (iv) $9q$ shillings.

41. (i) $\frac{N+8}{2}$, (ii) $5N+3$, (iii) $12-\frac{N}{3}$, (iv) $\frac{N}{2}-6$.

42. (i) 12 pence, (ii) $2c$ pence. **43.** (i) 8 miles, (ii) $4x$ miles.

44. (i) $3k$ pence, (ii) $2z$ pence, (iii) $3c$ shillings, (iv) xw pence, (v) $12Wb$ pence.

45. (i) $115°$; $(180-x)°$; $79°$; $(180-y)°$.
(ii) $63°$; $(180-p)°$; $110°$, $110°$, $70°$; $(180-z)°$, $z°$, $(180-z)°$.
(iii) $55°$, $(110-b)°$, $(180-2x)°$, $(180-x-y)°$.

46. (i) $5x$, (ii) $35y$. $\frac{K}{5}$ days. **47.** $\frac{54}{b}$.

48. (i) 46, (ii) $10p+q$, (iii) 237, (iv) $100p+10q+r$.

Exercise 2 B (Pp. 9, 10, 11)

1. (*a*) 3, (*b*) 7, (*c*) $\frac{b}{12}$, (*d*) $\frac{k}{12}$. **2.** (*a*) 64, (*b*) 96, (*c*) $16h$, (*d*) $48j$.

3. (*a*) 5, (*b*) 8, (*c*) $\frac{m}{16}$, (*d*) $\frac{n}{16}$. **4.** (*a*) 3, (*b*) 8, (*c*) $\frac{e}{12}$, (*d*) $\frac{7k}{12}$.

5. (*a*) 6, (*b*) 9, (*c*) $\frac{g}{60}$, (*d*) $\frac{k}{60}$. **6.** (*a*) 32, (*b*) 56, (*c*) $8h$, (*d*) $24p$.

7. (*a*) 600, (*b*) 2100, (*c*) $100s$, (*d*) $100w$.

8. (*a*) 4, (*b*) 7·5, (*c*) $\frac{x}{100}$, (*d*) $\frac{7n}{100}$. **9.** $(s+t)$ miles.

10. (i) 12, (ii) $a+b$. **11.** (*a*) 2 ft., (*b*) 4 ft. 1 in., (*c*) $(5-b)$ ft.

12. (*a*) $(y-3)$ ft., (*b*) $(y-\frac{11}{12})$ ft., (*c*) $(y-b)$ ft.

13. (i) 16 ft., 12 sq. ft.; (ii) 62 ft., 240 sq. ft.; (iii) 38 ft., 60 sq. ft.; (iv) $(2m+2n)$ ft., mn sq. ft.; (v) $(72d+2c)$ in., $36cd$ sq. in.

14. (i) 16, (ii) $3G$, (iii) $\frac{72}{K}$, (iv) $\frac{12P}{Q}$.

15. (i) 36 pence, (ii) $2c$ pence, (iii) $3d$ pence, (iv) ln pence.

16. (i) 19, (ii) $e-f$. **17.** (i) 19, (ii) $d-c$.

18. (i) 8 days, (ii) $\frac{250}{a}$ days, (iii) $\frac{h}{35}$ days, (iv) $\frac{s}{t}$ days.

19. (i) 14 years, (ii) $(17-2k)$ years, $(t-3)$ years, $(t-2k)$ years.

20. (i) $2x-2$, (ii) $2x+2$. **21.** $l-m$. **22.** 33.
23. 11, 34. **24.** 17, 12. **25.** 11. **26.** 8.
27. 7. **28.** 1. **29.** 60. **30.** 10.

31. 6. **32.** 15. **33.** 22. **34.** 11.
35. 12. **36.** 40. **37.** 8. **38.** 84.
39. 4. **40.** (i) $8k$, (ii) $56s$, $\frac{z}{8}$ days.
41. (i) 12 shillings, (ii) $3h$ shillings.
42. (i) $3Q+12$, (ii) $\frac{Q}{3}+7$, (iii) $4Q-6$, (iv) $180-2Q$.
43. (i) 2500 shillings, (ii) 10,000 shillings, (iii) $5x$ shillings, (iv) $15y$ shillings.
44. (i) $4c$ shillings, (ii) $6t$ shillings, (iii) $4m$ pence, (iv) lX pence, (v) eZ pence.
45. $\frac{360}{w}$. **46.** (i) $36°$, $r°$, $21°$, $r°$.
(ii) $105°$; $(180-r)°$; $121°$; $(180-z)°$, $z°$, $(180-z)°$
(iii) $40°$, $(65-x)°$, $158°$, $(360-y-z)°$.
47. (i) 93, (ii) $10x+y$, (iii) 758, (iv) $100x+10y+z$.
48. (i) 125 miles, (ii) $25v$ hours.

EXERCISE 3 A (P. 15)

2. 345. **3.** abc, acb, bca, bac, cab, cba.
4. (i) 63, (ii) 0, (iii) $7s$.
5. (i) $4\times c$, $c\times c\times c\times c$; (ii) $3\times c\times c$, $2\times c\times c\times c$. **6.** (i) h^6, (ii) n^q.
7. (i) $3xz$, (ii) $7c^2$, (iii) $9a^2$, (iv) $8b^3$, (v) $3n^3$, (vi) $9s^2$, (vii) $4x^3$.
8. (i) 24, (ii) 0. **9.** 12. **10.** 12. **11.** 12. **12.** 0.
13. 54. **14.** 12. **15.** 3. **16.** 9. **17.** 7. **18.** 0.
19. 27. **20.** 2. **21.** 14. **22.** 0. **23.** 100. **24.** 7000.
25. 810. **26.** 4. **27.** 9. **28.** 0. **29.** 36. **30.** 75.
31. $1\frac{1}{6}$. **32.** $\frac{1}{3}$. **33.** $\frac{2}{3}$. **34.** $12\frac{2}{9}$. **35.** $\frac{2}{3}$. **36.** $1\frac{1}{3}$.

EXERCISE 3 B (Pp. 15, 16)

1. 724. **2.** pqr, prq, qpr, qrp, rpq, rqp. **4.** (i) k^5, (ii) v^x.
5. (i) $5\times d$, $d\times d\times d\times d\times d$; (ii) $4\times e\times e\times e$, $3\times e\times e\times e\times e$.
6. (i) 21, (ii) $7t$, (iii) 0.
7. (i) $4yw$, (ii) $16c^2$, (iii) $3m^3$, (iv) $9d^2$, (v) $4t^3$, (vi) $27y^4$, (vii) $125t^3$.
8. (i) 6, (ii) 162, (iii) 0. **9.** 8. **10.** 0. **11.** 0. **12.** 10.
13. 6. **14.** 0. **15.** 14. **16.** 4. **17.** 6. **18.** 500.
19. 96. **20.** 3000. **21.** 64. **22.** 48. **23.** 0. **24.** 12.
25. 0. **26.** 34. **27.** 125. **28.** 1250. **29.** 64. **30.** 80.
31. $1\frac{1}{2}$. **32.** $18\frac{1}{2}$. **33.** 1. **34.** 9. **35.** $2\frac{1}{4}$. **36.** $1\frac{1}{8}$.

EXERCISE 3 C (P. 16)

1. 16.	**2.** 70.	**3.** 3.	**4.** 900.	**5.** 5.	**6.** 216.
7. 1.	**8.** 1.	**9.** 9.	**10.** 36.	**11.** 190.	**12.** 320.
13. 161.	**14.** 145.	**15.** 30.	**16.** 780.	**17.** 0.	**18.** 0.
19. 2.	**20.** $\frac{1}{4}$.	**21.** 0.	**22.** 80.	**23.** $1\frac{1}{4}$.	**24.** 30.
25. $4\frac{13}{36}$.	**26.** $4\frac{5}{6}$.	**27.** 2.	**28.** $\frac{1}{3}$.	**29.** $1\frac{1}{6}$.	**30.** $1\frac{1}{4}$.
31. 12.	**32.** 6.	**33.** 0.	**34.** 0.		

EXERCISE 4 A (P. 18)

1. $4x$.	**2.** $5y$.	**3.** $6x$.	**4.** $7p$.	**5.** $6m$.	**6.** $6m$.
7. $15m$.	**8.** $12s$.	**9.** $12z$.	**10.** $10t$.	**11.** $2h$.	**12.** $3X$.
13. $\frac{2Y}{3}$.	**14.** $7x$.	**15.** $10w$.	**16.** $12v$.	**17.** $6t$.	**18.** $8t$.
19. d.	**20.** $3d$.	**21.** 0.	**22.** w.	**23.** $6x$.	**24.** z.
25. 0.	**26.** $2w$.	**27.** $5x$.	**28.** $7b$.	**29.** $7d$.	**30.** $\frac{3p}{2}$.
31. $\frac{3h}{4}$.	**32.** $7k$.	**33.** $7n$.	**34.** 34.	**35.** $9d$.	**36.** 0.

EXERCISE 4 B (P. 18)

1. $7t$.	**2.** $3p$.	**3.** $4w$.	**4.** $10l$.	**5.** $8s$.	**6.** $8n$.
7. $3c$.	**8.** $12k$.	**9.** $15v$.	**10.** $2Z$.	**11.** $12u$.	**12.** $21s$.
13. $\frac{5X}{6}$.	**14.** $8c$.	**15.** $11x$	**16.** $14d$.	**17.** $13l$.	**18.** $9d$.
19. x.	**20.** 0.	**21.** $7d$.	**22.** d.	**23.** $4c$.	**24.** 0.
25. $6w$.	**26.** b.	**27.** $4h$.	**28.** $\frac{5x}{2}$.	**29.** $8d$.	**30.** $7t$.
31. $4t$.	**32.** $3v$.	**33.** 0.	**34.** $7c^2$.	**35.** 530.	**36.** $14s$.

EXERCISE 4 C (Pp. 18, 19)

1. 470. **2.** $11x^2$. **3.** $3lm$. **4.** $5cd$. **5.** 0. **6.** $7x^2y$.
7. $11a^2b^2$. **8.** $12c^3d$. **9.** $9abcd$. **10.** $9xyz$. **11.** $8x$. **12.** $24s$.
13. $8d$/-. **14.** $11t$ hours. **15.** $46x$. **16.** $9z$ miles E.
17. $30n$/-. **18.** $48x$ pence. **19.** $21a$ in. **20.** x/-.

EXERCISE 5 A (P. 20)

1. $2a+b$. **2.** $4a+3b$. **3.** $4x+3$. **4.** $2x+3y$. **5.** $c+3$.
6. $4a+3b+1$. **7.** $c+4d$. **8.** $2l+m$. **9.** $3lm$.
10, 11, 12. Not possible. **13.** $a-b$. **14.** $2a+b+9$.
15. Not possible. **16.** $7x+2$. **17.** $4d+7$. **18.** $3t$. **19.** $3k-3$.

20. $6s-t$. **21.** 3. **22.** $4m-2$. **23.** $9b+4$. **24.** $2x+3y$.
25. Not possible. **26.** $2s+3t$. **27.** $20x-20y$. **28.** $3s+2t$.
29. $2c$. **30.** $l+8m+1$. **31.** Not possible. **32.** $2xy+3yz$.
33. $9ab$. **34.** Not possible. **35.** $ab+3b$. **36.** $5ab-ac$.

Exercise 5 B (P. 20)

1. $3c+3d$. **2.** $5a+4$. **3.** $3u+v$. **4.** $3r+4s+5$.
5. $3l+4m$. **6.** $2x+4$. **7.** $3rs$. **8.** $u+6v$.
9. $2p+2q$. **10, 11, 12.** Not possible. **13.** $6m+5n$.
14. Not possible. **15.** $10k+4$. **16.** $4u+2v+3$. **17.** $2x-y$.
18. 5. **19.** Not possible. **20.** $3k-5$. **21.** $8m-2n$.
22. $5n$. **23.** $l+4m$. **24.** $3v$. **25.** $42t-42s$.
26. $3m+n$. **27.** Not possible. **28.** $2s+2t$. **29.** Not possible.
30. $2cd+4ad$. **31.** $2u+8v+3$. **32.** 0. **33.** $4lm-ls$.
34. $2cd+5c$. **35.** $7a^2$. **36.** Not possible.

Exercise 5 C (Pp. 20, 21, 22)

1. Not possible. **2.** $7x^2$. **3.** $6x^2$. **4.** Not possible.
5. 0. **6.** 0. **7.** Not possible. **8.** $\frac{7u}{3}$.
9. $\frac{3x}{4}$. **10.** $3k$. **11.** Not possible. **12.** st.
13. $11A$. **14.** $11z^3$. **15.** $2x^2y+2xy^2$.
16. $5st+2s+3t$. **17.** 0. **18.** $5bc-3ac-ab$. **19.** $3pqrs$.
20. $3x^2y^2$. **21.** $5a+2b$. **22.** $(2a+3b)$ shillings.
23. $(11c+d)$ hours. **24.** $\left(\frac{3a}{2}+2b\right)$ miles E.
25. $(8l+21m+36n)$ shillings. **26.** $(24l+24m)$ pence.
27. $(x+9y+5z)$ shillings. **28.** $(16r+30s)$ feet.
29. $19x$ pence. **30.** $(2x+y+3)$ in. **31.** $(2a+2b+4)$ in.
32. $(X+2x)$ lb., 54 lb. **33.** 28, $12x-7$, $12z-t$.
34. $(z+7)$ shillings. **35.** $(2c+4d)$ shillings. **36.** $(120-2s)$ pence.

Exercise 6 A (P. 23)

1. x^3+2x^2+4x. **2.** t^2+3t. **3.** $3a^2+a$.
4. m^3+6m^2+3m+4. **5.** c^4+3c^2+7c. **6.** $2l^2+3l+3$.
7. r^4+3r+7. **8.** $3t^3-5t^2+6t+5$. **9.** $3x+2x^2+5x^4$.
10. $5+4m+6m^2-m^3$. **11.** $3-2n+n^3$. **12.** $3+t+5t^2+3t^3$.
13. $5+9t+3t^2$. **14.** $8+2r+r^2$. **15.** $3+10d+5d^2$.
16. $3+2c-c^2+4c^3$.

EXERCISE 6 B (P. 24)

1. $4b^2+2b$. **2.** t^4+2t^2+3t. **3.** $3s^2+7s$. **4.** $2d^4+d^3+5d$.
5. $2n^2+6n+5$. **6.** $4k^3+3k^2+k+2$. **7.** $5s^3+9s^2-4$.
8. m^3+4m+5. **9.** $5-4x+3x^2$. **10.** $2z^2+3z^3+9z^4$.
11. $8+3t+5t^2-t^3$. **12.** $3+6c+5c^2$. **13.** $7+3n+n^3$.
14. $2+4x+3x^2+5x^3$. **15.** $7+5b-2b^2+6b^3$. **16.** $1+9h+4h^2$.

EXERCISE 6 C (P. 24)

1. $2t^3+4t^2-3$. **2.** $6x^2+2x+5$. **3.** $10a^2+11a+4$.
4. t^4+t^3+3t+2. **5.** $9+3y+10y^2+2y^3$. **6.** $4a+3a^3+a^5$.
7. (*a*) $11+s-2s^2+4s^4$, $4s^4-2s^2+s+11$;
(*b*) $2+5t+8t^2+t^3$, t^3+8t^2+5t+2;
(*c*) $1+13x^3+x^4$, x^4+13x^3+1;
(*d*) $3-4x+11x^2+x^3$, x^3+11x^2-4x+3;
(*e*) $7-3c+2c^2+5c^3$, $5c^3+2c^2-3c+7$;
(*f*) $3-2y+3y^2+8y^3$, $8y^3+3y^2-2y+3$;
(*g*) $4+3h^3+2h^6$, $2h^6+3h^3+4$;
(*h*) $5-7x-2x^2+3x^4$, $3x^4-2x^2-7x+5$.
8. (*a*) 3, 4, 4; (*b*) 12, 0, 2; (*c*) 7, 11, 1; (*d*) 0, 0, 7; (*e*) $2a$, $3a^3$, 3; (*f*) $6b$, $5b^3$, 2; (*g*) 4, 11, 3; (*h*) 0, 0, 2.

EXERCISE 7 A (P. 26)

1. 3. **2.** 9. **3.** 4. **4.** 6. **5.** 3. **6.** 11.
7. 14. **8.** 8. **9.** 10. **10.** 8. **11.** 16. **12.** 3.
13. 0. **14.** 0. **15.** $\frac{3}{5}$. **16.** 7. **17.** 0. **18.** 16.
19. 2. **20.** $1\frac{1}{2}$. **21.** 3. **22.** 4. **23.** 2. **24.** $5\frac{2}{3}$.

EXERCISE 7 B (P. 27)

1. 3. **2.** 5. **3.** 4. **4.** 8. **5.** 6. **6.** 5.
7. 11. **8.** 28. **9.** 14. **10.** 33. **11.** 56. **12.** 24.
13. 0. **14.** $\frac{7}{15}$. **15.** 0. **16.** 0. **17.** 23. **18.** 15.
19. 1. **20.** 0. **21.** 2. **22.** 5. **23.** 1. **24.** 3.

EXERCISE 8 A (P. 30)

1. (i) Subtract $2x$ from each side, (ii) add $5x$ to each side, (iii) add $2t$ to each side, (iv) subtract $2x$ from each side.
2. (i) Subtract $2x$ from each side, (ii) add $5x$ to each side, (iii) add $2t$ to each side, (iv) subtract $2y$ from each side.
3. (i) Subtract 17 from each side, (ii) subtract 4 from each side, (iii) add 15 to each side, (iv) add 8 to each side.

4. (i) Subtract 6 from each side, (ii) add 25 to each side, (iii) add 12 to each side, (iv) subtract 2 from each side.

5. 2. **6.** 2. **7.** 4. **8.** 10. **9.** 16. **10.** 0.
11. 54. **12.** 63. **13.** 9. **14.** 6. **15.** 1·2. **16.** $3\frac{1}{2}$.

EXERCISE 8 B (Pp. 30, 31)

1. (i) Subtract $3x$ from each side, (ii) add $10x$ to each side, (iii) subtract $3x$ from each side, (iv) add $5t$ to each side.

2. (i) Add $7x$ to each side, (ii) subtract $4x$ from each side, (iii) add $3y$ to each side, (iv) add $4t$ to each side.

3. (i) Add 11 to each side, (ii) subtract 7 from each side, (iii) subtract 11 from each side, (iv) add 3 to each side.

4. (i) Add 37 to each side, (ii) add 10 to each side, (iii) subtract 1 from each side, (iv) subtract 10 from each side.

5. 3. **6.** 2. **7.** 9. **8.** 0. **9.** 14. **10.** 4.
11. 25. **12.** $12\frac{2}{3}$. **13.** 56. **14.** 2. **15.** 15. **16.** 2·8.

EXERCISE 8 C (P. 31)

1. 3. **2.** 4. **3.** 4. **4.** 2. **5.** 3. **6.** 6.
7. $2\frac{2}{3}$. **8.** 3. **9.** 4. **10.** 3. **11.** 0. **12.** 2.
13. 5. **14.** 3. **15.** 3·7. **16.** $1\frac{3}{4}$. **17.** $1\frac{1}{5}$. **18.** $3\frac{1}{2}$.
19. $4\frac{3}{4}$. **20.** $\frac{1}{2}$. **21.** $\frac{5}{6}$. **22.** $\frac{7}{9}$. **23.** 0. **24.** $1\frac{2}{5}$.
25. 28. **26.** $10\frac{4}{5}$. **27.** 12. **28.** 3·6. **29.** 24. **30.** $\frac{4}{5}$.
31. $6\frac{3}{4}$. **32.** $19\frac{1}{6}$. **33.** $6\frac{2}{3}$. **34.** 0. **35.** 16. **36.** 6.

EXERCISE 9 A (Pp. 35, 36)

1. 24, 13. **2.** 8. **3.** 72. **4.** 24. **5.** 12.
6. 42, 43. **7.** 58, 60. **8.** 39. **9.** 16, 18, 20.
10. 9, 10, 11, 12.

EXERCISE 9 B (P. 36)

1. 33, 16. **2.** 6. **3.** 49. **4.** 36. **5.** 15.
6. 38, 39. **7.** 52, 54. **8.** 71, 73. **9.** 21, 22, 23.
10. 21, 23, 25, 27.

EXERCISE 9 C (Pp. 37, 38)

1. 37. **2.** 9. **3.** 13, 26. **4.** *A* £39, *B* £24, *C* £16.
5. *A* £25, *B* £35, *C* £75. **6.** 3. **7.** *A* £13 10s., *B* £12 15s.
8. 2′ 6″. **9.** 10. **10.** *A* 6s. 10d., *B* 13s. 2d.
11. 52°, 52°, 76°. **12.** 55°, 55°, 70° or 50°, 65°, 65°.
13. $A=84°$, $B=68°$, $C=28°$. **14.** 150°, 112°, 98°. **15.** 2 hours.
16. 400 miles away. **17.** 12 m.p.h., 9 m.p.h. **18.** 50.
19. *A* £11, *B* £48, *C* £35. **20.** 80°, 80°, 20°.

TEST PAPERS I (Pp. 38, 39, 40, 41, 42, 43)

A. 1. (i) 10, (ii) 8, (iii) 4, (iv) 16.

2. (i) $6a$, (ii) 0, (iii) $3b+7$, (iv) $3b$, (v) $3c$, (vi) 0, (vii) 0, (viii) $5x$, (ix) $7t$.

3. (i) 14, (ii) 4. **4.** 2″. **5.** $10c$. **6.** (i) $\frac{xy}{(x+z)}$ days, (ii) $\frac{xy}{20}$.

B. 1. $7a$ years, $4a$ years. **2.** (i) $15x^2y$, (ii) $4a$, (iii) $5a+3$.

3. (i) $4\frac{1}{3}$, (ii) 7. **4.** $12\frac{1}{2}$. **5.** $\frac{5p}{12}$ shillings, $\left(10-\frac{5p}{12}\right)$ shillings.

6. 3 goals, 9 tries.

C. 1. (i) 3, (ii) 3, (iii) 4.

2. (i) $\frac{2x}{y}$, (ii) c^3d, (iii) $2st$, (iv) and (v) no change, (vi) z^x, (vii) $4xy+x+2y$.

3. Right-angled. **4.** (i) 3, (ii) $\frac{4}{7}$. **5.** 8.

6. Son £175, daughter £525.

D. 1. (i) $\frac{7y}{60x}$, (ii) $\frac{60b}{a}$. **2.** $2+2x+7x^3+3x^4$, 7, $2x$. **3.** $14\frac{2}{5}$.

4. 46°, 115°, 19°. **5.** $4a^2$ sq. yd., $36a^2$ sq. ft.

E. 1. (i) $3a^2+a+2$, (ii) $3-3b$. **2.** (i) 0, (ii) 0.

3. (i) $180-x$, (ii) $\frac{7n}{60}$. **4.** $22\frac{1}{2}$. **5.** $\frac{13}{12}ab$ shillings.

6. £700.

F. 1. (i) $18y^2$, (ii) $10x+y$. **3.** 3. **4.** (i) 14, (ii) $14x$.

5. £1 7s. 6d. **6.** $4\frac{1}{5}$.

G. 1. (i) $25N$, (ii) $\frac{Wz}{x}$ pence. **2.** (i) $2a^4+2a^3+a+2$, (ii) $156r$ inches.

3. (i) 8, (ii) 30, (iii) 34, (iv) 21. **4.** 61°, 68°, 51°.

5. $5\frac{1}{2}$d., 2d. **6.** 1s. 11d.

H. 1. £$\left(x+\frac{7y}{20}\right)$. **2.** (i) $3a^2b+3a^2+2b$, (ii) $a+2a^2+3a^3$.

3. 66. **4.** (i) 62, (ii) 1. **5.** (i) 0, (ii) $\frac{2}{3}$. **6.** 13.

I. 1. (i) 25, (ii) 100, (iii) 1016. **2.** $(8a+6b)$ miles.

3. (i) $8\frac{16}{19}$, (ii) 13. **4.** (i) $6c^3+5c-4$, (ii) $2x^2+7x+3$.

5. 35. **6.** 15 cm., 6 cm., 90 sq. cm.

J. 1. (i) $(5x+y)$ in., (ii) $(x+y)$ in.

2. (i) $8+4n-7n^2+3n^3$, (ii) $2+7c+6c^2$. **3.** (i) 0, (ii) 6.

4. 25 m.p.h. **5.** £233 6s. 8d. **6.** 65°, 65°, 50° or 55°, 55°, 70°.

K. **1.** (i) 24, (ii) $\frac{1}{8}$, (iii) 6. **2.** (i) 7, (ii) 0. **3.** $49a^2$ sq. cm.
4. $b+c=a+2$. **5.** (i) $2uv$, (ii) $7lm+5l+7m$, (iii) $7yz-5xz-xy$.
6. $90°$, $70°$, $95°$, $105°$.

L. **1.** $(9k+2l)$ shillings. **2.** (i) £$\frac{61x}{240}$, (ii) $(16a+40b)$ ft.
3. (i) 6, (ii) 0. **5.** 3472. **6.** 64 sq. ft.

Exercise 10 A (Pp. 45, 46)

1. Three times the sum of b and 2 ; 21.
2. a times the sum of c and 3 ; 42.
3. 8 times the result of taking 4 from a ; 24.
4. Twice the sum of b and 3 ; 16.
5. b times the result of taking 4 from twice c ; 10.
6. b times the result of taking c from a ; 20.
7. c times the result of taking b from a ; 6.
8. 5 times the result of taking four times c from three times a ; 45.
9. The sum of a and b, divided by c ; 4.
10. The result of taking twice c from three times a, divided by b ; 3.
11. The result of taking twice c from a, divided by b ; $\frac{1}{5}$.
12. The result of taking b from a, multiplied by the result of taking from twice b ; 14.

13. 9. **14.** 10. **15.** 10. **16.** 9. **17.** 9. **18.** 2.
19. 4. **20.** 2. **21.** 2. **22.** 5. **23.** 3. **24.** 5.

25. $12t+6(12-t)$. **26.** $288+(n-144)$ lb. **27.** $\frac{10-c}{5}$ inches.

28. $6(12x+y)-240z$. **29.** $40x+18(7-x)$ miles.

30. $x(P-Q)$ lb., $Q+\frac{P-Q}{2}$ lb.

31. (*a*) $7(a-b+2c)$, (*b*) $(2R+S)-(P+3T)$, (*c*) $\frac{x+2y}{a-b}$.
(*d*) $\frac{x-2a}{7}$. (*e*) $(r+s)^2$. (*f*) $36(s-t)+12(2r+s)+k$ inches.

Exercise 10 B (Pp. 46, 47)

1. z times the result of taking 5 from x ; 6.
2. 7 times the result of taking 2 from y ; 7.
3. 3 times the sum of z and 7 ; 27.
4. x times the result of taking 12 from five times y ; 24.
5. z times the result of taking z from y ; 2.
6. 5 times the sum of x and 2 ; 50.
7. z times the result of taking z from x ; 12.
8. y times the result of taking seven times z from twice x ; 6.
9. The result of taking four times z from three times y, divided by x ; $\frac{1}{8}$.

10. The sum of x and twice z, divided by y; 4.

11. The result of taking six times z from three times x, divided by twice y; 2.

12. The sum of x and z, multiplied by the result of taking z from three times y; 70.

13. 13. **14.** 25. **15.** 25. **16.** 1. **17.** 7. **18.** 1.

19. $3\frac{1}{2}$. **20.** 4. **21.** $6\frac{3}{5}$. **22.** $4\frac{2}{3}$. **23.** $6\frac{1}{2}$. **24.** 100.

25. $2448+136(n-36)$. **26.** $12x+6y+(30-x-y)$ pence.

27. $(d-c)$ lb., $\frac{a}{d-c}$. **28.** $55t+15(15-t)$ miles.

29. $240r-(240x+12y+z)-16$ pence. **30.** $1536+4(n-168)$ ounces.

31. (*a*) $3c(2a-b)$. (*b*) $\frac{3}{5}(X+3Y)$. (*c*) $\frac{4d-3c}{5}$.

(*d*) $(2d-c)^3$. (*e*) $2(a+b)+(c-d)$ pints. (*f*) $(5a-2c)-(x-3y)$.

Exercise 11 a (P. 50)

1. 6. **2.** 10. **3.** 22. **4.** 26.
5. $a-b-c$. **6.** $a-b+c$. **7.** $a+2b-c$. **8.** $a+2b+3c$.
9. 8. **10.** 0. **11.** 14. **12.** 4.
13. $l+m-r-s$. **14.** $l+m-r+s$. **15.** $l-m+r-s$. **16.** $l-m-r+s$.
17. 5. **18.** 5. **19.** $3a+2b$. **20.** $4d$.
21. $2x+3y$. **22.** $2p-q$. **23.** $s+t$. **24.** $3h-k$.
25. $2b+11$. **26.** $m+8n$. **27.** $4a-2b$. **28.** $3x+3y$.
29. $3c-2d$. **30.** $5p+2q$. **31.** $a-b$. **32.** $4b-2a$.
33. $3a-2b-c$. **34.** $7x-5y-8z$. **35.** $y+6z-3x$. **36.** $10-c$.

Exercise 11 b (Pp. 50, 51)

1. 24. **2.** 40. **3.** 10. **4.** 26. **5.** $a+b-c$.
6. $a-b-2c$. **7.** $a+b+3c$. **8.** $a-3b+2c$.
9. 6. **10.** 12. **11.** 0. **12.** 10. **13.** $c-d+h-k$.
14. $c-d-h+k$. **15.** $c+d-h-k$. **16.** $c+d-h+k$.
17. 8. **18.** $3x+2y$. **19.** $8k$. **20.** $l+3$.
21. $2y$. **22.** $k-3l+2m$. **23.** 4. **24.** $3x+6y$.
25. $7u+2v$. **26.** $9x-5y$. **27.** $s+7b$. **28.** $9u-12$.
29. 0. **30.** $3a+20z$. **31.** $6c-4a$. **32.** $2a-5b$. **33.** $4S-2R$.
34. $x-5y+3z$. **35.** $2x+2y-7z$. **36.** $12x-8y-z$.

Exercise 11 c (P. 51)

1. $25-3k$. **2.** $3a-2b$. **3.** $2c-5d$. **4.** $2x-3y$. **5.** $2p+q$.
6. $2t-s$. **7.** $8k-h$. **8.** $17-4t$. **9.** $10t-17$. **10.** $m-2n$.
11. $2b-2a$. **12.** $3c-2d$. **13.** $8a-5b$. **14.** $9c-b$. **15.** $2a+6c$.
16. $4b-4c$. **17.** $2a-4b$. **18.** $4b+6c$. **19.** 0. **20.** $2x+y+z$.

21. $4a+3b-4$. **22.** a. **23.** $6a^2+4b^2-11c^2$.
24. $2x^2+2y^2+3z^2$. **25.** $2x+2y+z$. **26.** $2x-6y+7z$.
27. $2x+y+5z$. **28.** $2p-6q+5r$. **29.** $3a+4b+3c$.
30. $x-y-3z$.

Exercise 12 A (P. 53)

1. $5a+5b$. **2.** $7c+21$. **3.** $21-7b$. **4.** $3a+ab$.
5. $8a-6$. **6.** $xz-yz$. **7.** $15x+10y$. **8.** $24t-32s$.
9. $3a-6b-3c$. **10.** $8a+12b-8c$. **11.** $15a-20b+25c$.
12. $3p^2-pq-2pr$. **13.** $2ac+3bc$. **14.** $ad-2bd+3cd$.
15. $4xt-3yt$. **16.** $5kl+3km$. **17.** $\frac{1}{2}x-3$.
18. $\frac{2}{3}x-\frac{2}{3}y+4$. **19.** $2x^3-6x^2$. **20.** $15xy-21x^2$.
21. $5a+13$. **22.** $13a+18$. **23.** $47a+36b$. **24.** $7P+7Q$.
25. $5s+5t+2st$. **26.** $3xy+3xz$. **27.** $2y^2+13y+6$. **28.** $6x^2+15x$.
29. $2x+10$. **30.** $c+2d$.

Exercise 12 B (P. 53)

1. $8+4y$. **2.** $3c-3d$. **3.** $10+2k$. **4.** $4c-cd$. **5.** $ad+cd$.
6. $12-9l$. **7.** $15s-25t$. **8.** $28u+12v$. **9.** $7x+14y+21z$.
10. $12x-18y-24z$. **11.** $4x^2+xy-5xz$. **12.** $28x-24y-20z$.
13. $4s^2+5st$. **14.** $7au-6bu$. **15.** $4ms-3mu$.
16. $2ax-3ay+4az$. **17.** $3c^3-6c^2d$. **18.** $\frac{1}{4}x-2$.
19. $\frac{2c}{5}+4$. **20.** $8ac+8ab-12a^2$. **21.** $5x+31$.
22. $43x+75$. **23.** $48a+34b$. **24.** $34P+27Q$.
25. $6l+6m+5lm$. **26.** $5c^2+10cd$. **27.** $11t^2+8t$.
28. $43a^2+42a$. **29.** $9y+1$. **30.** $6u+6v$.

Exercise 12 C (P. 54)

1. $19t+9u$. **2.** $5x+4y$. **3.** $a-5b$. **4.** $2x-5y$. **5.** $5a-b$.
6. $8x-4y$. **7.** $12a^2+10a+10$. **8.** $20a^2-6b^2$.
9. $A+A^2$. **10.** $13-15m$. **11.** $14-16x$. **12.** $29-3a$.
13. $2a^2-9ab+b^2$. **14.** $2a^2-3ab-b^2$. **15.** 2. **16.** 4.
17. $8t$. **18.** $2a^2$.

Exercise 13 A (Pp. 55, 56)

1. $42x+23$. **2.** $\frac{23a}{3}-2$. **3.** $3x+20$. **4.** $9c+20$. **5.** $5c+4$.
6. 4. **7.** $x+1$. **8.** $x-1$. **9.** $3x-1$. **10.** $7x-1$.
11. $4a+15$. **12.** $a-6$. **13.** $23-14x$. **14.** $\frac{31}{4}c-2d$.
15. $18x-11y$. **16.** $10-8d$. **17.** $8u-10v$. **18.** $15v-6u$.
19. Identity. **20.** Equation. **21.** Equation. **22.** Equation.
23. Identity. **24.** Identity.

Exercise 13 B (P. 56)

1. $\frac{13}{2}c-3$. **2.** $19x+16$. **3.** $36z+4$. **4.** $2x+9$.
5. $4t+18$. **6.** $13x+4$. **7.** $a-3$. **8.** $8a-1$.
9. $2a+2$. **10.** $a+36$. **11.** $11a-4$. **12.** $16a-4$.
13. $\frac{17}{3}x-2y$. **14.** $46-t$. **15.** $27-18x$. **16.** $112d-17c$.
17. $11a-10b$. **18.** $3a-14b$. **19.** Equation. **20.** Identity.
21. Identity. **22.** Equation. **23.** Identity. **24.** Equation.

Exercise 13 C (P. 56)

1. $3-a$. **2.** $5b-2c$. **3.** $2s-4t$. **4.** $8t-2s$. **5.** $15d-6c$.
6. $24c-15d$. **7.** $3k+5l$. **8.** $3a-12b$. **9.** $10c^2+4c$. **10.** $4n-2l$.
11. $3b-3a$. **12.** 2. **13.** c^3+cd^2. **14.** x^3-xy^2. **15.** $2x-x^2$.
16. 0. **17.** $a^2+4ab-a$. **18.** $11x^2+6x-4xy$.

Exercise 14 A (P. 57)

1. 6. **2.** 4. **3.** 1. **4.** 1. **5.** 5. **6.** 5.
7. 3. **8.** 6. **9.** 3. **10.** 7. **11.** 4. **12.** 2.
13. 3. **14.** 6. **15.** 3. **16.** 2. **17.** 11. **18.** 3.

Exercise 14 B (Pp. 57, 58)

1. 7. **2.** 2. **3.** 2. **4.** 8. **5.** 2. **6.** 3.
7. 8. **8.** 3. **9.** 2. **10.** 3. **11.** 23. **12.** 15.
13. 3. **14.** 11. **15.** 5. **16.** 4. **17.** 2. **18.** 3.

Exercise 14 C (Pp. 58, 59)

1. $2\frac{1}{2}$. **2.** $\frac{2}{7}$. **3.** $\frac{1}{8}$. **4.** 0. **5.** $1\frac{1}{3}$. **6.** $2\frac{2}{5}$.
7. $\frac{8}{13}$. **8.** 8. **9.** $\frac{1}{7}$. **10.** $3\frac{1}{3}$. **11.** $2\frac{1}{7}$. **12.** $\frac{1}{2}$.
13. $1\frac{2}{7}$. **14.** $6\frac{2}{3}$. **15.** $9\frac{4}{7}$. **16.** 6. **17.** $1\frac{8}{13}$. **18.** $\frac{3}{13}$.
19. $2\frac{1}{5}$. **20.** $2\frac{1}{4}$. **21.** 0. **22.** $1\frac{1}{2}$. **23.** $3\frac{1}{3}$. **24.** $1\frac{1}{3}$.
25. $1\frac{3}{4}$. **26.** $\frac{5}{21}$. **27.** $1\frac{1}{2}$. **28.** $2\frac{1}{5}$. **29.** $5\frac{5}{6}$. **30.** $\frac{3}{8}$.
31. $1\frac{1}{9}$. **32.** 3. **33.** $3\frac{16}{17}$. **34.** $12\frac{1}{7}$. **35.** $1\frac{13}{17}$. **36.** $\frac{13}{17}$.

Exercise 15 A (P. 60)

1. 11. **2.** 12. **3.** 10, 11, 12. **4.** 24, 16.
5. 90, 6. **6.** 45 years. **7.** 8 years. **8.** 43 years.
9. 36 years. **10.** 42 years.

Exercise 15 B (P. 61)

1. 11. **2.** 4, 5, 6. **3.** 12. **4.** 44, 28.
5. 84, 8. **6.** 32 years. **7.** 30 years. **8.** 14 years.
9. 30 years. **10.** 48 years.

Exercise 15 C (Pp. 61, 62, 63)

1. 17. **2.** 12 turkeys, 38 chickens. **3.** 112.
4. 8. **5.** $2\frac{1}{2}$ hours. **6.** 14 ins., 22 receipt.
7. 5 miles. **8.** $2\frac{1}{2}$ hours. **9.** £51. **10.** *A* 4/8, *B* 3/10.
11. 1st 37/6, 2nd 21/-, 3rd 10/6. **12.** 116°, 116°, 116°, 96°, 96°.
13. 14. **14.** $1\frac{1}{2}$ hours.
15. 116°, 128°, 116°, 128°, 116°, 116°. **16.** One penny.
17. £36. **18.** 2 hr. 48 min. **19.** 26 years. **20.** 22.
21. 2s. **22.** 40. **23.** $3\frac{1}{4}$ hr.
24. 125°, 149°, 125°, 71°, 125°, 125°. **25.** 45 years.
26. $\frac{3}{4}$ mile. **27.** £210. **28.** *X* £1 5s.; *Y* £3 15s.
29. 125°, 116°, 125°, 58°, 116°. **30.** 2s. 6d.

Exercise 16 A (P. 71)

The independent variable is:
1. The date. **2.** The number of yards.
3. The time of day. **4.** The day of the month. **5.** H.P.
6. The age of the males. **7.** The speed of the train.
8. The day of the month.

Exercise 16 B (P. 71)

The independent variable is:
1. The date. **2.** The time of day. **3.** The number of shillings.
4. H.P. **5.** The number of years.
6. The day of the month. **7.** The date.
8. The length of the pendulum.

Exercise 17 A (P. 71)

Take 1 inch equal to:
1. 2 units. **2.** 10 units. **3.** 20 units. **4.** 100 units.
5. $\frac{1}{10}$ unit. **6.** $\frac{1}{100}$ unit. **7.** 1000 units. **8.** 20 units.

Exercise 17 B (P. 72)

Take 1 inch equal to:
1. 10 units. **2.** 10 units. **3.** 40 units. **4.** 50 units.
5. $\frac{1}{5}$ unit. **6.** $\frac{1}{50}$ unit. **7.** 500 units. **8.** 20 units.

EXERCISE 18 A (Pp. 72, 73, 74)

1. (i) June and July, (ii) Nov. to Feb., (iii) Jan. and Dec., Feb. and Nov., no.
2. (i) 4, (ii) 1928, (iii) 1932, 1933.
3. In millions approx. (i) 292, (ii) $318\frac{1}{2}$, (iii) 328.
4. (i) 7s. 3d., (ii) 13s. 3d., (iii) 29s., (iv) 34s., approx.
5. (i) June, (ii) Feb.
6. $P=26\cdot7$ is wrong, $P=28$ is correct. Yes. £26, £31, £36·5.
7. (i) June, July, Aug., (ii) Jan., Feb., Dec., (iii) Apr., Oct.
8. $P=$(i) 5·5, (ii) 9·4, (iii) 14·8 ; $E=$(i) 18·6, (ii) 23·4, (iii) 24 8 ; $W=$(i) 56, (ii) 38·5.

EXERCISE 18 B (Pp. 74, 75, 76, 77)

1. (i) July, Aug., Sept., (ii) Apr., May, Oct., Nov. Below freezing point. No.
2. (i) Jan. and Mar., June ; (ii) June, July, Aug.
3. In millions approx. 20, 45, 115. **4.** £138, £198, £242 ; 11.
5. (i) £156, (ii) just over 28 years.
6. Yes. $P=13\cdot3$ is wrong, $P=15$ is correct. £12·8, £22·5, £36·5.
7. (i) Dec., (ii) April. **8.** £68·6, £93·2, £168.

TEST PAPERS II (Pp. 77, 78, 79, 80, 81, 82, 83)

A. **1.** (i) 34, (ii) 18, (iii) 16. **2.** (i) $1\frac{11}{13}$, (ii) 8.
3. 72°, 72°, 36°. **4.** A 45s., B 15s.
5. Length 6″, breadth 2″. **6.** (i) $a+b-9c$, (ii) $60x-6y-21z$.

B. **1.** (i) 12, (ii) 36, (iii) 30. **2.** (i) 5, (ii) $\frac{5}{11}$. **3.** 49.
4. (i) $\frac{1}{2}x+\frac{1}{2}y$, (ii) $8x+8y-3z$. **5.** Cup is 1s. 9d., saucer 1s. 3d.
6. 42, 113 ; 119, 34.

C. **1.** $3a^4-15a^3-12a^2+3a$, 3, $3a$. **2.** (i) 7, (ii) 169. **3.** 0.
4. 46°. **5.** 12s. 3d. **6.** Yes ; 37·5, 30·9, 21 years approx.

D. (i) $3x+11y-13z$, (ii) $21a+b$. **2.** (i) $\frac{2}{3}$, (ii) $4\frac{5}{8}$.
4. 12 miles. **5.** 45. **6.** (i) $2a$ years, (ii) $(6a^2+4a-12)$ pence.

E. **1.** (i) $5\frac{1}{2}$, (ii) 13, (iii) $2\frac{2}{5}$. **2.** (i) $17x-4y-11z$, (ii) $2a^3-4a^2+9$.
3. (i) $2\frac{1}{2}$, (ii) 2. **4.** 160°. **5.** 72 years, 12 years.
6. £18, £35, £75·5.

F. **1.** y. **2.** 36. **3.** (i) 12, (ii) 1. **4.** (i) 1, (ii) $47\frac{1}{2}$. **5.** 10′ 10″.
6. (i) Identity, (ii) $x=6$, (iii) $x=5$, (iv) Identity.

G. **1.** (i) and (iii). **2.** (i) $1\frac{2}{3}$, (ii) $2\frac{1}{4}$. **3.** $(1008ac+1104c)$ pence.
4. $(4x+14)$ in., 2, 22 sq. in.
5. 271 men, 219 women, 56 children. **6.** 5s. 9d., 4s. 11d., 4s. 9d.

H 1 2. **2.** (i) $17x-27$, (ii) $31a^6$. **3.** (i) $\frac{1}{9}$, (ii) 1.

4. (i) $\dfrac{160x+8y}{K}$ shillings, (ii) 27, 29, 31, 33, 35. **5.** 19. **6.** 120.

I. 1. 3502, 735. **2.** (i) 15, (ii) 0·67. **3.** $(3u+7v)$ pence.
4. 14. **5.** (i) $9q-7p$, (ii) $6B+3B^2+9B^3$.
6. 8 years, 5 years, 2 years.

J. 1. $(cx+dy)$ miles, $\dfrac{cx+dy}{z}$ hours. **2.** (i) $7\frac{1}{13}$, (ii) 4.
3. (i) 16, (ii) 16. **4.** C £18, D £22.
5. (i) $23p-9q$, (ii) $5kl-2k$. **6.** $12\frac{1}{2}$ lbs., $17\frac{1}{2}$ lbs., 22 lbs.

K. 1. (i) $8c^6$, (ii) $34b+19c-7a$. **2.** (i) $5b-2a$, (ii) $6x^3+20xy^2$.
3. (i) 5, (ii) 2·4. **5.** (i) $x=2$, (ii) Identity, (iii) Identity, (iv) $t=2$.
6. 30s, 15s., 7s. 6d.

L. 1. $\dfrac{as+bs-at}{b}$. **2.** (i) $4a-14b+5c$, (ii) $3x-2y-2z$.
3. 99°, 141°, 99°, 141°, 99°, 141°. **4.** (i) 2, (ii) 0. **5.** 240 yards
6. (i) 28·2′, (ii) 42·2′.

Exercise 19 A (P. 85)

1. (i) Take £x out of the bank, (ii) a rise in the temperature of $x°$, (iii) x minutes before midnight.

2. (i) A rise in the temperature of $p°$, (ii) put the clock back p minutes. (iii) go on p yards.

3. (i) $(+6)$, (ii) (-8), (iii) $(+10)$, (iv) (-10).

4. (i) $(+3650)$, (ii) (-1300), (iii) $(+2840)$, (iv) (-2863).

5. (i) Bradford is 10 miles west of Leeds, (ii) the clock is 5 minutes slow, (iii) the barometer has fallen 2 mm.

Exercise 19 B (Pp. 85, 86)

1. (i) Put the clock on y minutes, (ii) go y miles East, (iii) a decrease in salary of £y.

2. (i) Put £t in the bank, (ii) sell t lb. of sugar, (iii) my employer owes me £t.

3. (i) $(+50)$, (ii) (-20), (iii) $(+110)$, (iv) (-42).

4. (i) $(+40)$, (ii) (-8), (iii) $(+24)$, (iv) (-336).

5. (i) The school is 42 ft. above sea-level. (ii) The price of tea has gone up 2 pence per lb. (iii) The population of England has increased by 2 per cent.

Exercise 20 A (Pp. 89, 90)

1. (i) $(-6°)$, (ii) $(+4°)$, (iii) $(+20°)$.

2. (i) $(+6°)$, (ii) $(-30°)$, (iii) $(-20°)$, (iv) $(-4°)$.

3. (i) $(-a)$ sh., $(+12a)$ sh. **4.** £(-9).

5. Gained £$(+30)$; lost £(-30).
6. $(+16)$ m.p.h., (-20) m.p.h., (-78) m.p.h.
7. (i) (-20), (ii) (-2), (iii) 0, (iv) (-4), (v) (-4), (vi) (-18).
8. (i) $(+2a)$, (ii) $(-20a)$, (iii) $(-4b)$, (iv) 0, (v) $(-3c)$, (vi) $(-12c)$.
9. (i) $(+2)$, (ii) (-5), (iii) (-7), (iv) (-14), (v) 0, (vi) (-4).
10. (i) (-2), (ii) (-12), (iii) $(+6)$, (iv) (-2), (v) $(+12)$, (vi) 0.
11. (i) $(-17x)$, (ii) $(+7x)$, (iii) $(-3t)$, (iv) $(-2t)$, (v) $(+14t)$, (vi) $(-t)$.
12. (i) $(+10)$, (ii) 0, (iii) (-21), (iv) $(+9)$, (v) $(+2)$, (vi) (-3).

Exercise 20 B (Pp. 90, 91)

1. (i) $(+12°)$, (ii) $(-44°)$, (iii) $(-23°)$, (iv) $(-9°)$.
2. (i) $(-12°)$, (ii) $(+9°)$, (iii) $(+23°)$. **3.** (i) £$(-5b)$, (ii) £$(+8b)$.
4. £(-20). **5.** Gained £(-35) ; lost £$(+35)$.
6. $(+17)$ m.p.h., (-18) m.p.h., (-45) m.p.h.
7. (i) (-6), (ii) $(+2)$, (iii) (-20), (iv) $(+4)$, (v) 0, (vi) (-29).
8. (i) $(-6x)$, (ii) $(+2t)$, (iii) $(-20t)$, (iv) 0, (v) $(-5d)$, (vi) $(-5d)$.
9. (i) (-11), (ii) $(+15)$, (iii) (-14), (iv) (-16), (v) 0, (vi) $(+7)$.
10. (i) $(+5)$, (ii) (-16), (iii) (-6), (iv) $(+18)$, (v) (-7), (vi) $(+13)$.
11. (i) $(-2u)$, (ii) $(+9l)$, (iii) $(+15z)$, (iv) $(-15x)$, (v) $(+2m)$, (vi) $(-17k)$.
12. (i) $(+52)$, (ii) (-11), (iii) $(+7)$, (iv) (-35), (v) 0, (vi) $(+7)$.

Exercise 20 C (P. 91)

1. (i) $(-2x)$, (ii) $(+10y^2)$, (iii) $(-3xy)$, (iv) $(+11xy)$, (v) $(-5ab)$, (vi) 0.
2. (i) -9, (ii) -4, (iii) 1, (iv) -4, (v) -4, (vi) -12.
3. (i) -12, (ii) -21, (iii) -8, (iv) 10, (v) -10, (vi) 0.
4. (i) $a-8b$, (ii) $6y-2x$, (iii) $2s-4t$, (iv) $2c^2-4c$, (v) $5t^2-10t$, (vi) $xy-3x^2$.
5. (i) $3x^2-10x-4$, (ii) $-2x^2-2x+5$, (iii) $-10x^2+6x-8$, (iv) $-3x^2+2x-13$, (v) $-3x^3-5x^2+7x+2$, (vi) $-3x^3-3x^2+11$.
6. (i) $12-14x-5x^2$, (ii) $5-2x+6x^2$, (iii) $-11+7x-11x^2$, (iv) $9-10x-2x^2$, (v) $11-12x+x^2-4x^3$, (vi) $3-4x^2+3x^3$.
7. (i) $-a-2$, (ii) $-d$, (iii) -3, (iv) $a-2b+9$, (v) $3-2z$, (vi) $-4t$.
8. (i) $-x^2-4x$, (ii) $-4x^2-3x-2$, (iii) $3x^2-3x$, (iv) $-a^2-6a-8$, (v) 0, (vi) $3b-5a-7c$.

Exercise 21 A (P. 93)

1. (-21).	**2.** $(+18)$.	**3.** (-15).	**4.** $(+12)$.	**5.** (-24).
6. $(+30)$.	**7.** (-30).	**8.** $(+24)$.	**9.** $(+34)$.	**10.** (-40).
11. (-18).	**12.** (-30).	**13.** $(+60)$.	**14.** (-5).	**15.** 0.
16. $(+6)$.	**17.** (-4).	**18.** (-3).	**19.** $(+12)$.	**20.** (-3).

21. $(+2)$. **22.** (-6). **23.** (-4). **24.** (-16). **25.** $(+6)$.
26. (-3). **27.** $(+6)$. **28.** (-18). **29.** (-12). **30.** $(+4)$.
31. $(+8)$. **32.** (-6). **33.** $(+16)$. **34.** 0. **35.** $(+3)$.
36. (-12). **37.** 0. **38.** (-24). **39.** (-8). **40.** (-8).
41. $(+9)$. **42.** $(+36)$.

EXERCISE 21 B (P. 94)

1. (-28). **2.** $(+28)$. **3.** (-28). **4.** $(+12)$. **5.** (-12).
6. $(+28)$. **7.** (-12). **8.** $(+20)$. **9.** (-20). **10.** (-48).
11. (-99). **12.** 0. **13.** $(+132)$. **14.** $(+14)$. **15.** $(+48)$.
16. (-8). **17.** $(+2)$. **18.** (-4). **19.** (-4). **20.** $(+16)$.
21. (-2). **22.** (-6). **23.** (-8). **24.** $(+6)$. **25.** $(+15)$.
26. $(+12)$. **27.** (-25). **28.** (-5). **29.** (-3). **30.** $(+20)$.
31. $(+4)$. **32.** (-13). **33.** $(+13)$. **34.** (-10). **35.** (-3).
36. $(+15)$. **37.** 0. **38.** (-20). **39.** (-64). **40.** 0.
41. $(+24)$. **42.** (-1).

EXERCISE 21 C (P. 94)

1. $(+9)$. **2.** (-8). **3.** (-60). **4.** (-1). **5.** 0.
6. $(+144)$. **7.** (-14). **8.** (-7). **9.** $(+2)$. **10.** 0.
11. (-3). **12.** (-2). **13.** $(+21)$. **14.** 0. **15.** $(+1)$.
16. 0. **17.** (-3). **18.** (-1). **19.** $(+1)$. **20.** 0.
21. (-2). **22.** $(+1\frac{1}{2})$. **23.** (-4). **24.** $12y-2x$. **25.** $-2s-2t$.
26. $8a-9b$. **27.** $7s+t$. **28.** $-s-7t$. **29.** $15b-3a$.
30. $11v-11u$. **31.** $5y-5x$. **32.** $5x^2-4x-3$. **33.** $9a-3b$.

EXERCISE 22 A (Pp. 96, 97)

1. $5a+4b+3c$. **2.** $15x+5y-7z$. **3.** $3b-2c$.
4. $13l-11m-8n$. **5.** $7xy-yz$. **6.** $3p-2q+2r+3s$.
7. $-K-4$. **8.** $10t^2+6$. **9.** $2l^2+7l-2$.
10. $7x^4+2x^2-8x$. **11.** $-2x^3-4x^2y+4xy^2$. **12.** $-4-2t+2t^2+t^4+t^5$.
13. (i) $2a-5b$. (ii) $3a-6b$. (iii) $5a+9b$. (iv) $a+4b$.
14. (i) $r-4s$. (ii) $4r+6s$. (iii) $5r-5s$. (iv) $6r-8s$.
15. (i) x. (ii) $-3x+2y$. (iii) $-5x$. (iv) $-5x+16y$.
16. (i) $-3a+3k$. (ii) $c-4d$. (iii) $-3x+4y$. (iv) $-y-3z$.
17. $-x+3y-2z$. **18.** $x-5y-3z$. **19.** $3p+7q-3r$.
20. $6a+3b-5c$. **21.** $6xy-5yz+7zx$. **22.** x^3-x^2-x-2.
23. x^3-x^2-x+2. **24.** $x^4+2x^3+5x^2-4x$. **25.** $-3a^2b+3ab^2+2b^3$.
26. $3+2a-2a^2-2a^3$. **27.** $y+2z$. **28.** $3a^2-4b^2+3c^2$.

EXERCISE 22 B (Pp. 97, 98)

1. $7x+6y$. **2.** $5t$. **3.** $a+b+4c$. **4.** $3a+2c$.
5. $-5ab+4ca$. **6.** $7p+r+s$. **7.** $2x^2+x$. **8.** $2t^3+3t^2-3$.
9. $6a^4+a^3$. **10.** $8c^4-5c^3+4c^2+2c$.
11. $2a^3b+2ab^3$. **12.** $3x^3+x^5$.
13. (i) $3a+14b$. (ii) $4a+3b$. (iii) $3a-b$. (iv) $7a-9b$.
14. (i) $-2x$. (ii) $-2x+14y$. (iii) $4x$. (iv) $-3x+3y$.
15. (i) $-5a+5h$. (ii) $2c-13d$. (iii) $-7x+7y$. (iv) $-3x-3z$.
16. (i) $3r-4s$. (ii) $4r-6s$. (iii) $3r+14s$. (iv) $2r-7s$.
17. $-2a+5b-4c$. **18.** $2q$. **19.** $8l-14m+3n$.
20. $4ab+2bc+2ca$. **21.** $ab-7bc$. **22.** $2cd-7$.
23. $3a^2b+6ab^2+b^3$. **24.** $3x^3-2x^2y-2xy^2+2y^3$.
25. $-3-5m-3m^2+2m^4$. **26.** $-x^4-7x^3+4x^2+5x+3$.
27. $x+2y+8z$. **28.** $-4x^2+2y^2+8z^2$.

EXERCISE 23 A (P. 99)

1. -1 **2.** -2. **3.** 6. **4.** -9. **5.** 5. **6.** 6.
7. -6. **8.** -2. **9.** 4. **10.** 7. **11.** -11. **12.** 0.
13. -15. **14.** -3. **15.** -8. **16.** 3. **17.** 2. **18.** -3.
19. 4. **20.** -2. **21.** -6. **22.** 0. **23.** $-\frac{2}{7}$. **24.** -4.

EXERCISE 23 B (Pp. 99, 100)

1. -2. **2.** 3. **3.** -1. **4.** 7. **5.** -6. **6.** 7.
7. -3. **8.** 5. **9.** -10. **10.** 0. **11.** 3. **12.** -5.
13. -2. **14.** -24. **15.** -12. **16.** 4. **17.** -8. **18.** 25.
19. -4. **20.** -8. **21.** 16. **22.** -18. **23.** 0. **24.** $\frac{5}{9}$.

EXERCISE 23 C (P. 100)

1. -6. **2.** $-1\frac{4}{5}$. **3.** -1. **4.** -4. **5.** -3. **6.** -7.
7. 0. **8.** -4. **9.** 2. **10.** -3. **11.** -2. **12.** -2.
13. -2. **14.** 2. **15.** 13. **16.** $-1\frac{2}{3}$. **17.** $3\frac{4}{5}$. **18.** -22.
19. -4. **20.** 2. **21.** 3. **22.** -1. **23.** $-\frac{1}{5}$. **24.** $-2\frac{1}{4}$.
25. -2. **26.** 9. **27.** 1. **28.** 2. **29.** -1. **30.** $-16\frac{1}{2}$.
31. -5. **32.** 5. **33.** -5. **34.** -3. **35.** $-9\frac{4}{5}$. **36.** 0.

EXERCISE 24 A (Pp. 104, 105)

1. (i) $3a^2$, (ii) b^3, (iii) $3c^3$, (iv) x^4, (v) x^5, (vi) x^6, (vii) z^4, (viii) $4a$, (ix) $5x$, (x) t^3, (xi) d^4-d, (xii) x^3y^2.
2. (i) x^7, (ii) a^9, (iii) a^7, (iv) $6xy$, (v) $12x^2y$, (vi) ab^2c, (vii) x^5, (viii) x^4, (ix) a^4, (x) $4x^3$, (xi) $2x^2$, (xii) $\frac{3}{2}x^5$.
3. (i) $12c^3$, (ii) $3c$, (iii) $4d^2e^4$, (iv) $p+p^2+p^3$, (v) p^6, (vi) $6p^3$, (vii) $2a^2$, (viii) 18, (ix) x^2, (x) x^3+2x, (xi) $4b^3$, (xii) $8b^3$.

4. $\frac{4x^2}{9}$, $\frac{x^2}{9}$, $\frac{z^4}{25}$, $\frac{4}{t^2}$, $4s^2t^2$, a^6, $\frac{d^8}{16}$, $x^6y^4z^2$.

5. $\pm\frac{x}{3}$, $\pm 2b$, $\pm\frac{a}{b}$, $\pm\frac{a}{bc}$, $\pm\frac{x^2}{yz}$, $\pm K^4$.

6. $64x^3$, $-27y^3$, $\frac{8z^3}{125}$, $-\frac{8}{t^3}$, $-a^9$, $8p^3q^3r^3$.

7. $2a$, $\frac{b}{4}$, $-3z$, $-\frac{y}{3}$, $-K^2$, K^3.

8. (i) $24a^5b^4c$, (ii) $15a^3l^2m^5$, (iii) $12a^2$, (iv) $-48a^2$, (v) $-18ax^2y^2z$, (vi) $-a^5bx^4y$, (vii) $21a^7b^3c^4$, (viii) $10a^3b^3c^4$, (ix) $6a^{10}b^9$, (x) $-24a^3t^8$, (xi) a^{13}, (xii) x^{15}.

Exercise 24 b (P. 105)

1. (i) $5c^2$, (ii) $6k^2$, (iii) t^4, (iv) x^6, (v) x^4, (vi) $4d+1$, (vii) x^8, (viii) c^3, (ix) v^2, (x) ab^2c, (xi) $5u^2$, (xii) a^3-a^2.

2. (i) t^6, (ii) t^9, (iii) $20cd^2$, (iv) a^5, (v) a^2c^2, (vi) $10s^2t^3$, (vii) x^3, (viii) t^3, (ix) l^6, (x) $3c^5$, (xi) $3a^4$, (xii) $\frac{5}{3}c^5$.

3. (i) $10c^4$, (ii) $2t^2$, (iii) $8x^2y^2$, (iv) r^3, (v) $6r^2$, (vi) $1+r+r^2$, (vii) u^4+5u^2 (viii) $\frac{u^3}{2}$, (ix) $10t$, (x) $4k^3$, (xi) $81s^4$, (xii) $9s^4$.

4. $\frac{1}{16}c^4$, $\frac{1}{4}a^8$, $\frac{z^2}{9}$, $\frac{9}{x^6}$, $9s^2t^4$, $81s^4t^4$, l^{10}, $a^4b^6c^8$.

5. $\pm\frac{a}{4}$, $\pm 5c$, $\pm\frac{3x}{2yz}$, $\pm\frac{x}{3t}$, $\pm R^5$, $\pm\frac{CD^2}{E^2F}$.

6. $125t^3$, $-8k^3$, $-\frac{27s^3}{64}$, $\frac{8}{a^9}$, $-b^{12}$, $27r^3s^6t^3$.

7. $3m$, $-\frac{a}{5}$, $-2z^2$, $\frac{UV}{2}$, $-4M^4$, M^5.

8. (i) $28a^4r^4s^2$, (ii) $15c^3$, (iii) $135c^3$, (iv) $24a^6b^4c^2$, (v) $-abc^4d^5$, (vi) $18a^6b^5c$, (vii) $21a^5b^6c^4$, (viii) $-10a^3b^2l^5$, (ix) $-15a^4t^9$, (x) $42a^{10}b^9$, (xi) z^{18}, (xii) c^{10}.

Exercise 24 c (P. 106)

1. (i) $4t^3$, (ii) x^{10}, (iii) $3t^3$, (iv) $-2t^2$, (v) x^2y^3z, (vi) $-x^3yz^2$, (vii) x^4y^3z, (viii) x^2y^4z, (ix) $2lmn$, (x) $\frac{2x}{3y}$, (xi) $\frac{4}{7t^2}$, (xii) $\frac{8a^3}{5bc^2}$, (xiii) x^4, (xiv) $\frac{1}{9}$, (xv) $\frac{1}{8}$, (xvi) $-\frac{l^2m}{n}$, (xvii) $6ax^4$, (xviii) $9a^3b^5c^8$.

2. (i) $5a^3b^2c$, (ii) $3a^2x^5$, (iii) $-t^{10}$, (iv) $2K^6$, (v) $-x^3y^4$, (vi) $-\frac{6t^6}{5s^3}$, (vii) $\frac{8t^2}{9c^5}$, (viii) x^3.

3. (i) $12x^7$, (ii) $3x^4+4x^3$, (iii) $\frac{3x}{4}$, (iv) $3x^4-4x^3$, (v) $2t^{12}$, (vi) 1, (vii) t^{24}, (viii) 0, (ix) $13x^7$, (x) $40x^{14}$, (xi) $\frac{8}{5}$, (xii) $3x^7$, (xiii) $32a^5$, (xiv) $-2a^5$,

(xv) $-32a^5$, (xvi) $2a^5$, (xvii) $\frac{3x^2}{2y^2}$, (xviii) $25x^2y^2$, (xix) $\frac{5}{2y}$, (xx) $\frac{4}{xy}$, (xxi) $\frac{1}{x^2}$.

4. (i) $7a^5b^2$, (ii) $-5b^3c^2$, (iii) $6ab^5c^4$, (iv) (a) 2, (b) 3, (c) 7, (d) 100, (v) $\pm 8a^9b^6$, $\pm 2a^3b^2$, (vi) $4a^6b^4$, $\pm 2a^3b^2$.

Exercise 25 A (Pp. 107, 108)

1. $4a+16b-12c$. **2.** $-15a+35b+10c$. **3.** $2x^2y+2x^3y^2-2x^4y^3$.
4. $-4a^2b+6ab^2+10b^3$. **5.** $30a^5-18a^4-36a^3+12a^2$.
6. $-12lm^3+9lm^4+21lm^5$. **7.** $-s^4t^4+s^3t^6-2s^2t^2$.
8. $-3a^4b+9a^3b^2-9a^2b^3+3ab^4$. **9.** $a-3b$. **10.** $-a+5b$.
11. $5x-7$. **12.** $-2x+9$. **13.** $2a-6b$. **14.** $2a-6b$.
15. $-3x^2+7y^2$. **16.** $-a+b-c$. **17.** $7x-11$. **18.** $11x-13$.
19. $2c^2-4d^2$. **20.** a^2-5ab.
21. $-a^4b-ab^4+a^2b^3-a^3b^2$. **22.** $-4a+6b+9c$.

Exercise 25 B (P. 108)

1. $6a+42b-30c$. **2.** $-6a+15b-9c$. **3.** $3a^3+6a^2b+3ab^2$.
4. $6xy^2+9x^2y^3-12x^3y^4$. **5.** $2a^4b+3a^5b-5a^6b$.
6. $40x^5-25x^4+20x^3-35x^2$. **7.** $-2a^4b+12a^3b^2-24a^2b^3+16ab^4$.
8. $-a^5b^6+a^6b^5-a^7b^4$. **9.** $2a-6b$. **10.** $-3a+12b$.
11. $-9x+5$. **12.** $7x-8$. **13.** $3a-12b$. **14.** $-2a+7b$.
15. $15x-9$. **16.** $-4a+3b-c$. **17.** $17x-23$. **18.** $-2x^2+7y^2$.
19. $2x^2-7xy$. **20.** $-2d^3+3c^2d$.
21. $-3a+2b+4c$. **22.** $-x^2-y^2+xy$.

Exercise 26 A (Pp. 110, 111)

1. ab. **2.** $3a^2$. **3.** xy. **4.** $2lm^2$. **5.** 8. **6.** $2ab$.
7. ab. **8.** xy. **9.** $3a^2$. **10.** x^2. **11.** $3l$. **12.** c.
13. x^2y. **14.** x^2yz. **15.** $2x^3y$. **16.** $6a^2b^2$. **17.** $15a^3b^4$. **18.** $10l^2mn$.
19. $60xyz$. **20.** $12a^2b^2$. **21.** $8r^2s^2$.
22. $60a^3b^3c^3$. **23.** $120t^4z^3$. **24.** $20u^4v^3$.

Exercise 26 B (P. 111)

1. xy. **2.** ab. **3.** $5x^3$. **4.** $2xy^2$. **5.** $3xy$. **6.** 11.
7. a^2b. **8.** $2l^2$. **9.** $8c^2$. **10.** z. **11.** x^2. **12.** $4b$.
13. a^3b^2. **14.** $3t^3u^2$. **15.** cd^2. **16.** $21a^2bc$. **17.** $20x^3y^2$.
18. $21x^2y^6$. **19.** $30lmn$. **20.** $24a^2b^2c^2$. **21.** $24x^2y^2$.
22. $36a^4$. **23.** $12a^3b^3$. **24.** $24x^3y^2z^4$.

EXERCISE 26 C (P. 111)

1. $a^2l^2m^2$. **2.** $7l^2mn^3$. **3.** $5ab$. **4.** x^4y^4. **5.** $7x^2$. **6.** $60k$.
7. $60x^2y^2z^2t$. **8.** $2lm^3n$. **9.** $60k^5$. **10.** $60a^3b^3c^3$.
11. $11x^2z^3$. **12.** $7xyz^2$. **13.** H.C.F. $5bc$, L.C.M. $75ab^3c^2$.
14. H.C.F. y, L.C.M. $28x^2y^3z^2$. **15.** H.C.F. az, L.C.M. $a^3xy^2z^3$.
16. H.C.F. $2a^2d^2$, L.C.M. $120a^5b^3c^2d^5$.

EXERCISE 27 A (P. 114)

1. 3. **2.** $\frac{3y}{4z}$. **3.** $\frac{3a}{b}$. **4.** $\frac{c^2}{3}$. **5.** l. **6.** $\frac{a}{3b}$.
7. $\frac{3}{2l}$. **8.** 1. **9.** $\frac{1}{2y^2}$. **10.** $\frac{2}{3a^4}$. **11.** $\frac{b^2}{c}$. **12.** $\frac{3}{c^2d}$.
13. $\frac{3lm}{4n}$. **14.** $\frac{xy^4}{3}$. **15.** $\frac{ac}{b}$. **16.** $\frac{3y^2}{z}$. **17.** $\frac{2ac}{3bd}$. **18.** $\frac{2ad}{bc}$.
19. 1. **20.** lm. **21.** $\frac{3}{2}$. **22.** $2xy$. **23.** $3xz$. **24.** $\frac{a^3}{c}$.
25. $\frac{1}{z^7}$. **26.** $\frac{4y}{3x}$. **27.** $\frac{a^{10}}{2}$. **28.** x^3y.

EXERCISE 27 B (P. 114)

1. k. **2.** $\frac{2s^2}{9}$. **3.** 1. **4.** $\frac{x^4}{4}$. **5.** $\frac{a}{2c}$. **6.** $\frac{l}{2m}$.
7. $\frac{1}{3m^3}$. **8.** $6ab$. **9.** $\frac{1}{2xy}$. **10.** $\frac{3}{2d^2}$. **11.** $\frac{5}{6c^2}$. **12.** $\frac{m^4}{l}$.
13. $\frac{3a}{b^2}$. **14.** $\frac{yz}{x}$. **15.** $\frac{r^2s^3}{5}$. **16.** $\frac{rv}{6su}$. **17.** $\frac{5t^3}{v}$. **18.** $\frac{3xy}{4tz}$.
19. a^2b^2. **20.** $\frac{5t}{2u}$. **21.** 1. **22.** $7bc$. **23.** $\frac{x^4}{yz}$. **24.** $3yz$.
25. ab^3. **26.** $\frac{x^8}{2}$. **27.** $\frac{1}{3x^7}$. **28.** $\frac{5c^3}{6a^2}$.

EXERCISE 27 C (P. 115)

1. $\frac{l}{c}$. **2.** $\frac{4mx}{5y}$. **3.** $\frac{z}{3cx}$. **4.** $\frac{3v^2}{20a^3b}$. **5.** $\frac{21d^4m}{c^4l}$. **6.** $\frac{sz}{r^2x}$.
7. $\frac{3abc}{2l^2}$. **8.** $\frac{3rQ}{4s^2P}$. **9.** $\frac{xy}{l}$. **10.** $\frac{2b^2x}{3a}$. **11.** $\frac{t^2y^2}{2z}$.
12. $\frac{c^2}{3b^2}$. **13.** $\frac{9b^2}{2a^2d}$. **14.** $\frac{a}{6cx}$. **15.** $\frac{5z^3}{7ty}$. **16.** $\frac{14m}{45dl}$.

EXERCISE 28 A (P. 117)

1. (i) $\frac{3}{4}$, (ii) $\frac{3a}{4}$, (iii) $\frac{3}{4x}$, (iv) $\frac{3a}{4x}$. **2.** (i) $\frac{1}{6}$, (ii) $\frac{a}{6}$, (iii) $\frac{1}{6x}$, (iv) $\frac{a}{6x}$.

3. (i) $\frac{5}{8}$, (ii) $\frac{5k}{8}$, (iii) $\frac{5k}{8l}$.

4. (i) $\frac{13}{8}$, (ii) $\frac{8+5k}{8}$, (iii) $\frac{8l+5}{8l}$, (iv) $\frac{8l+5k}{8l}$.

5. (i) $\frac{11}{7}$, (ii) $\frac{14-3t}{7}$, (iii) $\frac{14t-3}{7t}$, (iv) $\frac{14y-3x}{7y}$.

6. (i) $\frac{5}{7}$, (ii) $\frac{14-3a}{7}$, (iii) $\frac{14b-3}{7b}$, (iv) $\frac{14b-3a}{7b}$.

7. (i) $\frac{c+d}{cd}$, (ii) $\frac{d-c}{cd}$, (iii) $\frac{c^2+c+1}{c}$, (iv) $\frac{c^2+c+1}{c^2}$.

8. (i) $\frac{2a+3}{15}$, (ii) $\frac{3-a}{6}$, (iii) $\frac{3-5a}{6}$, (iv) $\frac{6b-11a}{6}$.

9. (i) $\frac{3c^2+4c+5}{c^2}$, (ii) $\frac{x^2-2x+1}{x}$, (iii) $\frac{x^2-2xy+y^2}{xy}$, (iv) $\frac{9x^2+18x+1}{3x}$.

10. (i) $\frac{5-a}{3}$, (ii) $\frac{13+x}{3}$, (iii) $\frac{19-2z}{2}$, (iv) $\frac{19-6z}{15}$.

11. (i) $\frac{2t-2}{3}$, (ii) $\frac{7t-15}{12}$, (iii) $\frac{9-4t}{6}$, (iv) $\frac{t-24}{20}$.

12. (i) $\frac{7-3x}{4}$, (ii) $\frac{19-6x}{4}$, (iii) $\frac{5-7x}{6}$, (iv) $\frac{37-14x}{12}$.

EXERCISE 28 B (P. 118)

1. (i) $\frac{2}{15}$, (ii) $\frac{2c}{15}$, (iii) $\frac{2}{15d}$, (iv) $\frac{2c}{15d}$.

2. (i) $\frac{22}{35}$, (ii) $\frac{22k}{35}$, (iii) $\frac{22}{35l}$, (iv) $\frac{22k}{35l}$. **3.** (i) 1, (ii) a, (iii) $\frac{a}{X}$.

4. (i) $\frac{13}{5}$, (ii) $\frac{15-2u}{5}$, (iii) $\frac{15v-2}{5v}$, (iv) $\frac{15v-2u}{5v}$.

5. (i) $\frac{22}{9}$, (ii) $\frac{18+4h}{9}$, (iii) $\frac{18k+4}{9k}$, (iv) $\frac{18k+4h}{9k}$.

6. (i) $\frac{29}{9}$, (ii) $\frac{36-7x}{9}$, (iii) $\frac{36y-7}{9y}$, (iv) $\frac{36y-7x}{9y}$.

7. (i) $\frac{4c^2+7c-2}{c^2}$, (ii) $\frac{3t-t^2-4}{t}$, (iii) $\frac{4a^2-30ab-9b^2}{6ab}$,
(iv) $\frac{8l^2-20l-1}{4l}$.

8. (i) $\frac{2n-3m}{mn}$, (ii) $\frac{5n+4m}{mn}$, (iii) $\frac{d^2+15d-25}{5d}$, (iii) $\frac{6w^2+w+8}{2w^2}$.

9. (i) $\frac{3x+10}{40}$, (ii) $\frac{16-3x}{40}$, (iii) $\frac{15-11x}{15}$, (iv) $\frac{15y-26x}{15}$.

10. (i) $\frac{17-8a}{5}$, (ii) $\frac{14-3a}{5}$, (iii) $\frac{-4a-1}{12}$, (iv) $\frac{9-14a}{12}$.

11. (i) $\frac{19-3a}{5}$, (ii) $\frac{23-t}{2}$, (iii) $\frac{27-5K}{7}$, (iv) $\frac{13-25K}{30}$.

12. (i) $\frac{3m+6}{4}$, (ii) $\frac{2m+28}{15}$, (iii) $\frac{1}{10}$, (iv) $\frac{2m-56}{63}$.

Exercise 29 A (P. 121)

1. 10. **2.** 15. **3.** −3·5. **4.** −12. **5.** 1. **6.** −11.
7. −7. **8.** 11. **9.** 18. **10.** 4. **11.** −2. **12.** 3.
13. 12. **14.** 2. **15.** 5. **16.** −1. **17.** −16. **18.** −2.

Exercise 29 B (Pp. 121, 122)

1. 3·5. **2.** 48. **3.** −6. **4.** −5. **5.** −4. **6.** 12.
7. 9. **8.** −1. **9.** −1. **10.** −7. **11.** −9. **12.** 3.
13. 1·5. **14.** −8. **15.** 9. **16.** −9. **17.** −17. **18.** −9.

Exercise 30 A (Pp. 123, 124)

1. 60. **2.** 28. **3.** 45. **4.** 5 miles. **5.** 6.
6. 33 miles. **7.** 3·5 miles. **8.** 7·5 miles.
9. £20 at $3\frac{1}{2}$ per cent., £40 at 5 per cent. **10.** 36 miles.
11. £75 at 6 per cent., £110 at 5 per cent. **12.** 960.

Exercise 30 B (Pp. 124, 125)

1. 72. **2.** 84. **3.** 55. **4.** 33 miles. **5.** 9.
6. 10 miles. **7.** £60 at 3 per cent., £40 at 5 per cent. **8.** $2\frac{5}{8}$ miles.
9. 45 miles. **10.** 360. **11.** 400.
12. £160 at $2\frac{1}{2}$ per cent., £180 at $4\frac{1}{2}$ per cent.

Exercise 30 C (Pp. 125, 126)

1. 6 lb. **2.** 12s. **3.** 110 **4.** 560. **5.** 32.
6. 264. **7.** 2s. 10d. per lb. **8.** 132. **9.** 240.
10. 252. **11.** 48. **12.** £1 7s.

Test Papers III (Pp. 126, 127, 128, 129, 130, 131, 132, 133, 134)

A. 1. (i) 0, (ii) $3x$, (iii) $3x^2$. **2.** (i) −24, (ii) 72, (iii) −32.
3. H.C.F. $5ab^2$, L.C.M. $525a^3b^3c^2$. **4.** $-2p+4q$.
5. (i) $-\frac{2}{7}$, (ii) −4. **6.** 16.

B. **1.** (i) -21, (ii) -9, (iii) -212. **2.** (i) $3n$, (ii) $4x$, (iii) $-\frac{1}{2}xy$.
3. (i) $7-4x-3x^2$, (ii) $12x-y-z$.
4. Yes; 41·1, 31·9, 24·8 years. **5.** (i) -4, (ii) $-\frac{1}{2}$. **6.** $\frac{4}{5}$ mile.

C. **1.** (i) $5x^2+26xy-10y^2$, (ii) $2b^4$.
2. $y=ux$. (i) 5 hours after leaving P he is 150 miles N. of P.
(ii) 3 hours after leaving P he is 90 miles S. of P.
(iii) $\frac{1}{2}$ hour before reaching P he is 25 miles S. of P.
(iv) $\frac{1}{4}$ hour before reaching P he is 10 miles N. of P.
3. (i) -1, (ii) $1\frac{1}{3}$. **4.** $\left(\frac{25}{2}a-\frac{15}{2}b+2c\right)$ shillings.
5. (i) $\frac{15y-2x}{5x^2y}$, (ii) $\frac{xy-x^2}{y^2}$. **6.** (i) -78, (ii) -198, (iii) 38.

D. **1.** (i) $8a$, (ii) $x-y-2$. **2.** (i) $6b$, (ii) $13x-5z$.
3. (i) 13, (ii) 11, (iii) 289. **4.** (i) $\frac{9}{10}$, (ii) -3.
5. 60 gallons. **6.** (*a*) No, (*b*) 477 lb.

E. **1.** (i) $6a$, (ii) -3, (iii) $-16a^2$. **2.** (i) $11-6x-3x^2$, (ii) $4y+6xy$.
3. (i) 14, (ii) -31. **4.** (i) $-4\frac{7}{8}$, (ii) $-\frac{3}{7}$.
5. (i) $-9x^2+9x-9$, (ii) $-5x^4-5x+17$ **6.** $43\frac{3}{4}$ miles.

F. **1.** (i) $\frac{17x}{30}$, (ii) y^6, (iii) $\frac{2x+y}{x}$. **2.** (i) -, (ii) $-\frac{1}{5}$, (iii) $1\frac{2}{3}$.
3. $4a^3-14a^2b-6ab^2+10b^3$, $-6b^2$, -6, 6. **4.** $5b-a-8c$.
5. $\frac{73}{121}$. **6.** 16.

G. **1.** $6y$. **2.** H.C.F. $=8a^2b$, L.C.M. $=336a^3b^2c^2$.
3. (i) $2a-2x$, (ii) $2+k$. **4.** (i) 7, (ii) -15.
5. After $4\frac{1}{2}$ hours. **6.** (i) 6·45 c. in., (ii) 21·7 lb. per sq. in.

H. **1.** (i) $2a-7b+2c$, (ii) $7l^3mn^7$. **2.** (i) $8\frac{1}{2}$, (ii) $9\frac{1}{4}$, (iii) $15\frac{1}{4}$, (iv) $6\frac{1}{4}$.
3. H.C.F. $3axy^2$, L.C.M. $36a^2x^3y^3$. **4.** $1\frac{1}{2}$ hours.
5. $-1\frac{1}{3}$. **6.** $37rs$ pence, $(120-37rs)$ pence.

I. **1.** $(3a-b+15c)$ shillings). **2.** (i) $\frac{1}{4}$, (ii) $-12s^2$, (iii) $-\frac{1}{81}$, (iv) $246c^5$.
3. (i) $12y-3x-z$, (ii) $7-7x-7x^2$. **4.** (i) $1\frac{1}{4}$, (ii) 0.
5. 12 geese, 90 rabbits. **6.** (i) 11·5″, (ii) 3·7 gallons.

J. **1.** (i) $29a-50b$, (ii) $\frac{1}{x^2}$, (iii) $\frac{x-1}{x}$. **2.** $13y-48x$.
3. $20x+12y+24z$. **4.** (i) 2, (ii) $39\frac{2}{3}$. **5.** (i) -1, (ii) 13.
6. 4 miles.

K. **1.** (i) $9a^4b^2$, $4x^2y^4$, (ii) $-5x^3-7x^2+11x+8$. **2.** (i) -35, (ii) -18.
3. (i) $\frac{8}{11c^4}$, (ii) $\frac{3}{8r^3s^2}$, (iii) $\frac{25a-36b}{30a^2}$.

4. (i) 169, (ii) -19, (iii) $\frac{16}{81}$, (iv) $\frac{8}{27}$, (v) -65.
5. C £18, D £22. **6.** (i) 7,875 lb., (ii) 63·5 m.p.h.

L. 1. (i) $\frac{4l^2}{3m}$, (ii) $\frac{b^3}{ac^2}$, (iii) $-\frac{1}{3x}$.
2. (i) H.C.F. ab, L.C.M. $14a^2b^2x^2$, (ii) $12a^2-28ab+9b^2$, 113.
3. (i) $3a+3b+2c$, (ii) $-2ab$. **4.** $3\frac{1}{2}$ hours.
5. (i) $-\frac{5}{8}$, (ii) $\frac{3}{37}$. **6.** 65.

M. 1. (i) $Q-6P-4R$, (ii) $2c-d+3e$.
2. H.C.F. $9n^2$, L.C.M. $2268m^3n^4$. **3.** (i) abc, (ii) l^2+3m^2.
4. (i) 0·67 ft., (ii) 0·87 ft. **5.** X 9s. 2d., Y 2s. 10d., Z 4s. 7d.
6. 4.

N. 1. (i) $(36a-85b+62c)$ pence, (ii) $\left(s-\frac{t}{2}\right)$ feet.
2. (i) $3s^2-22st+2t^2$, (ii) $6z-2x$. **3.** 120. **4.** (i) -2, (ii) $\frac{1}{3}$.
5. (i) $7xy^2z^3$, (ii) $12x^4y^4$. **6.** (i) $-\frac{5}{24}$, (ii) $3-25n$.

P. 1. (i) $2p+2q-8$, (ii) $9-2t-t^2+t^3+t^4+t^5$, -1. **2.** $2x+y$.
3. (i) 2, (ii) $\frac{3}{5}$. **4.** 1st horse £44, 2nd horse £42.
5. (i) $-x^4$, (ii) $-3x^{10}$, (iii) $-2a^3c^6$. **6.** 255 sq. in.

Q. 1. (i) $2a+3b+5c$, (ii) £$\left(\frac{6b-5a}{20}\right)$.
2. (i) $-3t^5$, (ii) $-3x^4$, (iii) $\frac{4x^2z}{5y^2}$.
3. (i) H.C.F. xy, L.C.M. $72a^2p^2x^2y^3$, (ii) $-4a^3b^4$, (iii) 20.
4. (i) -3, (ii) $21\frac{1}{23}$. **5.** Father 58 years, X 30 years, Y 28 years.
6. (i) $\frac{7c-165}{18}$, (ii) $a+43b$.

PART II

EXERCISE 31 A (P. 138)

(The values of the unknowns are given in alphabetical order)

1. 11, 3. **2.** 12, 12. **3.** 3, 2. **4.** 3, 2. **5.** 7, 9.
6. 2, 5. **7.** 12, 15. **8.** -4, 1. **9.** 3, 2. **10.** 12, 4.
11. $1\frac{1}{4}$, $1\frac{1}{2}$. **12.** 9, $1\frac{1}{2}$.

EXERCISE 31 B (P. 138) (See note at head of Ex. 31 a)

1. 14, 3. **2.** 6, 1. **3.** 7, 7. **4.** 2, 4. **5.** 4, 3.
6. 5, 1. **7.** 2, -4. **8.** 9, 1. **9.** -5, 1. **10.** 8, 2.
11. $1\frac{1}{2}$, $3\frac{1}{2}$. **12.** 5, -2.

EXERCISE 32 A (P. 141) (See note at head of Ex. 31 a)

1. 2, 3. **2.** -3, 5. **3.** 7, 2. **4.** -2, -3. **5.** $-\frac{1}{2}$, 3.
6. 3, -4. **7.** $1\frac{1}{3}$, $3\frac{1}{2}$. **8.** $1\frac{1}{2}$, -10. **9.** $1\frac{1}{4}$, $2\frac{1}{3}$. **10.** $\frac{2}{5}$, $-1\frac{2}{3}$.
11. $\frac{2}{3}$, $\frac{1}{2}$. **12.** $\frac{2}{5}$, $-\frac{3}{5}$. **13.** 3, -1. **14.** 25, 9. **15.** $2\frac{2}{3}$, $-\frac{8}{9}$.

EXERCISE 32 B (P. 142) (See note at head of Ex. 31 a)

1. 4, 5. **2.** $3\frac{1}{2}$, $5\frac{1}{2}$. **3.** 5, 1. **4.** -9, -2. **5.** 12, -15.
6. 5, -3. **7.** $-1\frac{1}{2}$, $1\frac{1}{3}$. **8.** $-\frac{2}{3}$, $-1\frac{1}{2}$. **9.** $-\frac{2}{3}$, $2\frac{1}{2}$.
10. $\frac{1}{2}$, $-\frac{2}{3}$. **11.** $-\frac{4}{5}$, $\frac{1}{3}$. **12.** $\frac{1}{2}$, $-\frac{3}{4}$. **13.** $1\frac{2}{3}$, $-1\frac{1}{2}$. **14.** 2, $-\frac{1}{4}$.
15. 1, -3.

EXERCISE 32 C (Pp. 142, 143) (See note at head of Ex. 31 a)

1. 3, -5. **2.** -2, 3. **3.** $-\frac{1}{2}$, 3. **4.** $1\frac{2}{3}$, $5\frac{1}{2}$. **5.** 8, 2.
6. -2, -5. **7.** 16, -4. **8.** 3, 0. **9.** $\frac{1}{3}$, $-\frac{1}{5}$. **10.** 15, $6\frac{1}{2}$.
11. -3, $2\frac{1}{2}$. **12.** $-8\frac{1}{2}$, $\frac{2}{3}$. **13.** $1\frac{1}{3}$, $-\frac{1}{4}$. **14.** $9\frac{1}{2}$, $\frac{2}{19}$. **15.** 1·4, 1·2.
16. 3, 7. **17.** 2·14, 0·63. **18.** 1·90, 1·41. **19.** -4·25, -1·32.
20. 3·87, 1·44. **21.** -0·28, 0·81. **22.** -24·71, 24·14. **23.** 6.
25. 0. **26.** 2, -9. **28.** 3, 13. **30.** 2, -5; -11, $5\frac{1}{2}$.
32. 3, -7; 3, -16.

EXERCISE 33 A (Pp. 144, 145)

1. 63, 29. **2.** 45, 36. **3.** Cow £18 ; Sheep £5.
4. Table £7 10s. ; Chair £1 10s. **5.** 5d.
6. Tea 2s. 6d. ; Coffee 2s. **7.** 36. **8.** 27. **9.** 45.
10. A 43 years, B 36 years. **11.** 43 years, 8 years.
12. 43 years, 8 years, 6 years. **13.** $\frac{4}{7}$. **14.** $\frac{9}{16}$.
15. $a=1\frac{1}{2}$, $b=4\frac{1}{2}$. **16.** 12, 190.

EXERCISE 33 B (Pp. 145, 146)

1. 73, 22. **2.** 55, 42. **3.** Horse £25 ; Cow £14.
4. Tea 1s. 6d. ; Coffee 2s. 4d. **5.** $\frac{3}{4}$d.
6. Apples, 4d. ; Plums 3d. **7.** 95. **8.** 35. **9.** 27.
10. A 16 years, B 5 years. **11.** 43 years, 7 years. **12.** 45 years.
13. $\frac{9}{13}$. **14.** $\frac{11}{13}$. **15.** X 105°, Y 60°, Z 15°.
16. A £20, B £48.

EXERCISE 33 C (Pp. 146, 147)

1. A 15s., B 18s. **2.** P 105°, Q 35°, R 40°.
3. 12 shillings, 4 half-crowns. **4.** 30 gallons. **5.** $2\frac{1}{2}$ hours.
6. 23, 277. **7.** $l=35$, $m=1\frac{1}{4}$. **8.** 8d. **9.** £15 10s.
10. $4\frac{1}{2}$ m.p.h., $3\frac{3}{4}$ m.p.h. **11.** 40 apples, 20 pears.
12. 42 pears, 72 bananas. **13.** $1\frac{1}{4}$ hours. **14.** 36 eggs, 48 oranges.
15. 3d., 5d. **16.** 80.

EXERCISE 34 A (Pp. 154, 155)

Some of the following answers have been obtained by calculation, and have been given to a degree of accuracy which cannot be expected from the graphs.

5. (i) 400, 200, $133\frac{1}{3}$, 100, 80, $66\frac{2}{3}$, $\frac{200}{t}$ m.p.h. ; (iii) 76·9 ; (v) 1·8 ; (viii) (a) very small, (b) very large.
6. (i) 653, 114 ; (ii) 2·3, 4·6 ; (iii) 655, 3·4 ; (iv) 1·8, 5·3.
7. (i) −5·3, 0·42. −15·2 ; (ii) 3·6, −0·6 ; (iii) $\frac{1}{2}$; (iv) 2·8 or 0·2, 4·46 (Yes, −1·46), no solutions (No).
8. (i) 2·6 and 0·4 ; (ii) 3·8 and −0·8 ; (iii) 1·5 and 1·5 ; (iv) 4·7 (−1·7 is outside the range).
9. (i) 2·3 and −0·8 ; (ii) $\frac{3}{4}$ and $\frac{3}{4}$; (iii) 2·8 and −1·3 ; (iv) 1·9 and −0·4.
10. 3·3, −2·8. **11.** No solutions. **12.** 2, −1·3.
13. 0·24, −0·84. **14.** 1·85, −1·35. **15.** −3·4, −0·6.

Exercise 34 b (Pp. 155, 156) (See note at head of Ex. 34 a)

5. (i) 29·4, 98·6, −16·6 ; (ii) 4·87 and 0·13, 3·29 and 1·71, 5·15 and −0·15 ; (iii) 150 ft. above the ground, 2·5 sec. ; (iv) 5·56 sec. ; (v) 5·05 and −0·05, $t=5\cdot05$ and −0·05 ; (vi) 4·2 sec. approx.

6. (i) 23, −0·95, 15 ; (ii) 0, 3 ; (iii) $-1\frac{1}{4}$; (iv) 2·86, 0·14 ; no solutions ; 5·6 (Yes, −2·6).

7. (i) 940π c.c., (ii) 4·8 cm., (iii) $1333\frac{1}{3}\pi$ c.c., 20 cm., (iv) 7·2. No.

8. (i) ±2·74, (ii) ±3·87, (iii) ±1·87, (iv) ±2·74.

9. (i) −1·9, 1·5, (ii) −0·86, 0·46, (iii) −0·2 and −0·2, (iv) 0·29, −0·69.

10. 1·7, −7·7. **11.** −2, 2·6. **12.** 0·2, −1·2.

13. 2·6, −2·1. **14.** −2·4, −0·1. **15.** 1·32, −0·57.

Exercise 34 c (Pp. 156, 157) (See note at head of Ex. 34 a)

1. 3·45, −1·45. **2.** (2, 0·4) ; (−1·3, 1·3).

3. (*a*) −2·14, 1·64 ; (*b*) from $x=-2$ to 1·5.

4. (*a*) From $x=-1\cdot87$ to 0·54 ; (*b*) from $x=-2$ to 0; (*c*) −2·52, 1·19.

5. −2·686, 0·186. **6.** Between 2·86 and −1·19.

7. (*a*) 5·36, (*b*) 6, (*c*) between −1·45 and 3·45.

8. (i) ±1·90, (ii) from $x=-2$ to −0·78 and from $x=1\cdot28$ to 2.

9. 0·79 and −3·79. **10.** (3·37, 1·495) ; (0·83, −0·535).

11. $9\frac{1}{4}$; between 0·54 and −5·54.

12. Negative between $x=-2\cdot67$ and $x=1\cdot87$; −5·16.

13. 6·46, −0·46 ; 12. **14.** $A=x^2+(6-x)^2$; 0·88″, 5·12″.

16. 3·46 ft.

Exercise 35 a (P. 162)

7. $\frac{3}{4}$. **8.** $-\frac{2}{5}$. **9.** 0. **10.** $\frac{3}{5}$. **11.** 0. **12.** $-\frac{m}{6}$.

13. $\frac{4}{3}$. **14.** −2. **15.** $\frac{3}{5}$. **16.** 0. **17.** −3. **18.** $-\frac{3}{5}$.

Exercise 35 b (P. 163)

7. $-\frac{3}{4}$. **8.** $\frac{5}{6}$. **9.** $-\frac{4}{5}$. **10.** 0. **11.** 0. **12.** $\frac{a}{2}$.

13. $\frac{3}{4}$. **14.** $-\frac{2}{5}$. **15.** 0. **16.** $-\frac{3}{11}$. **17.** $-\frac{2}{3}$. **18.** $-\frac{1}{2}$.

Exercise 36 a (Pp. 167, 168)

1. $16\frac{2}{3}$ miles from *A*, at 12.35 p.m.

2. 36 miles from the starting point at 1.24 p.m. **3.** 54 miles.

4. After $2\frac{1}{4}$ hours ; after $2\frac{1}{8}$ hours and again after $2\frac{3}{8}$ hours.

5. (*a*) 100 miles, (*b*) 4·3 min., (*c*) 5·3 miles, (*d*) 74 m.p.h.

6. 15° C., 113° F. **7.** 10 m.p.h., 8 m.p.h., 11.30 a.m.
8. 390 units. **9.** 10 miles. **13.** 40, 61.
14. $8\frac{1}{9}$ miles from A at $12.1\frac{2}{3}$ p.m.

Exercise 36 b (Pp. 168, 169)

1. 11 miles at $11.22\frac{1}{2}$ a.m. **2.** 50 miles.
3. (i) 10 miles from the start, (ii) 5 m.p.h., (iii) 2 hours and again 3 hours after the start.
4. 36 miles. **5.** (i) £104, (ii) £329.
6. 7 ft. per sec., 59 ft. per sec.; 14 m.p.h., 33 m.p.h.
7. (*a*) About 55 min., (*b*) about 4·6 Km., (*c*) about 19 Km. per hr.
8. 100, 72. **9.** 20 m.p.h. **13.** 41 miles.

Exercise 37 a (Pp. 170, 171)

1. $-8x^4+6x^3+14x^2-2x$. **2.** $-4t^2+10t+6$.
3. $3a^3b-5a^2b^2-2ab^3$. **4.** $-6x^4y+9x^3y^2+15x^2y^3-9xy^4$.
5. $5x^2-36x-32$. **6.** $25t^2-4$. **7.** $10x^2-48x+54$.
8. $3x^2+2xy-y^2$. **9.** $3a^2-7ab-6b^2$. **10.** $6-11u+3u^2$.
11. $a^2-b^2c^2$. **12.** $2a^4-a^2b^2-10b^4$. **13.** $6x^4-x^2y^2-12y^4$.

Exercise 37 b (P. 171)

1. $8x^3-6x^2-10x$. **2.** $-15l^2+35l-5$. **3.** $-2c^3d+c^2d^2-3cd^3$.
4. $-6c^4d+4c^3d^2-8c^2d^3+2cd^4$. **5.** $15x^2+x-6$.
6. $3x^2-25x+28$. **7.** $4t^2-49$. **8.** $5a^2+7ab-6b^2$.
9. $2x^2-5xy+3y^2$. **10.** $15-13u+2u^2$. **11.** $x^2-4y^2z^2$.
12. $10x^4-11x^2y^2+3y^4$. **13.** $3a^4-7a^2bc-6b^2c^2$.

Exercise 37 c (P. 171)

1. $x^3-7x^2+13x-4$. **2.** c^3+27. **3.** $x^3-3x^2y+3xy^2-y^3$.
4. $18c^3-27c^2d+16cd^2-4d^3$. **5.** $9s^3-36s^2t+47st^2-20t^3$.
6. $4-5x-23x^2+6x^3$. **7.** $2a^4-7a^3+15a^2-11a+10$.
8. $15x^4-18x^3-16x^2+30x-15$.
9. $-16t^5+24t^4-30t^3+21t^2+4t-3$.
10. $21x^4-23x^3-23x^2+11x-10$. **11.** $5x^3+17x^2-9x-36$.
12. $3a^2+7ab+b^2$. **13.** $3x^2-8x+15$. **14.** x^3+y^3.

Exercise 38 a (P. 173)

1. x^2+5x+6. **2.** $t^2+4t-21$. **3.** $t^2+3t-10$.
4. $c^2-4c-21$. **5.** $l^2-3l-10$. **6.** $d^2-10d+21$.
7. $b^2-6b-55$. **8.** $km+9k-8m-72$. **9.** $tw+5t-5w-25$.

10. $n^2 - 10n + 25$. **11.** $6xy - 9x + 10y - 15$. **12.** $6x^2 - 19x + 15$.
13. $6x^2 - x - 15$. **14.** $12c^2 + 7cd - 10d^2$. **15.** $35s^2 - 11st - 10t^2$.
16. $6l^2 - 31lm + 18m^2$. **17.** $35s^2 - 39st + 10t^2$. **18.** $12c^2 - 23cd + 10d^2$.
19. $35s^2 + 11st - 10t^2$. **20.** $6l^2 + 23lm - 18m^2$. **21.** $6l^2 - 23lm - 18m^2$.
22. $35s^2 + 39st + 10t^2$. **23.** $12c^2 - 7cd - 10d^2$. **24.** $6a^4 + 5a^2b^2 - 6b^4$.
25. $2a^4 - 7a^2bc + 5b^2c^2$. **26.** $6a^4 - 5a^2b^2 - 6b^4$. **27.** $4a^4 + 4a^2bc - 15b^2c^2$.
28. $6a^4 - 11a^2bc - 10b^2c^2$. **29.** $-2a^4 + 5a^2bc + 25b^2c^2$.
30. $6a^4 - 13a^2b^2 + 6b^4$.

Exercise 38 b (P. 173)

1. $x^2 + 10x + 21$. **2.** $x^2 + 3x - 10$. **3.** $a^2 + 3a - 18$.
4. $y^2 - 3y - 10$. **5.** $t^2 - 3t - 18$. **6.** $x^2 - 5x + 6$.
7. $u^2 - 16$. **8.** $ab - 3a + 10b - 30$. **9.** $m^2 - 6m + 9$.
10. $yz - 9y + 8z - 72$. **11.** $6a^2 + 13a - 5$. **12.** $6cd - 2c - 15d + 5$.
13. $6a^2 - 13a - 5$. **14.** $9a^2 - 34ab - 8b^2$. **15.** $7x^2 + xy - 6y^2$.
16. $12x^2 + 8xy - 15y^2$. **17.** $9a^2 + 34ab - 8b^2$. **18.** $12x^2 - 8xy - 15y^2$.
19. $9a^2 + 38ab + 8b^2$. **20.** $7x^2 - 15xy - 18y^2$. **21.** $9a^2 - 38ab + 8b^2$.
22. $12x^2 - 28xy + 15y^2$. **23.** $21x^2 - 32xy + 12y^2$.
24. $8x^4 - 22x^2y^2 + 15y^4$. **25.** $-6x^2 + 25xyz - 24y^2z^2$.
26. $9x^2 + 21xyz - 8y^2z^2$. **27.** $8x^4 - 2x^2y^2 - 15y^4$.
28. $-6x^2 + 7xyz + 24y^2z^2$. **29.** $8x^4 + 2x^2y^2 - 15y^4$.
30. $15x^2 + 46xyz + 16y^2z^2$.

Exercise 38 c (Pp. 173, 174)

1. -26. **2.** -29. **3.** 8. **4.** 6. **5.** -10. **6.** 21.
7. 47. **8.** 6. **9.** 0. **10.** $-14a^2$. **11.** $9c^8 - 16c^4 - 8c^2 - 1$.
12. $1 - 5k + 10k^2 - 10k^3 + 5k^4 - k^5$. **13.** $-35 + 22y + 23y^2 - 10y^3 - 3y^4$.
14. $3x^7 + 9x^6 + 20x^5 - 34x^4 - 34x^3 - 19x^2 + 41x + 20$.
15. $4 - 12x + 9x^2 - 8x^3 + 12x^4 + 4x^6$. **16.** $6x^5 + x^4 - 71x^3 + 78x^2 + 79x - 105$
17. 1. **18.** -5. **19.** 9. **20.** $\frac{1}{2}$.
21. $-\frac{2}{3}$. **22.** $\frac{1}{3}$. **23.** $\frac{5}{3}$. **24.** $-\frac{1}{2}$.

Exercise 39 a (P. 175)

1. $a^2 + 2ax + x^2$. **2.** $4 + 4x + x^2$. **3.** $c^2 + 6c + 9$.
4. $4x^2 + 12x + 9$. **5.** $x^2 - 6x + 9$. **6.** $b^2 - 2bc + c^2$.
7. $36 - 12x + x^2$. **8.** $9l^2 - 6l + 1$. **9.** $16x^2 - 24x + 9$.
10. $4c^2 - 12cd + 9d^2$. **11.** $9a^2 + 30ab + 25b^2$. **12.** $9x^2 - 42xy + 49y^2$.
13. $81l^2 + 72lm + 16m^2$. **14.** $9x^2 - 60xy + 100y^2$. **15.** $4x^4 + 20x^2y^2 + 25y^4$.
16. $25a^4 - 20a^3 + 4a^2$. **17.** $49 - 42c^2 + 9c^4$. **18.** $9c^4 + 6c^2d + d^2$.
19. $a^4 - 4a^2b^2 + 4b^4$. **20.** $9x^4 - 12x^2y^2 + 4y^4$. **21.** $4a^2 - 20abc + 25b^2c^2$.

22. $4a^6+12a^3b^3+9b^6$. **23.** $16x^6-40x^3+25$. **24.** $36-60x^4+25x^8$.
25. $9a^4b^2-42a^2bc^3+49c^6$. **26.** $81a^4+72a^5+16a^6$.
27. $25c^2+60c^5+36c^8$. **28.** $25c^2-120cd^2+144d^4$. **29.** 40804.
30. 168100. **31.** 39601. **32.** 10100·25. **33.** 9900·25.
34. 1010025. **35.** 2520·04. **36.** 2490·01.

Exercise 39 B (P. 175)

1. $b^2+2bh+h^2$. **2.** $25+10x+x^2$. **3.** $t^2+14t+49$.
4. $9t^2+6t+1$. **5.** $16-8a+a^2$. **6.** $64-16t+t^2$.
7. $s^2-2rs+r^2$. **8.** $4a^2-12a+9$. **9.** $25x^2-20x+4$.
10. $x^2-8xy+16y^2$. **11.** $9a^2-30ab+25b^2$. **12.** $4x^2+28xy+49y^2$.
13. $64l^2-80lm+25m^2$. **14.** $16x^2+24xy+9y^2$. **15.** $4x^4-4x^2y^2+y^4$.
16. $9x^4-6x^3+x^2$. **17.** $4-36x^2+81x^4$. **18.** $25c^2-10cd^2+d^4$.
19. $r^4-2r^2s^2+s^4$. **20.** $25a^2+20abc+4b^2c^2$. **21.** $c^4-6c^3d+9c^2d^2$.
22. $25n^6-70n^3+49$. **23.** $x^6+4x^3z^3+4z^6$. **24.** $4x^6-20x^3y^3+25y^6$.
25. $4r^2s^4-36rs^2t^3+81t^6$. **26.** $9-42a^4+49a^8$.
27. $25l^2-70lm^2+49m^4$. **28.** $4x^4+20x^5+25x^6$.
29. 93025. **30.** 41209. **31.** 9801. **32.** 420·25.
33. 380·25. **34.** 1002001. **35.** 1648·36. **36.** 1584·04.

Exercise 40 A (P. 177)

1. $-7a+9b$. **2.** $-2c^2+3cd$. **3.** $-7x^3+2x$. **4.** $3l^2-2m^2$.
5. $t+7$. **6.** $a-12$. **7.** $x+13$. **8.** $k+4$.
9. $a+2$, rem. 2. **10.** $2x+1$, rem. -2. **11.** $2t-1$, rem. 3.
12. $x-3$. **13.** $-7c+5$, rem. 10. **14.** $5a-2b$.
15. t^2+6t+5. **16.** $x^2+4x-21$, rem. -5.

Exercise 40 B (P. 177)

1. $-a+3b$. **2.** $-4c+3d$. **3.** $-p-q+r$. **4.** $-5x^2+3x$.
5. $d+3$. **6.** $c-4$. **7.** $t-13$. **8.** $3x+2$.
9. $5x+1$, rem. -6. **10.** $3a-2$. **11.** $5t+8$.
12. $2x+3$, rem. -12. **13.** $3c-7$. **14.** $-2a-7$.
15. x^2-5x+6, rem. -4. **16.** $9c^2+9c+5$, rem. 20.

Exercise 40 C (P. 177)

1. $2x-1$. **2.** $1-3x$. **3.** $3a+5$. **4.** x^2+3x+2.
5. $4x^2+3x-2$. **6.** z^2-z+1. **7.** t^2+2t+3, rem. $31t+15$.
8. $7a^2+5a-3$, rem. $-39a+19$. **9.** $3x^2+5x-1$.
10. $5c^2+2c-3$, rem. $-10c+10$. **11.** $3x^2-4x+1$, rem. -2.
12. $8x^2+3x+2$, rem. $2x+2$. **13.** $6a^2+7ab-5b^2$.
14. $2x^2+2x$, rem. $-9x+5$. **15.** $3a^2-4b^2$, rem. $3ab^3-b^4$.

Exercise 41 a (P. 179)

1. $3(3a+b)$. **2.** $2c(5c-6)$. **3.** $z(z-1)$. **4.** No factors.
5. $3c(c+d)$. **6.** $x(x+y)$. **7.** $x^2(1-2x)$. **8.** $6x(x+3y)$.
9. $x^3(x+2)$. **10.** $5c^2(1-4d)$. **11.** $12a(1-3ab)$. **12.** No factors.
13. No factors. **14.** $2(x-2y+z)$. **15.** $6a(a^2-2a+4)$.
16. 351. **17.** 10,500. **18.** 42,000. **19.** 16,500. **20.** 300.
21. £36.

Exercise 41 b (P. 179)

1. $3(6c-5d)$. **2.** $2t(7t-10)$. **3.** $l(3l-1)$. **4.** No factors.
5. $r(r+5s)$. **6.** $x(x-2y)$. **7.** $3(3t^2-k)$. **8.** $5l(l-2m)$.
9. $s^4(2s-1)$. **10.** $b^2(24a^2-35b^2)$. **11.** $7x^2(1+6y)$.
12. $17t(1-4tw)$. **13.** $7(k-3l-4m)$. **14.** No factors.
15. No factors. **16.** 19,500. **17.** 14,040. **18.** 11,000.
19. 127,600. **20.** 400. **21.** £18.

Exercise 41 c (Pp. 179, 180)

1. $7x(x^3-2x^2+3)$. **2.** $a(a^2-ab-b^2)$. **3.** $3c(5bc-3b^2-c^2)$.
4. $3x^2y(y-3-y^2)$. **5.** $2x^3(x^2-3xy-y^2)$. **6.** No factors.
7. $7c^2d(c^2-2cd-7d^2)$. **8.** $ax(ax-by-cy)$. **9.** $7(ab-2bc-3ac)$.
10. $a^2(a^3-5a^2b+10ab^2-10b^3)$. **11.** No factors.
12. $xy(8x-5x^2+3y^2)$. **13.** $3a(a^2-3a-30)$. **14.** No factors.
15. $7a^3(2a-b+4b^4)$.

Exercise 42 a (P. 182)

1. $(c+d)(s+t)$. **2.** $(s+t)(a-b)$. **3.** $(a-b)(3-c)$.
4. $(c-3d)(2a^2+7)$. **5.** No factors. **6.** $(s+t)(xy-z)$.
7. $(u+v)(a+b)$. **8.** $(x-y)(c-d)$. **9.** $3(x^2+y^2)(1+2a)$.
10. No factors. **11.** $(a-b)(k+l)$. **12.** $(d-4)(c-7)$.
13. $(x+y)(x+z)$. **14.** $(5l+3)(l+m)$. **15.** $(s+t)(4-t)$.
16. $(ac-1)(b+1)$. **17.** $(c^2+d^2)(4c-3)$. **18.** $(x-y)(2x+z)$.
19. $(3x-y)(x^2+2y^2)$. **20.** $(2x^3-3)(3x-5)$. **21.** No factors.
22. $(a^3+2)(a-1)$. **23.** $(6+x)(4-y)$. **24.** $(2-3x)(5-y)$.

Exercise 42 b (P. 182)

1. $(l-m)(x+y)$. **2.** $(c-d)(a-b)$. **3.** $(u+4v)(3k^2-2)$.
4. $(x+y)(5-t)$. **5.** $(d+5)(2ab-9)$. **6.** No factors.
7. $(u-v)(x+y)$. **8.** $(p+q)(c+l)$. **9.** $(t-2)(m-5)$.
10. $(l^2+m^2)(5+4x)$. **11.** No factors. **12.** $(y-z)(m+n)$.
13. $(b+c)(b-d)$. **14.** $(km+l)(km+n)$. **15.** $(2-c)(d+c)$.

16. $(b+d)(2b+c)$. **17.** No factors. **18.** $(x+1)(1-ty)$.
19. $(2a-b)(a^2+b^2)$. **20.** $(4l+5m)(3l^2-2m^2)$.
21. $(t-1)(2t^3+3)$. **22.** No factors. **23.** $(5-x)(3-y)$.
24. $(7+x)(2-y)$.

EXERCISE 42 C (Pp. 182, 183)

1. $(2x^2-x+5)(3x^2+7)$. **2.** $(3x^2-2x+4)(3y-5z)$.
3. $(x^2-2x-7)(3a-2b+4c)$. **4.** $(4x-3y+7z)(l-3m-2n)$.
5. $(x+7)(3x^2-5)$. **6.** $(7x-5)(x^2-2)$. **7.** $(cl-d)(dl-c)$.
8. $(4ax-b)(bx-4a)$. **9.** No factors. **10.** $(x^2+y^2)(a^2-b)$.
11. $(a+b)(al+bm+c)$. **12.** $(2c+1)(c-d+x)$. **13.** $(a-b)(al-bm+c)$.
14. $(2c-1)(c+d+x)$. **15.** No factors. **16.** $(2x-5z)(x-y+3)$.

EXERCISE 43 A (Pp. 185, 186)

1. 5, 3. **2.** 8, 3. **3.** 11, 2. **4.** −8, −3. **5.** None.
6. −11, −2. **7.** 9, −7. **8.** −7, 6. **9.** −9, 5. **10.** 7, −6.
11. −9, 7. **12.** 9, −5. **13.** None. **14.** 16, 3. **15.** 8, −5.
16. 16, −6. **17.** 15, 3. **18.** 3, −3. **19.** 4, −14. **20.** 11, 8.
21. −22, −4. **22.** 11, −6. **23.** 3, −6. **24.** −6, −3. **25.** 3, −16.
26. 16, 6. **27.** None. **28.** 14, 4.

EXERCISE 43 B (P. 186)

1. 9, 7. **2.** 6, 7. **3.** 9, 5. **4.** −6, −7. **5.** −9, −5.
6. −9, −7. **7.** 3, −5. **8.** −3, 8. **9.** 11, −2. **10.** 3, −8.
11. −11, 2. **12.** 5, −3. **13.** 6, 3. **14.** 8, 5. **15.** 6, −16.
16. 5, −8. **17.** 9, 1. **18.** 3, −15. **19.** None. **20.** −16, −3.
21. None. **22.** −16, −6. **23.** 1, −9. **24.** −3, 6. **25.** 16, −3.
26. −3, −15. **27.** 11, −8. **28.** 14, −4.

EXERCISE 44 A (P. 189)

1. $(2x+1)(x+1)$. **2.** $(2d+1)(d+3)$. **3.** $(3t+2)(t+1)$.
4. $(2a-3)(a-1)$. **5.** $(2x-3y)(x-2y)$. **6.** $(3a-2)(a-3)$.
7. $(2c-1)(c+5)$. **8.** $(2x-5y)(x+y)$. **9.** $(3c-2)(c+3)$.
10. $(3z+4)(z-2)$. **11.** $(5p+q)(3p-2q)$. **12.** $(3a+8)(a-1)$.
13. $(2+3c)(1-2c)$. **14.** $(3-z)(1-3z)$. **15.** $(3+x)(1-3x)$.
16. No factors. **17.** $(a+2)(a-5)$. **18.** $(a+5)(a+2)$.
19. $2(x-9y)(x-7y)$. **20.** No factors. **21.** $(2x^2+1)(x^2-2)$.
22. $(2a+bc)(a-bc)$. **23.** $3(k-8)(k+7)$. **24.** $3(a-7)(a+9)$.
25. $(x-5y)(x+8y)$. **26.** $9(t-2)(t+6)$. **27.** $(4+5st)(1-2st)$.
28. $5(a-8b)(a+7b)$. **29.** No factors. **30.** $(4-5xy)(1-2xy)$.
31. No factors. **32.** $(8l+7m^2)(l-m^2)$. **33.** $(2x+5yz)(x-3yz)$.

Exercise 44 b (Pp. 189, 190)

1. $(3x+1)(x+1)$. **2.** $(3t+2)(t+2)$. **3.** $(2c+1)(c+4)$.
4. $(2x-y)(x-5y)$. **5.** $(2c-1)(c-2)$. **6.** $(2a-5)(a-2)$.
7. $(3t-1)(t+2)$. **8.** $(5t-1)(t+6)$. **9.** $(3a-2b)(a+b)$.
10. $(2d-3)(d+1)$. **11.** $(15s-2)(s+1)$. **12.** $(5a-b)(3a+2b)$.
13. $(2-3x)(1-2x)$. **14.** $(2+5t)(1+t)$. **15.** $(2-5k)(1+k)$.
16. $(x+1)(x+7)$. **17.** No factors. **18.** $(-x+4)(x+5)$.
19. No factors. **20.** $(2ab-c)(ab+2c)$. **21.** $(2x^2-1)(x^2+1)$.
22. $(2x^2-5y^2)(x^2-3y^2)$. **23.** $7(x+2)(x-6)$. **24.** No factors.
25. $4(a+8b)(a+7b)$. **26.** $(2c+7d)(c-4d)$. **27.** $(4a-7b)(a+2b)$.
28. $(8-p)(5-p)$. **29.** $5(p-7q)(p-8q)$. **30.** $(4a+7bc)(a+2bc)$.
31. $(8-7ab)(1-ab)$. **32.** $(4-5z^2)(1+2z^2)$. **33.** $3(c^2-c-63)$.

Exercise 44 c (P. 190)

1. $(2a-1)(a-8)$. **2.** $(3c-5)(c+6)$. **3.** No factors.
4. $3(6x+1)(x+7)$. **5.** $3(5a-2)(a+7)$. **6.** $(21+2t)(1-t)$.
7. $(3t+5)(t+6)$. **8.** $(7l-m)(3l-2m)$. **9.** $(4c-5d)(3c-2d)$.
10. $3(5k+7)(2k+3)$. **11.** $(2a+1)(a-8)$. **12.** $(4x+5y)(7x-y)$.
13. $(3+5xy)(5-3xy)$. **14.** No factors. **15.** $(7p+q)(3p-2q)$.
16. $(4x-5)(3x+2)$. **17.** $(2t+3)(12t-7)$. **18.** $(5-3c)(3-5c)$.
19. $(7a+b)(4a+5b)$. **20.** No factors. **21.** $(5n+1)(n-21)$.
22. $7(3x-7y)(2x+y)$. **23.** $4(2a-3bc)(5a-6bc)$.
24. $(2x-3)(12x-7)$. **25.** $(5+4z)(4-5z)$. **26.** $(21a-2b)(a-b)$.
27. $5(3-2p)(7+5p)$. **28.** No factors. **29.** $2(3x+7)(2x-5)$.
30. $2(6d+1)(d-7)$. **31.** $2(7a^2-1)(a^2+7)$. **32.** $(x+9)(x-10)$.
33. $4(5x+7)(x-3)$. **34.** $2(5t^2+4)(4t^2+5)$. **35.** No factors.
36. $(5x+2y)(x+7y)$. **37.** $3(5m+6)(2m-3)$. **38.** $5(3a+2)(4a-7)$.
39. $3(14k+1)(k-2)$. **40.** $(4x^2-3)(x^2+10)$. **41.** $5(2x-y)(3x-7y)$.
42. No factors. **43.** $(7c+d)(c+7d)$. **44.** $3(5m-7n)(2m-n)$.
45. $4(5c+7)(2c-3)$. **46.** No factors. **47.** $2(5x-y)(x-21y)$.
48. No factors.

Exercise 45 a (Pp. 190, 191)

1. $(a+1)(a+2)$. **2.** $(a+4)(a+3)$. **3.** $(c+6d)(c+3d)$.
4. $(s-15t)(s-t)$. **5.** $(d-4)(d-1)$. **6.** $(x-4)(x-5)$.
7. $(a-2)(a+1)$. **8.** $(h-2)(h+5)$. **9.** No factors.
10. $(n-5)(n+4)$. **11.** $(m-6)(m+7)$. **12.** $(x-4)(x+3)$.
13. $(a+4)^2$. **14.** $(7b+c)(2b+c)$. **15.** $(z-7)(z+3)$.
16. $(x-4y)(x-7y)$. **17.** $(1-12xy)(1+2xy)$. **18.** No factors.
19. $(a-8b)(a+3b)$. **20.** $(1-7k)(1-9k)$. **21.** No factors.

22. $(11-t)(1+t)$. **23.** $(x-8)(x-17)$. **24.** No factors.
25. $(y-6z)(y-17z)$. **26.** No factors. **27.** No factors.
28. $(a+9c)(a-5c)$. **29.** $(1+5xyz)(1+7xyz)$. **30.** $(t-13)(t+6)$.

EXERCISE 45 B (P. 191)

1. $(b+3)(b+1)$. **2.** $(n+4)(n+2)$. **3.** $(d+6k)(d+7k)$.
4. $(y-2)(y-3)$. **5.** $(x-5)(x-2)$. **6.** $(m-5n)(m-3n)$.
7. $(z-4)(z+1)$. **8.** $(y-5)(y+3)$. **9.** $(x+3)(x-1)$.
10. $(t+3)(t-2)$. **11.** $(x-6y)(x+3y)$. **12.** $(b-15)(b+1)$.
13. $(x+3)(x+7)$. **14.** $(d-5)^2$. **15.** $(z-6)(z+4)$.
16. $(x+3y)(x+8y)$. **17.** No factors. **18.** $(x+7y)(x-4y)$.
19. $(7+xy)(2-xy)$. **20.** $(cd+4)(cd+11)$. **21.** $(c-12d)(c-2d)$.
22. $(a+23b)(a-3b)$. **23.** No factors. **24.** $(1+8k)(1-17k)$.
25. $(1-13x^2)(1+5x^2)$. **26.** No factors. **27.** $(c+27)(c-3)$.
28. $(x+13yz)(x-7yz)$. **29.** $(h+11)(h-8)$. **30.** $(t^3+5)(t^3-7)$.

EXERCISE 46 A (Pp. 192, 193)

1. $(a+5)(a-5)$. **2.** $(cd+4n)(cd-4n)$. **3.** $(3+7b)(3-7b)$.
4. $(7+5t)(7-5t)$. **5.** $(x+1)(x-1)$. **6.** $(2d+7)(2d-7)$.
7. $25(x+2)(x-2)$. **8.** $(11a+8b^3)(11a-8b^3)$.
9. $(12+5t^2)(12-5t^2)$. **10.** $(2x^2y^2+3z^3)(2x^2y^2-3z^3)$.
11. $(9l+10pq)(9l-10pq)$. **12.** $(5a^2+3x)(5a^2-3x)$.
13. None. **14.** $3(k+5)(k-5)$. **15.** $(13a+11b)(13a-11b)$.
16. $2(2+t)(2-t)$. **17.** None. **18.** $5(x+4y^2)(x-4y^2)$.
19. 313. **20.** 153,200. **21.** 68,400. **22.** 6,200. **23.** 9,200.
24. 84,800. **25.** 42·55. **26.** 998,000. **27.** 34. **28.** 1·7.
29. 175,000. **30.** 35. **31.** $a^2-4b^2+4bc-c^2$.
32. $9a^2+6ab+b^2-4c^2$. **33.** $16a^2-b^2-6bc-9c^2$.
34. $25x^4-x^2-4x-4$. **35.** $c^4-9c^2+24c-16$. **36.** $4x^4-25x^2+16$.

EXERCISE 46 B (P. 193)

1. $(c+3)(c-3)$. **2.** $(1+7n)(1-7n)$. **3.** $(x+6)(x-6)$.
4. $(10+xyz)(10-xyz)$. **5.** $(3k+8)(3k-8)$.
6. $(6a+5bc)(6a-5bc)$. **7.** $b^2(2ab+3c^2)(2ab-3c^2)$.
8. $36(2a+b)(2a-b)$. **9.** $(7x^2+3y^3)(7x^2-3y^3)$.
10. $(6mn+7x^2)(6mn-7x^2)$. **11.** $(9+5p^3)(9-5p^3)$.
12. $(11a^2+1)(11a^2-1)$. **13.** $2(3x+5y)(3x-5y)$.
14. $7(3a+b^2)(3a-b^2)$. **15.** None. **16.** $5(3+2c)(3-2c)$.
17. None. **18.** $3(3b+c)(3b-c)$. **19.** 10,200. **20.** 1,840.

21. 9,400. **22.** 2,800. **23.** 570,000. **24.** 60,800. **25.** 100.
26. 1·6. **27.** 20. **28.** 2·874. **29.** 3·2. **30.** 0·5571.
31. $4a^2 - b^2 + 2bc - c^2$. **32.** $4a^2 - 9b^2 - 6bc - c^2$.
33. $25a^2 + 20ab + 4b^2 - 9c^2$. **34.** $9x^4 + 17x^2 + 9$.
35. $4x^4 - x^2 - 10x - 25$. **36.** $t^4 - 21t^2 + 4$.

EXERCISE 46 C (P. 194)

1. $(a+2b+c+d)(a+2b-c-d)$.
2. $(3m+2n+2x-y)(3m+2n-2x+y)$.
3. $(a-b+c-d)(a-b-c+d)$.
4. $(4b-3c+2m+n)(4b-3c-2m-n)$.
5. $(5b-c+m-2n)(5b-c-m+2n)$.
6. $(m+4n+3x-y)(m+4n-3x+y)$.
7. $(a-b+c+d)(a-b-c-d)$. **8.** $(a+b+c-d)(a+b-c+d)$.
9. $(a+b+n)(a+b-n)$. **10.** None.
11. $(c+y+z)(c-y-z)$. **12.** $(a-b+x)(a-b-x)$.
13. $(4c-3d+2l)(4c-3d-2l)$. **14.** $(3a+b-c)(3a-b+c)$.
15. $(3c+2d+4l)(3c+2d-4l)$. **16.** $(9x-4y)(3x-10y)$.
17. $(5x+y-z)(5x-y+z)$. **18.** $(4b-c)(2b-7c)$.
19. $(15l-5m+4r+8s)(15l-5m-4r-8s)$.
20. $(5c+4x+2y)(5c-4x-2y)$. **21.** $(3x+4y+8z)(3x-4y-8z)$.
22. $(2k+7m-14n)(2k-7m+14n)$.
23. $(12l-9m+6r-2s)(12l-9m-6r+2s)$.
24. $(8t+10x+15z)(8t-10x-15z)$. **25.** $(7x-13y)(y-x)$.
26. $(7l-13m)(l+m)$. **27.** $(8x+7y)(2x-7y)$. **28.** $7d(10c-7d)$.
29. $(7x-5y)(x+5y)$. **30.** $3d(8c-3d)$. **31.** $(4a-11b)(7b-2a)$.
32. $(3x+y)(5x-y)$. **33.** $(5l+m)(l+5m)$.
34. $-3x(5x-8y)$ or $3x(8y-5x)$. **35.** $(9t-4)(7t+4)$.
36. $(17c-d)(9d-13c)$.

EXERCISE 47 A (P. 195)

1. $a+2$. **2.** $a-5$. **3.** $2a+1$. **4.** $2a-7$.
5. $3a-1$. **6.** $3d+5$. **7.** $3x-4y$. **8.** $4x-1$.
9. Not a square. **10.** $5l+2m$. **11.** Not a square.
12. $4x+7y$. **13.** $5m+3n$. **14.** Not a square.
15. $5t^2-6$. **16.** $2a^2-11$. **17.** Not a square.
18. $5c+11d$.

EXERCISE 47 B (P. 195)

1. $a+4$. **2.** $a-3$. **3.** $2a-3$. **4.** $2a+5$.
5. $3x-2$. **6.** $3c-7$. **7.** $3a+8b$. **8.** Not a square.

9. $4x+3$. **10.** $4x-5y$. **11.** $5c-d$. **12.** Not a square.
13. $5m-8n$. **14.** Not a square. **15.** $5x^2-4$.
16. $3a^2+10$. **17.** $4x-9y$. **18.** Not a square.

EXERCISE 48 A (P. 196)

1. $(x-y+z)(x-y-z)$. **2.** $(x+y+z)(x-y-z)$.
3. $(x-y+5)(x-y-5)$. **4.** $(2+y-z)(2-y+z)$.
5. $(1+k-3l)(1-k+3l)$. **6.** $(2a+b+8)(2a+b-8)$.
7. $(2l+3m+9)(2l+3m-9)$. **8.** $(a+b+2)(a-b-2)$.
9. $(6+3y-z)(6-3y+z)$. **10.** $(c-4+3d)(c-4-3d)$.
11. No factors. **12.** $(x^2+x-4)(x-4-x^2)$.
13. $(x+y+l+m)(x+y-l-m)$. **14.** $(1+2c+x-y)(1+2c-x+y)$.
15. $(a-3b+p-2q)(a-3b-p+2q)$. **16.** No factors.
17. $(a-7+l-3m)(a-7-l+3m)$. **18.** $(2c-3+x+3y)(2c-3-x-3y)$.
19. $(2l-3m+x+y)(2l-3m-x-y)$.
20. $(2l+m+3y-4z)(2l+m-3y+4z)$. **21.** $7(a-b+3c)(a-b-3c)$.
22. $(3a+4x+2y)(3a-4x-2y)$.
23. $(2c^2-3d^2+a^2-5b^2)(2c^2-3d^2-a^2+5b^2)$.
24. $(2m-7n+5t)(2m-7n-5t)$.

EXERCISE 48 B (Pp. 196, 197)

1. $(x+y+z)(x+y-z)$. **2.** $(x+y-z)(x-y+z)$.
3. $(x+y+4)(x+y-4)$. **4.** $(7+y+z)(7-y-z)$.
5. $(x+y+5)(x-y-5)$. **6.** $(5+k+3l)(5-k-3l)$.
7. $(5+y+7z)(5-y-7z)$. **8.** $(2a-b+1)(2a-b-1)$.
9. $(c+3+2d)(c+3-2d)$. **10.** $(2y-3z+6)(2y-3z-6)$.
11. $(x^2+x-3)(x^2-x+3)$. **12.** No factors.
13. $(c-d+x-y)(c-d-x+y)$. **14.** $(7a+2b-2)(7a-2b)$.
15. $(x-5+2c+3d)(x-5-2c-3d)$.
16. $(2l+m+p-5q)(2l+m-p+5q)$.
17. $(p-q+a-b)(p-q-a+b)$. **18.** No factors.
19. $(3x^2+y-3z)(3x^2-y+3z)$. **20.** $(x^2+2y-z)(x^2-2y+z)$.
21. $(2x+7+6y)(2x+7-6y)$. **22.** $5(x-y+3)(x-y-3)$.
23. $(3c+2l-4m)(3c-2l+4m)$.
24. $(x^2-y^2+l^2+m^2)(x^2-y^2-l^2-m^2)$.

EXERCISE 49 (Pp. 198, 199)

1. $9x(2x^2-y^2)$. **2.** $(3x+1)(x+2)$. **3.** $(4a+7x)(4a-7x)$.
4. $(y-c)(y+5)$. **5.** $3(5x+4)(5x-4)$. **6.** $(t+13)(t-6)$.
7. $(3a-1)(a-6)$. **8.** $t(4t^2-t+1)$. **9.** $(2c+d+l-m)(2c+d-l+m)$.

10. $(3l-s)(2n-1)$. **11.** $2a(a+5)(a-5)$.
12. $(2y-1)(y+6)$. **13.** $8m(3l-2m)$. **14.** $7(1+2c)(1-2c)$.
15. $(11+a)(10-a)$. **16.** $(3a-4b)(a+b)$. **17.** $(cx-yd)(2c+3d)$.
18. $(4x+y)(2x-3y)$. **19.** $2y^3(y^2-3yz+z^2)$. **20.** $(3-l)(1-15l)$.
21. $5x(4y^2+3xy+x^2)$. **22.** $(3x+5)(x-2)$. **23.** $3(a^2+10)(a^2-10)$.
24. $(3a-5b)^2$. **25.** $13(1+2z)(1-2z)$.
26. $(5x+2)(x+3)$. **27.** $x(x^2+5)(x^2-5)$. **28.** $9a(a^2-ab+3b^5)$.
29. $(1+13x^2)(1-5x^2)$. **30.** $(l+13m)(l+2m)$. **31.** $(r^2+s^2)(l^2-c)$.
32. $(1-7a)^2$. **33.** $ab^2c(c-a)$. **34.** $20y(5x+y)(5x-y)$.
35. $(s-9t)(s+5t)$. **36.** $(5x+6)(x-1)$. **37.** $(c-1)(1+2c)(1-2c)$.
38. $3d(3d-1)$. **39.** $(7a-3)(a^2-3)$. **40.** $(4x-5)(x+1)$.
41. $4(5a-b)(4a+b)$. **42.** $(2d^2+7)^2$. **43.** $(c+1)(c^2+1)$.
44. $(y+12)(y-3)$. **45.** $(10x-7)(x+3)$. **46.** $x(5x+1)(3x+7)$.
47. $(12t+5)^2$. **48.** $(y+1)(x-1)$. **49.** $(z+7)(2z^2-3)$.
50. $(l-m)(t-1)$. **51.** $3(a+1)(a-1)(b+1)(b-1)$.
52. $(a-b+3)(a-b-3)$. **53.** $2(3r^2+2s^2)(3r^2-2s^2)$.
54. $a^3(3-a)(2-a)$. **55.** $2(2x-7)(x-5)$.
56. $(2l-m+7)(2l-m-7)$. **57.** $(3t+2)(t-1)$.
58. $2(10c-21)(c-1)$. **59.** $4(4x-5)(x-2)$.
60. $(4+a+3b)(4-a-3b)$. **61.** $(x+y)(x-y+2a)$.
62. $(7x+2)(7x+3)(7x+1)$. **63.** $(a+b+13c)(a+b+6c)$.
64. $(x+y-4)(x-y+4)$. **65.** $(3x^2-10y-10z)(2x^2+y+z)$.
66. $(5x+29)(x+9)$. **67.** $(3x-1)(3x+2)(3x-4)$.
68. $3c^3(3c^6-1)$. **69.** $(5a-5c+3t)(a-c+t)$.
70. $(a-5x+5y)(a+x-y)$. **71.** $(d+3)(d+1)(d-3)(d-1)$.
72. $(x+2)(x^2+6x+4)$. **73.** $(x-5l+5m)(x-2l+2m)$.
74. $(x+y)(a+b-c)$. **75.** $49x(3x-2)$. **76.** $(l-m)(a-b-c)$.
77. $(l-m)(l-m-1)$. **78.** $(x+3)(x-1)(x^2-2x+3)$.
79. $(x+y+z)(y+z-x)(x+y-z)(x-y+z)$. **80.** $(x+y)(x-y)^3$.
81. $(a-2b+l+3n)(a-2b-l-3n)$. **82.** $(l+1)(l+2)(l-2)(l-3)$.
83. $(a^2+a-1)(a^2-a+1)$. **84.** $(2x-1)(2x+1)(2x-5)(4x+1)$.
85. $2(x+2)(x+4)(x+5)$. **86.** $6lmn(6l-9m+8n-3lmn)$.
87. $(2a+b-c)(2a-b+c)(4a^2+b^2-2bc+c^2)$.
88. $(a-5c-5d)(a-c-d)$. **89.** $-5(1+t^2)(1+t)(7-5t)$.
90. $(3l-n+4k-d)(3l-n-4k+d)$. **91.** -1. **92.** 10.
93. -3. **94.** 2. **95.** 4. **96.** -3. **97.** ± 2, ± 14.
98. $6, 9, 12, 13$. **99.** $2, 5$.
100. ± 2, ± 7, ± 14, ± 19, ± 26, ± 37, ± 58, ± 119.

Test Papers IV (Pp. 199, 200, 201, 202, 203, 204, 205)

A. 1. (i) $\frac{7a}{100}$, (ii) $\frac{100x}{y}$. **2.** (i) $-3\frac{11}{21}$, (ii) $x=\frac{8}{65}$, $y=\frac{32}{117}$.

3. (i) $-3l^2-16m^2$, (ii) $\frac{9x}{y}$, (iii) $\frac{-2(c+4d)}{7}$.

4. (i) $27(3p-2)$, (ii) $(x-6)(x+4)$, (iii) $(5c+4d)(5c-4d)$.

5. $4'\ 10''$.

B. 1. (i) $\frac{100d}{K}$, (ii) $£A\left(1+\frac{x}{100}\right)$. **2.** (i) $7{\cdot}5$, (ii) $x=-16{\cdot}5$, $y=8{\cdot}5$.

3. (i) $3m-3m^2$, (ii) $\frac{2AQX^3}{P^4}$.

4. (i) $(l+m)(5+m)$, (ii) $(5x+1)(3x-7)$. **5.** 34 m.p.h. ; $2\frac{1}{2}$ hrs.

6. $6{\cdot}5''$.

C. 1. (i) $\frac{bc}{100}$, (ii) $£\frac{100x}{100+y}$. **2.** (i) $\frac{x^2y^2}{15}$, (ii) $\frac{-24c-23}{12}$.

3. (i) 3, (ii) $x=3$, $y=-0{\cdot}5$.

4. (i) $2d(d^2-2d-1)$, (ii) $(c-1)(c^2+1)$, (iii) $(3-ab)(1-10ab)$.

5. 44 m.p.h. **6.** (i) $7a^2bc^3$, (ii) $60a^2$.

D. 1. (i) $£(q-p)$, $\frac{(q-p)100}{p}$; (ii) $100-x$.

2. (i) $-\frac{25}{44}$, (ii) $x=127$, $y=-8$.

3. (i) $cz(cz-3)$, (ii) $(5x+12)(x-1)$, (iii) $(2x+y-z)(2x-y+z)$.

4. (i) -125, (ii) $9c-6a$, (iii) $\frac{s-3t}{3}$. **5.** 48 ft.

6. $44\frac{1}{2}$ min. from A, $33\frac{1}{3}$ miles from A.

E. 1. (i) $£\left(c+\frac{p}{20}\right)$, $\frac{5p}{c}$; (ii) $\frac{25y}{3x}$. **2.** (i) $5x^2-12x-43$, (ii) 0.

3. (i) 7, (ii) $x=-1{\cdot}25$, $y=-1$. **4.** 94.

5. (i) $(2d-3a)(c+5a)$, (ii) $(1-3b)(1+8b)$, (iii) $(2xy-z)^2$.

6. 2 ; $3{\cdot}41$, $0{\cdot}59$.

F. 1. $\frac{(100+x)(100+y)(100+z)}{10000}$. **2.** (i) $3\frac{1}{2}$, (ii) $x=24$, $y=-12$.

3. (i) $3x^3y(x^2-2xy+3y^2)$, (ii) $(x-3)(10x-7)$, (iii) $2(x+12)(x-12)$.

4. (i) a^4c^3, (ii) $14a-13$. **5.** 32 lbs.

6. (i) $19r^2s^2$, (ii) $12x^2y^2z^2t^3$.

G. 1. $\left(a+\frac{1}{12e}\right)^2$ sq. ft., $\frac{25(24ae+1)}{36a^2e^2}$.

2. (i) $29x^2-16x+14$, (ii) $-5+y+3y^2$.

3. (i) 1·45, (ii) $x=\frac{5}{14}$, $y=\frac{23}{15}$. **4.** (i) $\frac{4x^4y}{112z^3}$, (ii) $\frac{3x^2-9x-27}{x^3}$.

5. (i) $(a+b)(n^2-5)$, (ii) $(p-q^2)(3p-10q^2)$, (iii) $-(5p+q)(p+5q)$.

6. $\frac{2}{3}$.

H. 1. $\frac{lk}{l+m}$ lb., $\frac{mk}{l+m}$ lb. **2.** (i) 13, (ii) $x=0·6$, $y=-0·3$.

3. (i) $16x^4-2x^3-21x^2-x+11$, (ii) $6x^3+8x^2-9x-12$.

4. (i) $(l-m)(l-m-k)$, (ii) $(2+x)(1+x)(1-x)$, (iii) $(2lm+3n)^2$.

5. 34 m.p.h. **6.** Loss of £19.

I. 1. $\frac{x(p+q)}{p}$ lb. **2.** (i) 9, (ii) $x=8$, $y=-15$.

3. (i) $l(l+6)(l-7)$, (ii) $(4+3x)(5-2x)$, (iii) $6(3a^2+2c)(3a^2-2c)$.

4. (i) $2K^6$, (ii) 1. **5.** 18s. **6.** 5·6 m.p.h.

J. 1. $\frac{AB}{A+B}$ days. **2.** (i) $\frac{60l+104m-72}{9}$, (ii) $\frac{3rt-2st-rs}{12rst}$.

3. (i) $-1·83$, (ii) $x=10·5$, $y=-2$.

4. (i) $(x+y)^3(x-y)$, (ii) $(a-6b)(10a-3b)$. **5.** 63.

6. (i) $2xy^2$, (ii) $24x^3y^5z$.

K. 1. $\frac{ax+by}{a+b}$ years. **2.** (i) $\frac{21}{68}$, (ii) $x=-6$, $y=-0·25$.

3. (i) $\pm x^{12}y^3$, $\pm x^4y$, (iii) $120l^4m^3n^4$.

4. (i) $x^2(x+7)(x-9)$, (ii) $(lx+my)(mx+ly)$, (iii) $9(7c-4d)(4d-c)$.

5. $11\frac{3}{5}$ sec., 12 sec. **6.** $1\frac{3}{4}$; 1·62, −0·62.

L. 1. $\frac{100(B-A)}{AC}$ per cent. per annum.

2. $15\frac{9}{14}$, (ii) $x=\frac{2}{3}$, $y=-\frac{3}{2}$.

3. (i) $12a^4-16a^3b-27a^2b^2+46ab^3-15b^4$, (ii) $3x^3-x-4$.

4. (i) $s^2\div(18m^2)$, (ii) $256K-510L$.

5. (i) $(a^2+3)(a+1)(a-1)$, (ii) $(a+xy)(lm-k)$, (iii) $(2x+z)(2y-5z)$.

6. $\frac{11}{17}$.

Exercise 50 A (P. 208)

1. (i) 10, (ii) 0, (iii) 0. **2.** (i) $y=0$, (ii) nothing, (iii) $y=0$.

3. (i) Nothing, (ii) $x=0$ if $a\neq5$; otherwise nothing, (iii) $x=3$ if $y\neq4$; otherwise nothing.

4. 2, 5. **5.** −3, 6. **6.** −6, −7. **7.** 0, 5.

8. 0, −2. **9.** $-1\frac{1}{2}$, $-1\frac{2}{3}$. **10.** $1\frac{2}{5}$, twice. **11.** $\frac{3}{4}$, −11.

12. $\frac{5}{6}$, $-\frac{1}{8}$. **13.** 1, 2, 3. **14.** 0, twice. **15.** −2, −3, −4.

16. 0, −7, $\frac{1}{3}$, $\frac{3}{4}$. **17.** $5\frac{1}{2}$, $-2\frac{2}{3}$, $\frac{1}{5}$. **18.** $-3\frac{1}{4}$ twice.

EXERCISE 50 B (P. 208)

1. (i) 63, (ii) 0, (iii) 0.
2. (i) $x=0$, (ii) nothing, (iii) $x=0$.
3. (i) Nothing, (ii) $y=0$ if $c\neq -6$; otherwise nothing, (iii) $y=-7$ if $x\neq -2$; otherwise nothing.
4. 3, 8. **5.** $-9, -5$. **6.** $1, -2$. **7.** 0, 5.
8. $-4\frac{1}{4}, \frac{2}{9}$. **9.** $0, -4$. **10.** $-1\frac{1}{5}, \frac{3}{11}$. **11.** $12\frac{1}{2}$ twice.
12. $\frac{2}{3}, \frac{5}{11}$. **13.** $-1, 2, -3$. **14.** $3, 4, -5$. **15.** $4\frac{1}{3}$ twice.
16. 0, three times. **17.** $0, -9, 5\frac{1}{2}, -1\frac{2}{3}$. **18.** $3\frac{2}{3}, -2\frac{1}{2}, \frac{3}{17}$.

EXERCISE 51 A (P. 209)

1. $-2, -3$. **2.** $-4, 3$. **3.** 8, 8. **4.** $-\frac{1}{3}, -1$. **5.** 4, 5.
6. $\frac{3}{2}, 1$. **7.** $-2, -1\frac{1}{2}$. **8.** $2, -\frac{1}{3}$. **9.** 10, 4. **10.** $\frac{1}{2}, -5$.
11. $3, \frac{2}{3}$. **12.** $\frac{2}{3}, -3$. **13.** $\frac{3}{5}, -2$. **14.** $\frac{1}{5}, -\frac{2}{3}$. **15.** $7, -3$.
16. $-\frac{5}{2}, -\frac{5}{2}$. **17.** $\frac{2}{3}, -\frac{1}{2}$. **18.** $4, -5$. **19.** $8, -5$. **20.** $-4, -3\frac{1}{2}$.
21. $-3, -27$. **22.** $\frac{4}{5}, -\frac{1}{2}$. **23.** $5, -7$. **24.** $2, \frac{5}{2}$.
25. $\frac{1}{2}, \frac{7}{3}$. **26.** $-\frac{1}{3}, -\frac{3}{5}$. **27.** $\frac{5}{2}, -\frac{1}{2}$. **28.** $2, \frac{1}{2}$.
29. $4, -\frac{7}{2}$. **30.** $3, -\frac{7}{10}$. **31.** $\frac{7}{4}, -2$. **32.** $\frac{5}{3}, \frac{4}{5}$.
33. $1, -\frac{5}{4}$. **34.** $-1, -\frac{6}{5}$. **35.** $\frac{5}{3}, -2$. **36.** $\frac{2}{3}, \frac{3}{7}$.
37. $x^2-10x+21=0$. **38.** $x^2+4x-32=0$. **39.** $3x^2-25x-18=0$.
40. $4x^2+11x+6=0$. **41.** $x^3+9x^2-22x=0$. **42.** $x^3-5x^2-4x+20=0$.

EXERCISE 51 B (P. 209)

1. $-1, -4$. **2.** $2, -1$. **3.** $2, -5$. **4.** 2, 2.
5. $-6, -\frac{5}{3}$. **6.** 15, 1. **7.** $-2, \frac{4}{7}$. **8.** $3, -\frac{7}{5}$.
9. $2, \frac{3}{5}$. **10.** $4, -6$. **11.** $9, -10$. **12.** $2, \frac{15}{4}$.
13. $-\frac{7}{3}, -\frac{7}{3}$. **14.** $2, \frac{4}{7}$. **15.** $1, -\frac{21}{2}$. **16.** $-\frac{2}{3}, -\frac{1}{7}$.
17. $12, -3$. **18.** $\frac{2}{3}, \frac{5}{4}$. **19.** $\frac{1}{7}, -\frac{7}{2}$. **20.** $21, -\frac{1}{5}$.
21. $1, \frac{2}{21}$. **22.** $-6, -7$. **23.** $\frac{3}{2}, \frac{7}{12}$. **24.** $18, -4$.
25. $-7, -\frac{2}{5}$. **26.** $-10, \frac{3}{4}$. **27.** $\frac{16}{5}, -\frac{1}{3}$. **28.** $\frac{5}{4}, \frac{7}{2}$.
29. $-3, -\frac{7}{5}$. **30.** $\frac{3}{5}, \frac{5}{3}$. **31.** $\frac{5}{2}, \frac{1}{2}$. **32.** $\frac{5}{4}, -\frac{2}{3}$.
33. $-3, \frac{1}{10}$. **34.** $\frac{2}{3}, -\frac{3}{7}$. **35.** $\frac{3}{10}, -6$. **36.** $\frac{5}{4}, -\frac{7}{2}$.
37. $x^2-10x+9=0$. **38.** $x^2-x-20=0$. **39.** $3x^2-17x-6=0$.
40. $6x^2+7x+2=0$. **41.** $x^3-4x=0$. **42.** $x^3-6x^2=0$.

EXERCISE 52 A (Pp. 214, 215)

1. $7, -3$. **2.** $-1, -15$. **3.** $2, -\frac{2}{3}$. **4.** $1, -2$.
5. $-1\frac{2}{9}, -2\frac{1}{9}$. **6.** $4\frac{1}{7}, 2\frac{6}{7}$. **7.** $2\frac{1}{2}, -1\frac{1}{2}$. **8.** $7\frac{1}{3}, -8\frac{2}{3}$.
9. $-\frac{1}{3}, -3$. **10.** $1, \frac{1}{5}$. **11.** $1\frac{3}{8}, \frac{1}{8}$. **12.** $-1\frac{1}{4}, -1\frac{3}{4}$.

13. 4·45, −0·45. **14.** 1·47, −7·47. **15.** 2·87, −12·87.
16. 14·83, 3·17. **17.** 14·89, −2·89. **18.** −3·68, −10·32.
19. 4·92, −0·92. **20.** 1·74, −3·07. **21.** 1·96, −0·32.
22. 3·36, −0·56. **23.** 0·66, −1·99. **24.** −1·70, −0·55.
25. 16, $x-4$. **26.** 9, $x+3$. **27.** $6\frac{1}{4}$, $c-\frac{5}{2}$.
28. $12\frac{1}{4}$, $y+\frac{7}{2}$. **29.** $\frac{16}{81}$, $x-\frac{4}{9}$. **30.** $\frac{25}{144}$, $x+\frac{5}{12}$.
31. $\frac{4}{49}$, $y-\frac{2}{7}$. **32.** $1\frac{9}{16}$, $y+\frac{5}{4}$. **33.** $1\frac{13}{36}$, $z-\frac{7}{6}$.
34. $\frac{16}{121}$, $x+\frac{4}{11}$. **35.** $\frac{4a^2}{25}$, $x+\frac{2a}{5}$. **36.** $\frac{c^2}{9}$, $x-\frac{c}{3}$.
37. 15, −3. **38.** −6, −2. **39.** 3, −6.
40. 14, −7. **41.** $\frac{1}{2}$, −2. **42.** $-\frac{2}{3}$, 3.
43. $3\frac{1}{2}$, −3. **44.** $\frac{1}{4}$, −3. **45.** $\frac{1}{3}$, $-1\frac{1}{2}$.
46. $\frac{3}{4}$, $-1\frac{1}{3}$. **47.** $-\frac{3a}{2}$, $2a$. **48.** $\frac{4c}{3}$, $-2c$.
49. 4·41, 1·59. **50.** 1·10, −9·10. **51.** 4·30, 0·70.
52. 0·30, −3·30. **53.** 2·69, −0·19. **54.** 0·72, −1·12.
55. 1·32, −0·57. **56.** 2·87, 0·46. **57.** 1·40, −0·24.
58. 0·62, −0·32. **59.** 1·26, 0·45. **60.** 0·27, −0·82.

Exercise 52 b (Pp. 215, 216)

1. 7, 3. **2.** 1, −15. **3.** $-\frac{2}{3}$, −2. **4.** 2, 1.
5. $1\frac{8}{15}$, $1\frac{2}{15}$. **6.** $5\frac{3}{8}$, $3\frac{5}{8}$. **7.** $3\frac{2}{3}$, $-2\frac{1}{3}$. **8.** $8\frac{1}{4}$, $-9\frac{3}{4}$.
9. $2\frac{3}{4}$, $\frac{1}{4}$. **10.** $-\frac{1}{6}$, $-1\frac{1}{6}$. **11.** $1\frac{7}{15}$, $\frac{2}{15}$. **12.** $-\frac{14}{15}$, $-1\frac{11}{15}$.
13. 12·09, −6·09. **14.** 5·63, −7·63. **15.** 1·92, −9·92.
16. 10·45, 5·55. **17.** −14·29, −3·71. **18.** 10·94, −6·94.
19. 2·06, −0·39. **20.** 1·04, −2·24. **21.** −0·35, −1·21.
22. 2·26, −0·55. **23.** 0·67, 0·06. **24.** 1·29, −2·29.
25. 25, $x+5$. **26.** 4, $x-2$. **27.** $2\frac{1}{4}$, $y-\frac{3}{2}$. **28.** $20\frac{1}{4}$, $z+\frac{9}{2}$.
29. $\frac{25}{256}$, $x+\frac{5}{16}$. **30.** $\frac{9}{49}$, $x-\frac{3}{7}$. **31.** $\frac{16}{121}$, $y+\frac{4}{11}$. **32.** $\frac{9}{16}$, $t-\frac{3}{4}$.
33. $\frac{81}{100}$, $c+\frac{9}{10}$. **34.** $\frac{36}{169}$, $z-\frac{6}{13}$. **35.** $\frac{9a^2}{49}$, $x-\frac{3a}{7}$. **36.** $\frac{9c^2}{100}$, $x+\frac{3c}{10}$.
37. 12, −4. **38.** −3, −9. **39.** 12, −7. **40.** −13, 4.
41. $\frac{1}{2}$, 2. **42.** $-\frac{2}{3}$, −2. **43.** $-3\frac{1}{4}$, 4. **44.** $-1\frac{1}{5}$, 3.
45. $\frac{1}{3}$, −3. **46.** $\frac{1}{15}$, 5. **47.** $\frac{19a}{3}$, a. **48.** $-\frac{9d}{2}$, $4d$.
49. −0·27, −3·73. **50.** 7·47, −1·47. **51.** 7·53, −0·53.
52. −0·23, −8·77. **53.** 0·37, −2·70. **54.** 1·30, −0·13.
55. 2·28, 0·22. **56.** 1·53, 0·19. **57.** 0·44, −0·57.
58. 1·37, 0·40. **59.** −0·36, −1·39. **60.** 0·12, −0·82.

EXERCISE 52 C (P. 216)

1. $-\frac{5}{6}, \frac{1}{3}$. **2.** 3·73, 0·27. **3.** $5\frac{1}{3}, -3\frac{2}{5}$. **4.** −2·37, 0·70.
5. −2·58, 0·58. **6.** 5·58, −1·08. **7.** $-3, \frac{3}{5}$. **8.** 1·27, −0·47.
9. $\frac{5}{6}, -\frac{3}{8}$. **10.** 1·26, −0·16. **11.** $-3\frac{1}{2}, -1\frac{1}{6}$.
12. 1·84, −0·41. **13.** 3, −4. **14.** $\frac{2}{5}, -\frac{3}{5}$.
15. −1·17, 0·47. **16.** $\frac{2}{5}, -\frac{4}{15}$. **17.** 2·14, −2·64.
18. $1\frac{1}{2}, 2\frac{1}{7}$. **19.** $4, -\frac{1}{4}$. **20.** 3·06, 1·27. **21.** $-3\frac{3}{7}, -\frac{2}{3}$.
22. 0·27, −0·37. **23.** 3·91, 0·95. **24.** $1, -\frac{7}{8}$. **25.** 1·62, −0·62.
26. $2, \frac{1}{6}$. **27.** $1\frac{4}{5}, -2\frac{3}{5}$. **28.** −0·32, −1·15.

29. 0·52, 0·25. **30.** $2\frac{1}{4}, \frac{1}{4}$. **31.** $a\left(\frac{-7\pm\sqrt{45}}{2}\right)$.

32. $-2\pm\sqrt{-6}$. **33.** $a\left(\frac{-1\pm\sqrt{-27}}{2}\right)$. **34.** $\frac{3\pm\sqrt{105}}{8}$.

35. $\frac{-1\pm\sqrt{-14}}{3}$. **36.** $\frac{-2\pm\sqrt{154}}{20}$. **37.** $\frac{-1\pm\sqrt{-47}}{8}$.

38. $\frac{1\pm\sqrt{-39}}{10}$. **39.** $a\left(\frac{3\pm\sqrt{33}}{4}\right)$. **40.** $\frac{3\pm\sqrt{-31}}{4}$.

41. $\frac{-25\pm\sqrt{433}}{32}$. **42.** $\frac{7\pm\sqrt{-7}}{2}$. **43.** $\frac{3\pm\sqrt{65}}{4}$.

44. $\frac{1\pm\sqrt{-1}}{4}$. **45.** $a\left(\frac{1\pm\sqrt{821}}{82}\right)$. **46.** $\frac{-3\pm\sqrt{3}}{12}$.

47. $\frac{-3\pm\sqrt{-15}}{12}$. **48.** $\frac{-1\pm\sqrt{19}}{3}$.

EXERCISE 53 A (Pp. 218, 219, 220)

1. 7, 11 ; −11, −7. **2.** 13, 11 ; −5, −7. **3.** 11, 12.
4. 12 or −1. **5.** 14. **6.** $11, -11\frac{1}{5}$. **7.** 42 years.
8. 6. **9.** 41. **10.** 10.
11. 26 yd. by 30 yd. **12.** 28 yd. and 13 yd.
13. $2\frac{1}{4}$ sec. After $4\frac{1}{2}$ sec. **14.** 72 years.
15. 11, 12, 13, 14, 15, 16. **16.** 3·66 cm., 6·34 cm.
17. 12 yd. by 96 yd. or 48 yd. by 24 yd. **18.** 18. **19.** 18 ft.
20. 9″. **21.** 7″ by 7″ by $5\frac{1}{2}$″. **22.** 360 ft.
23. 30 yd. **24.** 26.
25. $3\frac{1}{2}$ m.p.h., $10\frac{1}{2}$ m.p.h. 2nd solution not valid. **26.** 2.

EXERCISE 53 B (Pp. 220, 221)

1. 8, 9. **2.** 17, 19. **3.** 4, 13; −13, −4. **4.** 7.
5. 7, 10; −4, −1. **6.** 3, $-2\frac{4}{7}$. **7.** 29″, 22″.
8. 13 years. **9.** 17 yd., 6 yd. **10.** 25. **11.** 21. **12.** 5.
13. 3. **14.** $1\frac{1}{2}$ sec., $5\frac{1}{4}$ sec. **15.** 15 ft.
16. Each part 4″. **17.** 22. **18.** 10 ft. **19.** 9 cm., 4 cm.
20. 19·1 ft., 16·1 ft. **21.** 60 yd. **22.** 10.
23. 8″ by 7″ by 5″. **24.** 12 ft. **25.** 20 ft.
26. $3\frac{1}{2}$ m.p.h., 14 m.p.h. 2nd solution not valid.

EXERCISE 54 A (Pp. 226, 227)

1. (2, 1). **2.** $(1\frac{1}{2}, 4)$. **3.** $(2, -\frac{1}{2})$. **4.** $(4, \frac{2}{3})$.
5. (2·5, −3·6). **6.** $(-\frac{3}{4}, 2)$. **7.** $(2, \frac{1}{5})$; $(1\frac{4}{7}, \frac{1}{14})$.
8. (0, −4); $(-1\frac{1}{2}, -2\frac{1}{2})$. **9.** (2, −1) twice.
10. 2·73, −0·73. **11.** $x^2 - x - 2 = 0$; 2 and −1. **12.** $4\frac{1}{3}$, $2\frac{1}{2}$.
13. 4, −1·5; $2y - 5x = 12$. **14.** 1·78, −0·28; $2y - 3x = 1$.
15. 2, −2·4; $5y + 2x = 24$. **16.** −1·72, −1·28; $5y + 15x + 11 = 0$.
17. 1·6, −1·7; $10y + x = 27{\cdot}2$.
18. No solutions; $11y - 18x + 8 = 0$. **19.** −3, 6·84, −1·17.
20. 3·87, 0·13; except between $x = 1{\cdot}38$ and $x = 3{\cdot}62$.
21. (*a*) 3·13, −0·80, (*b*) −1, $1\frac{2}{3}$.
22. 1, $-1\frac{1}{2}$. Find the values of x for which $y = \frac{1}{2}$.
23. Find the values of x where $y = 1\frac{1}{2}$; 2·6, −2·1. **24.** 2, 0·6, −2·6.

EXERCISE 54 B (Pp. 227, 228)

1. $(\frac{3}{5}, 3)$. **2.** $(1\frac{1}{2}, -2)$. **3.** $(-2, \frac{1}{2})$. **4.** $(3, \frac{4}{5})$.
5. (3·2, −2·4). **6.** $(\frac{4}{5}, -2)$. **7.** (2·1, −0·05); (−0·36, −1·28).
8. (5, 3); (−2, −4).
9. (0, 1); (1, 0); (0·71, 0·71); (−0·71, −0·71).
10. 3·30, −0·30. **11.** 2·38, −0·66. **12.** −1, 0, 3.
13. −1·31, −0·19; $4y + 6x + 1 = 0$. **14.** 3, 1·6; $23x = 24 + 5y$.
15. 1, −0·8; $5y - x = 4$. **16.** No solutions; $4y = x - 1$.
17. −2, $1\frac{1}{3}$; $3y + 2x = 8$. **18.** 2, $-\frac{3}{4}$; $5x + 6 = 4y$.
19. 60·7. **20.** (1) $2x^2 - x - 12 = 0$, (2) $4y^2 - 33y + 62 = 0$.
21. 0, 0·6, 3·4; $x^3 - 4x^2 + 2x = 0$.
22. Find the values of x where $y = 1$; 2·6, −2·1.
23. 2·85, −0·35. **24.** 1·68, −0·48; $a = -1{\cdot}8$.

Exercise 55 A (Pp. 230, 231)

1. $y=2x+3$. **2.** $2y=5x+1$. **3.** $2y+3x=0$.
4. $y+3x=8$. **5.** $3y=x+7$. **6.** $2y+x=10$.
7. $s=100t-25t^2$. **8.** $s=2+2t-t^2$.

Exercise 55 B (Pp. 231)

1. $y=3x-1$. **2.** $3y=4x$. **3.** $4y+x=11$.
4. $5y=3x-8$. **5.** $2y=5x+3$. **6.** $7x+8y=5$.
7. $y=5x^2$. **8.** $y=\frac{1}{4}x^2+5$; (6·3, 16) is probably wrong.

Exercise 56 A (P. 234)

1. $\frac{2}{xy-1}$. **2.** $\frac{a}{b}$. **3.** $\frac{2x}{3y}$. **4.** $\frac{3a}{2a+b}$.
5. $\frac{4x+3y}{2x(4x-3y)}$. **6.** $\frac{2x+7}{2x-7}$. **7.** $\frac{(a-1)(a+2)}{2a-1}$.
8. $\frac{a^2-1}{a^2-4}$. **9.** $\frac{y+7}{x+y}$. **10.** -2.
11. $-\left(\frac{2a+1}{4a+1}\right)$. **12.** $\frac{3}{25}$. **13.** $\frac{x-7}{x+6}$.
14. $\frac{(a+2)(a+1)}{a^2+1}$. **15.** $\frac{x-1}{x-5}$. **16.** $\frac{a+6}{a-3}$.
17. $\frac{r+2s+t}{t}$. **18.** $\frac{c+3}{c-5}$. **19.** $2c^2$.
20. $\frac{1}{(2s-t)(s-t)}$. **21.** $x+z$.

Exercise 56 B (Pp. 234, 235)

1. $\frac{3}{4cd-5}$. **2.** $\frac{b}{a}$. **3.** $\frac{c}{d}$. **4.** $\frac{x-3y}{x}$.
5. $\frac{3a-2b}{2a^2}$. **6.** $\frac{a-9}{3a-1}$. **7.** $\frac{(x+1)(x+4)}{(x-2)(x+7)}$. **8.** $\frac{3y}{4x^2}$.
9. $\frac{3(y+3)}{2(y+4)}$. **10.** $-\frac{2}{5}$. **11.** $-\left(\frac{2c+d}{2c+3d}\right)$. **12.** $-\frac{3}{x}$.
13. $-\frac{1}{2}$. **14.** 1. **15.** $\frac{1}{a+3}$. **16.** $\frac{2a^2b(a-3)}{2a-3}$.
17. $2x-z$. **18.** $\left(\frac{3x+2y-5z}{3x-2y-5z}\right)^2$. **19.** $\frac{z-2y}{y^2}$.
20. 1. **21.** $x^2(x^2+y^2)(c-d)$.

EXERCISE 57 A (P. 235)

1. $(x-2)(x+5)(3x-1)$.
2. $36(x-2)^2(x+2)$.
3. $(a+2b)(a-2b)(a+3b)$.
4. $3(x+3)(x-3)$.
5. $(4t+3k)(4t-3k)(2t-3k)$.
6. $3x^2(x+2)(x-2)$.
7. $(x-11)(x+3)(x-2)$.
8. $12(x-7)(2x+1)(x+3)$.
9. $(x-a)(x+a)(x-b)^2$.
10. $20x^2(x+4)(3x-1)(2x-3)$.
11. H.C.F. $2(x-2)$; L.C.M. $20x(x-2)^2(x-5)(3x-1)$.
12. H.C.F. $(x+1)$; L.C.M. $12x^2(x+1)(3x-1)(5x-2)(2x+1)$.

EXERCISE 57 B (Pp. 235, 236)

1. $(2x+1)(x-4)(3x+2)$.
2. $30(7x-2)^2(x+6)^2$.
3. $(x+3)^2(x+4)$.
4. $(2x+5)(2x-5)(x-3)$.
5. $a(a-1)^2(a+1)$.
6. $2(3c+4d)(3c-4d)$.
7. $(3x-5)(x+1)(7x-6)$.
8. $24x(x+2y)(x-2y)(x-6y)(x+y)$.
9. $30(2x-3)^2(x+5)(x-5)$.
10. $2a(2a+b)(a-3b)(3a-b)$.
11. H.C.F. $(3a-5b)$; L.C.M. $a(3a-5b)(a+2b)(2a+3b)(2a-3b)$.
12. H.C.F. $x(2x+1)$; L.C.M. $4x^2(2x+1)(x-1)(x-3)$.

EXERCISE 58 A (P. 237)

1. $\dfrac{2x}{(x+2)(x-2)}$.
2. $\dfrac{2}{(x+7)(x+9)}$.
3. $\dfrac{2x+14}{(x-3)(x+1)}$.
4. $\dfrac{2a-3b}{(a+6b)(a+3b)}$.
5. $\dfrac{13x+20y}{(4x-3y)(3x+2y)}$.
6. $\dfrac{2ab-3ac}{(a-2b)(a-3c)}$.
7. $\dfrac{2a-1}{(a+3)(a-3)}$.
8. $\dfrac{a^2+4b^2}{(a-2b)^2(a+2b)}$.
9. $\dfrac{2xy}{(x+3y)(x-2y)}$.
10. $\dfrac{1}{(y-3)(y-2)}$.
11. $\dfrac{4}{(a+8)(a+4)}$.
12. $\dfrac{5y^2}{(x+y)(x-2y)(x-3y)}$.
13. $\dfrac{4x^2+27}{6(2x-3)}$.
14. $\dfrac{12(2x+7)}{(x+3)(x+4)(x+7)}$.
15. $\dfrac{2(21x+8)}{3(3x+4)^2(x-2)}$.
16. $-\dfrac{9a^2}{(1-3a)^2}$.
17. $-\dfrac{(x+3)}{3x(3x-8)(x-3)}$.
18. $\dfrac{1}{2x+1}$.
19. 0.
20. $\dfrac{19-x}{(x^2-9)(x-2)}$.
21. $\dfrac{2}{x+3}$.
22. $-\dfrac{(2x+7)}{2(x+3)^2}$.
23. $\dfrac{3}{x-1}$.
24. $\dfrac{20}{9a^2-4b^2}$.

EXERCISE 58 B (P. 238)

1. $\frac{3}{(3x+5)(3x+8)}$. **2.** $\frac{4x}{(2x+7)(2x-7)}$. **3.** $\frac{3a+13b}{(a+2b)(a+3b)}$.

4. $\frac{3c+34}{(c-2)(c+3)}$. **5.** $\frac{3xy-5xz}{(x-6y)(x-10z)}$. **6.** $\frac{l+17m}{(5l-2m)(7l+3m)}$.

7. $\frac{x-9}{x(2x-3)(2x+3)}$. **8.** $\frac{9x+10y}{(3x-7y)(3x+2y)}$. **9.** $\frac{5a-b}{(a+b)(a-b)^2}$.

10. $\frac{1}{(a-4)(a-3)}$. **11.** $\frac{1}{5x-1}$. **12.** $\frac{2x+1}{x(x-1)(x+1)}$.

13. $\frac{9x^2+1}{3x}$. **14.** $\frac{7x-6}{7x(7x+2)(7x+3)}$. **15.** $\frac{3x}{(x-1)(x+1)^2}$.

16. $\frac{1}{1-x}$. **17.** $\frac{a-4b}{a-2b}$. **18.** $\frac{5}{x+5}$. **19.** $-\frac{4}{(a^2-1)^2}$.

20. $\frac{2(x-8)}{(x-3)(x-2)^2(x+2)}$. **21.** $\frac{8d}{3c+2d}$. **22.** $\frac{2}{3x+2}$.

23. $-\frac{7}{4c^2-9y^2}$. **24.** $-\frac{36ab}{(3a-4b)(3a+4b)(3a+2b)}$.

EXERCISE 58 C (Pp. 238, 239)

1. $\frac{2x}{x^2-1}$. **2.** 0. **3.** $\frac{14}{x^2-49}$. **4.** $\frac{7y}{4}$.

5. $\frac{x}{1-x^2}$. **6.** $\frac{x^2}{x^2-4}$. **7.** $\frac{1}{3c}$. **8.** $\frac{1}{(1-x)(1-y)}$.

9. 0. **10.** $\frac{1}{a+1}$. **11.** 1. **12.** $\frac{y}{(x-y)^2}$.

13. $\frac{3x}{(1-x)(1+2x)}$. **14.** $-\frac{a^2}{(a+2b)(a+b)^2}$.

EXERCISE 59 A (P. 240)

1. $\frac{5x}{10x^2+3y}$. **2.** $x-2y$. **3.** $\frac{c}{b}$. **4.** $\frac{a+2}{a+5}$.

5. $\frac{3x-1}{3}$. **6.** $\frac{3x+1}{3x-1}$. **7.** $\frac{2x^2+9x-1}{4x^2-1}$. **8.** $\frac{4ab}{b^2-4a^2}$.

9. $\frac{2(2x-1)}{(2x-3)}$. **10.** x^2+6x+8. **11.** 1. **12.** $\frac{2(x-1)}{4x-1}$.

13. $\frac{2}{3x-4}$. **14.** $\frac{1}{2x(x+y)}$. **15.** $\frac{x}{b}$.

16. $\frac{4(9m^2-1)}{m}$. **17.** $\frac{-2(x+6)}{x(x+10)}$. **18.** $\frac{15(y+1)}{(y+4)}$.

EXERCISE 59 B (P. 241)

1. $\frac{3a(3-c)}{2b}$. 2. $\frac{5y}{7x}$. 3. $-\frac{ac}{bd}$. 4. $\frac{3x(x-8)}{2(x+7)}$.

5. $x-\frac{1}{x}$. 6. $\frac{3b}{2a+3b}$. 7. $\frac{x-3}{x+12}$. 8. 1.

9. $\frac{2a+3b}{2a-3b}$. 10. $\frac{4x}{x-b}$. 11. $\frac{2(x-2)}{3(2x-1)}$. 12. x.

13. $\frac{25}{196}$. 14. $x+y$. 15. $\frac{3x-2y}{3x+2y}$.

16. $\frac{1}{7(7a-10)}$. 17. $\frac{54a^2+120a-25}{81a^2+105a+25}$. 18. $\frac{a(a-b+c)}{c(a+b+c)}$.

EXERCISE 60 A (Pp. 242, 243, 244)

1. $\frac{5}{8t}$. 2. $\frac{ab}{3}$. 3. $\frac{2d}{3c}$. 4. $\frac{16X}{3}$ weeks.

5. $\frac{2c}{3}$ days. 6. $\frac{7u}{18}$ miles. 7. $2h(l+d)$ sq. ft. 8. $\frac{3n+5}{2}$ pence.

9. (i) $(u+v)$ m.p.h., (ii) $(u-v)$ m.p.h., $\frac{n}{u-v}$ hours.

10. (i) $\left(x-\frac{z}{6}\right)\left(y-\frac{z}{6}\right)$ sq. ft., (ii) $\frac{6xz+6yz-z^2}{36}$ sq. ft.,

(iii) $\frac{n(6xz+6yz-z^2)}{27}$ pence.

11. (i) $\frac{240}{n}$ pence, (ii) $\frac{190+n}{n}$ pence. 12. $\frac{47a^2}{4}$ sq. ft.

13. $£\left(y-\frac{z}{240}\right)$, $\frac{100z}{240y-z}$. 14. $\frac{25(12x-y)}{3x}$.

15. $\frac{Ax}{x+y+z}$ lb., $\frac{Ay}{x+y+z}$ lb., $\frac{Az}{x+y+z}$ lb. 16. $\frac{aA}{a+b+c}$ cwt.

17. $\frac{x+y}{a+1}$. 18. $\frac{xyz}{5}$ shillings. 19. $100-a-b-c$; $\frac{100X}{c}$, $\frac{aX}{c}$.

20. $100\left(1-\frac{a^2}{10000}\right)$. 21. $\frac{xy}{x+y}$ sec. 22. $x=30$, $y=90$, $z=45$.

23. $£\frac{bA}{100}$. 24. $£\frac{PB}{A}$. 25. $\frac{lp}{3m}$. 26. $£\frac{B(C-D)}{100}$.

28. $\frac{100}{x}$ years. 29. $z=x+y$. 30. $(180-x-y)^\circ$, $\left(180-\frac{x}{2}-y\right)^\circ$.

EXERCISE 60 B (Pp. 245, 246, 247)

1. $\frac{k}{45t}$. **2.** $\frac{3}{20}$. **3.** $\frac{x}{2}$ weeks. **4.** $2b$.

5. $\frac{d}{u}$ hours. **6.** $\frac{9az}{xy}$ shillings. **7.** $\frac{10-n}{10}$. **8.** $\frac{xyz}{4}$ cu. ft.

9. (i) 6d, (ii) $(n+4)$d., (iii) 1s. It must be sent otherwise than by post

10. $\frac{x-2y}{z}$ in. **11.** $\frac{3k}{20n}$. **12.** $80x^2$.

13. $k\left(1-\frac{c}{100}\right)$ shillings, $\frac{ck}{100}$ shillings. **14.** $X\left(1-\frac{y}{100}\right)$.

15. $2p+3q+5r$. **16.** $\frac{x+c}{a+1}$. **17.** $\frac{p(x+y+z)}{x}$ lb.

18. $\frac{100Q}{Py}$. **19.** $\frac{lx+my}{l+m}$ shillings. **20.** $\frac{10{,}000\,Y}{(100-A)(100-X)}$.

21. £$\frac{QP}{x}$, £$\frac{100P}{x}$. **22.** £$\frac{100C}{B}$. **23.** $\frac{ab}{b-a}$ min.

24. $\frac{bcy}{ax}$. **25.** $a+b+c=360$. **26.** $\frac{7gxz}{y}$.

28. £$\frac{X}{100}\left(p\sim\frac{qY}{Z}\right)$. **29.** $x(c-x)=b(a-b)$.

30. z°, $(x-z)^\circ$ [or $(180-y-z)^\circ$].

EXERCISE 60 C (Pp. 247, 248)

1. $\left(z-\frac{a}{bxy}\right)$ ft. **2.** £$x\left(1-\frac{y}{20}\right)$, $5y$. **3.** $\frac{2a}{x(4x^2-1)}$ hours.

4. $\left(\frac{pa}{20}+\frac{qb}{144}\right)$ shillings; $\frac{20}{p+q}\left(\frac{pa}{20}+\frac{qb}{144}\right)$ shillings.

5. $\frac{363xyz}{224}$ tons. **6.** $\frac{2xy}{x+y}$ m.p.h. **7.** £$\frac{NF(a+3b)}{4b}$.

8. $b=a-\frac{(x-y)z}{60}$; after $\frac{60(a-c)}{x-y}\left[\text{or } \frac{z(a-c)}{(a-b)}\right]$ min.

9. $\frac{10xz}{3(x+y)}$. **10.** (i) £$\left(\frac{20x}{yz}-x\right)$, (ii) $100\left(\frac{20}{yz}-1\right)$ per cent.

11. $\frac{144\times 112np}{xz}$ ft. **12.** $50\left[abc-\left(a-\frac{d}{6}\right)\left(b-\frac{d}{6}\right)\left(c-\frac{d}{12}\right)\right]$ lb.

13. $xyz\times\frac{144}{p}\left(1-\frac{k}{100}\right)$.

14. (i) $\frac{12px}{5}-3py$ $2pz$ shillings, (ii) $\frac{4px}{5}-py$ shillings, (iii) nothing.

EXERCISE 61 A (Pp. 249, 250)

1. 88, 22·5. **2.** 120, 12·3. **3.** 1840, 42. **4.** 9, 24.
5. 84. **6.** 0·6, ±0·25. **7.** 3·5, 6 or 4·5. **8.** 20.
9. 1, $29\frac{52}{81}$, −60.
10. (i) 1·5 m.p.h, 22·5 m.p.h., (ii) 176 yd. per min., 616 yd. per min.
11. 100 cu. ft., 1 ft.
12. (i) 104° F., 32° F., 167° F., (ii) 5° C., −40° C., 100° C.
13. 125,250 ; 240,200 ; 100. **14.** 18,496 ; 41,075.
15. 4 per cent. **16.** 22·5. **17.** £882, £500. **18.** 26 cm.

EXERCISE 61 B (Pp. 250, 251)

1. 15, −3·5. **2.** 38·808, 1·4. **3.** 70·95, 19. **4.** 122, 5.
5. 2·1, 2. **6.** 331·5, 4·5. **7.** 156, 12. **8.** 35, 12 or −9.
9. −5, $-\frac{20}{9}$. **10.** 98·56 sq. in., 3·5 cm.
11. 18 miles, 45 m.p.h. **12.** 400 ft., 30 sec.
13. Yes. Nearly $1\frac{2}{3}$ tons. **14.** 2870 ; 12,685. **15.** 7·2.
16. £210. **17.** 37·5 lb., 9600 lb. **18.** 15 miles.

EXERCISE 62 A (Pp. 254, 255)

1. $n=\frac{A+14}{2}$. **2.** $k=\pm\sqrt{\frac{C}{5}}$. **3.** $n=\frac{5L}{3x^2}$.

4. $x=\frac{y-z}{7}$. **5.** $l=\frac{32\cdot2t^2}{4\pi^2}=\frac{8\cdot05t^2}{\pi^2}$. **6.** $r=\pm\sqrt{\frac{A}{\pi}}$.

7. $\frac{QR-PV}{PVT-QRS}$. **8.** $\frac{l^2-27k^2}{l}$. **9.** $-\frac{Ly}{2lm+L}$.

10. $\frac{3-3a}{2+a}$. **11.** $\pm\sqrt{\frac{S^2-\pi^2r^4}{\pi^2r^2}}$.

12. (i) $\frac{(2x+3y)(p+q)}{6y}$, (ii) $\frac{6ty-2xq-3yq}{2x+3y}$. **13.** $\frac{p}{mt-pt-ms}$; $\frac{1}{250}$.

14. $\frac{a(b^2-p^2)}{b^2+c^2-p^2}$, $\frac{1}{4}$. **15** (a) $\frac{\sqrt{y^2-a^2x^2}}{bx}$, (b) $\frac{1}{2}$, (c) $\frac{3}{4}$.

16. $\frac{273(pv-cd)}{cd}$. **17.** $\frac{fv}{f-v}$; 2 : 1. **18.** $\frac{45l^2}{9l^2-8m^2}$.

19. $u=\frac{2s-ft^2}{2t}=v-ft$; $v=\frac{s}{t}+\frac{1}{2}ft$; $s=66$.

20. (i) 5875, (ii) $\frac{3B}{5}+\frac{188T}{B^2}$, (iii) $349\frac{1}{3}$ ft.

EXERCISE 62 B (Pp. 255, 256)

1. $h=\frac{2A}{b}$. **2.** $x=\pm\sqrt{\frac{7L}{3n}}$. **3.** $c=2s-a-b$.

4. $l=\frac{t^2}{9}$. **5.** $C=\frac{F-32}{1\cdot 8}$. **6.** $P=\frac{100I}{rt}$.

7. $\frac{3bK}{K+2a}$. **8.** $\frac{6a-9}{3a-2}$. **9.** $\sqrt[5]{\frac{3c-U}{2d}}$.

10. $h\left(1-\frac{v}{l\sqrt{k}}\right)$. **11.** $h=\frac{x-k}{x+3k}$. **12.** (i) $\frac{2by(1+c)}{a(1-c)}$, (ii) $\frac{ax(1-c)}{2y(1+c)}$.

13. $\frac{ab-3b}{ac-c}$; $\frac{2}{3}$. **14.** (i) $\frac{S}{n}-\frac{d(n-1)}{2}$, (ii) $23\frac{1}{2}$.

15. $\frac{a[b^2+(p-q)^2]}{b^2-(p-q)^2}$, $9\cdot 65$. **16.** $\frac{l^2+n}{6(1+l^2n)}$. **17.** $\frac{5(y^2+5)}{(y^2-2)}$.

18. $\frac{V}{2\pi lt}+\frac{t}{2}$; $3\cdot 27$. **19.** 36.

20. (i) $\frac{y-c}{m}$, (ii) $\frac{2-3y}{y+3}$, (iii) $3y^2+2y+1$.

TEST PAPERS V (Pp. 256, 257, 258, 259, 260, 261, 262)

A. 1. $\frac{x(pq+qr+rp)}{6}$ shillings, $\frac{108pqrn}{7y}$.

2. (i) $2x(3x+8)(x-9)$, (ii) $(s+t)(t+3)(t-3)$.

3. (i) $x-4$, (ii) $\frac{x}{(x+17)(x+5)}$. **4.** (i) -30, (ii) 1, $-1\frac{1}{3}$.

5. 5, 6, 7.

6. They meet at $(-2, 0)$ and $(4, 18)$; $-2\frac{1}{4}$, $x^2-2x=8$.

B. 2. (i) $\frac{a-2}{b}$, (ii) $\frac{-x^2+15x+99}{(x-3)(x+12)}$.

3. (i) $x=-0\cdot 5$, $y=4$, (ii) $\frac{1}{6}$, $-3\frac{1}{2}$. **4.** 918.

5. $3(2x-1)(2x+5)(x+1)$. **6.** $(1, 4)$; none.

C. 1. $x=180-2y$.

2. (i) $(x-y)(x-y+1)(x-y-1)$, (ii) $(7+t^3)(5-t^3)$, (iii) $2(7a-5)^2$.

3. (i) $\frac{c(5-c)}{3c+2}$, (ii) $\frac{27t^3}{(2-9t(1-3t)^2}$. **4.** (i) $9\cdot 9$, (ii) $-0\cdot 27$, $7\cdot 27$.

5. $2\cdot 1'$, $4\cdot 7'$. **6.** Between $7\frac{7}{9}$ and 10 miles.

D. 1. £$\frac{A}{c+1}$, £$\frac{Ac}{c+1}$. **2.** (i) $x=1$, $y=-0\cdot 6$, (ii) $0\cdot 9$, $-1\cdot 9$.

3. $\frac{3}{x(2x-3)}$. **4.** (i) $\frac{2a-1}{4a(2a+1)}$, (ii) $\frac{x^2+4y^2}{4y^2}$.

5. $12(2x-3)^2(x+1)$. **6.** 100 miles, 2 hr. 32 min.

E. **1.** (i) $x(3x+2y)(3x-2y)$, (ii) $(5x-3y)(3y-x)$.

2. $x(a-2x)(b-2x)$ cu. in. ; $(ab-4x^2)$ sq. in.

3. (i) $\frac{1}{2(a-3b)}$, (ii) $\frac{ab}{a^2+b^2}$. **4.** (i) $x=3\frac{2}{3}, y=2$, (ii) $0{\cdot}69, -1{\cdot}26$.

5. $69{\cdot}5$ gallons, $20{\cdot}7$ cm. **6.** $16, -34$.

F. **1.** $x+Zy+Z$. **2.** $-\frac{1}{2}, -\frac{1}{2}$. **3.** (i) $2{\cdot}67$, (ii) $4, -5$.

4. $12(2x-1)^2(x+1)^2$. **5.** 12 miles.

6. The roots are the values of x for which $y=3$. $3{\cdot}84, -2{\cdot}34$.

G. **1.** (i) $(z^2+9)(z+3)(z-3)$, (ii) $(a+b)(a+2b)(a-3b)$.

2. $\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{x}$. **3.** (i) $x=\frac{1}{4}, y=-\frac{1}{9}$, (ii) $2{\cdot}3, -0{\cdot}9$.

4. (i) $\frac{(a-1)}{a(1-2a)}$, (ii) $\frac{2}{a}$. **5.** $1{\cdot}70, 4{\cdot}70$. **6.** $\left(-4\frac{2}{13}, -2\frac{6}{13}\right)$.

H. **1.** $\frac{1200y}{12a+b}$. **2.** (i) $\frac{5x+3}{(5x-2)^2}$, (ii) $-\frac{2b}{1-a^2b^2}$.

3. (i) $(0{\cdot}98, 2{\cdot}38)$, (ii) 8, 3. **4.** $\frac{1}{2}$ mile.

5. $(3a-2b)^2(3a+2b)(2a-5b)$.

6. $a=1\frac{1}{2}, b=81$. y pos. if $x>-54$, x pos. if $y>81$.

I. **1.** He spends £$\frac{A(x-y)+By}{100}$ more.

2. (i) $d(c+2d)(c+3d)(4c+13d)$, (ii) $(5c+3d)(3c-5d)$.

3. (i) $23{\cdot}19$, (ii) $0{\cdot}18, -0{\cdot}09$. **4.** $1''$. **5.** $4(1-x)$. **6.** £41.

J. **1.** $\frac{x^2}{100}$ per cent. gain. **2.** 33 years.

3. (i) $\frac{7(x-1)}{(3x-1)(4x+1)}$, (ii) $\frac{2a}{(a+x)(a+2x)}$.

4. (i) $(-\frac{1}{8}, \frac{1}{8})$, (ii) $-1{\cdot}31, 0{\cdot}51$. **5.** £50. **6.** $0{\cdot}18$.

K. **1.** (i) $(2x+1)(x+3)(x-3)$, (ii) $(l+m)(l-m)(d+c)(d-c)$.

2. $\frac{xyk}{z}$ days. **3.** (i) $(3, -0{\cdot}8)$, (ii) $-22, -\frac{8}{11}$.

4. (i) $\frac{2}{x-2}$, (ii) $\frac{a+5b}{a+6b}$. **5.** $40'', 42'', 58''$. **6.** $2, -1, \cdot 1$.

L. **1.** $\frac{a(100-x)+b(100-y)}{a+b}$. **2.** £36. **3.** x^3.

4. (i) -2, (ii) -3 or $2\frac{1}{5}$. **5.** $2\frac{1}{2}$d.

6. $6(2x+9y)^2(2x-9y)(x-y)$.

Exercise 63 A (Pp. 266, 267)

1. $3\frac{1}{4}$. **2.** $-19\frac{7}{27}$. **3.** $-0{\cdot}06$. **4.** $\frac{4}{37}$.
5. 4. **6.** $1{\cdot}76$. **7.** $6{\cdot}20$. **8.** $0{\cdot}86$.
9. $-\frac{2}{5}$. **10.** $3{\cdot}30$, $-0{\cdot}30$. **11.** $\frac{25}{28}$. **12.** $5\frac{1}{9}$.
13. $\frac{1}{4}$, 1. **14.** $1\frac{3}{5}$, $-\frac{4}{5}$. **15.** $-4\frac{3}{4}$, $-2\frac{1}{4}$. **16.** $-1\frac{3}{5}$, $\frac{4}{5}$.
17. 0, $-2\frac{1}{3}$. **18.** $-\frac{2}{5}$, $3\frac{1}{3}$. **19.** $-3\frac{3}{4}$, $-1\frac{2}{3}$. **20.** $-\frac{1}{7}$, $\frac{3}{7}$.
21. $1\frac{1}{2}$, $-\frac{3}{5}$. **22.** 7, $\frac{1}{2}$. **23.** $-0{\cdot}81$, $0{\cdot}31$. **24.** No solution.
25. $0{\cdot}65$, $-0{\cdot}81$. **26.** $\frac{12}{13}$. **27.** 2.
28. Any value except $x=0$. **29.** $\frac{1}{4}$. **30.** $-1{\cdot}42$, $-2{\cdot}58$.
31. $0{\cdot}39$, $-0{\cdot}17$. **32.** $1\frac{2}{3}$, 0. **33.** 3 or $6\frac{1}{2}$. **34.** 2.
35. $-1{\cdot}30$, $0{\cdot}59$. **36.** $2\frac{21}{29}$. **37.** $0{\cdot}13$, $-0{\cdot}88$. **38.** 4, $1\frac{1}{3}$.
39. 4, $-\frac{2}{3}$. **40.** $1\frac{11}{17}$.

Exercise 63 B (Pp. 268, 269)

1. $21\frac{23}{27}$. **2.** $-2\frac{39}{47}$. **3.** $5{\cdot}5$. **4.** 2.
5. 19. **6.** $5{\cdot}17$. **7.** $-1{\cdot}35$. **8.** $3{\cdot}33$.
9. $-1\frac{3}{4}$. **10.** 2. **11.** $7{\cdot}15$ or $0{\cdot}51$. **12.** $1{\cdot}22$, $-0{\cdot}72$.
13. 2, -2. **14.** $-\frac{3}{4}$, $-3\frac{1}{2}$. **15.** $11{\cdot}4$, 10. **16.** $2\frac{2}{3}$, $4\frac{1}{2}$.
17. $\frac{10}{37}$, $\frac{5}{24}$. **18.** $-\frac{1}{4}$, $\frac{1}{3}$. **19.** $3\frac{7}{27}$, $-\frac{16}{27}$. **20.** $2\frac{1}{4}$, $\frac{1}{6}$.
21. $-1\frac{1}{3}$. **22.** No solution. **23.** $1{\cdot}42$, $0{\cdot}27$. **24.** $4\frac{2}{3}$.
25. $-\frac{1}{9}$. **26.** $2\frac{1}{2}$ twice. **27.** 1, $-\frac{14}{15}$.
28. Any value except $x=-1\frac{1}{2}$. **29.** 0. **30.** $-3\frac{1}{5}$.
31. $-2{\cdot}28$, $-0{\cdot}22$. **32.** $2{\cdot}69$, $-0{\cdot}19$. **33.** $-1\frac{1}{2}$.
34. -2, $-4\frac{1}{2}$. **35.** $5\frac{1}{2}$. **36.** $1{\cdot}46$, $-5{\cdot}46$. **37.** 6.
38. $\frac{1}{4}$ or $-\frac{1}{6}$. **39.** $4\frac{1}{2}$. **40.** -5, $-\frac{1}{3}$.

Exercise 64 A (P. 270)

1. $3l-2m+5$. **2.** $\dfrac{8a+3pq}{p}$. **3.** $\dfrac{b-6}{a-3}$. **4.** $\dfrac{m}{l-4}$.
5. $6c-1$. **6.** $\dfrac{2}{3-a}$. **7.** $2a$. **8.** $-cd$.
9. $\dfrac{2-a+b}{3c}$. **10.** $\dfrac{3l}{2m-5n}$. **11.** $\dfrac{ac+b^2-abd}{3a-2b}$.
12. $\dfrac{2ab}{a+b}$. **13.** $\dfrac{ab}{a^2-b^2}$. **14.** $\dfrac{V+\pi lt^2}{2\pi lt}$. **15.** 1.
16. $p+2q$. **17.** $a-b$. **18.** $5l-4m$.

Exercise 64 b (P. 271)

1. $\frac{l-3n}{2m}$. 2. $\frac{2qr}{p+6q}$. 3. $2c-3d$. 4. $\frac{bc+ad}{an-bm}$.
5. $\frac{h}{c+3d}$. 6. $-\frac{a^2}{b}$. 7. $\frac{m^2-mn-n^2}{2n}$. 8. 0.
9. $\frac{2cd}{3c+d}$. 10. $\frac{c-d}{a-b}$. 11. $-\frac{5b}{2a}$. 12. $\frac{k^2-k-1}{k^2-1}$.
13. $\frac{2ab}{a+b}$. 14. $-\frac{20b^2}{c}$. 15. $\frac{a+b}{3}$. 16. lmn.
17. $a-b$. 18. $\frac{5l}{4}$.

Exercise 64 c (P. 271)

1. $2a$. 2. $2b$. 3. $-l$. 4. $-5a$.
5. $\frac{a+2b}{2}$. 6. $\frac{m-l}{2}$. 7. $-\frac{3a}{2}$. 8. $-8l$.
9. $mn+nl+ln$. 10. $\frac{ac}{b}$. 11. $m-n$. 12. $4(a^2+ab+b^2)$.

Exercise 65 a (P. 273)

The values of x are placed first and the values of y second.

1. l, l. 2. p, $2q$. 3. $3l+m$, $3l-m$. 4. 0, 0.
5. $a+b$, $a-b$. 6. $\frac{7l-4m}{33}$, $-\frac{(4l-7m)}{66}$. 7. $\frac{1-b}{1+a}$, $\frac{b-1}{1+a}$.
8. $\frac{2c(a+b)}{a^2+b^2}$, $\frac{3c(a-b)}{a^2+b^2}$. 9. $6l$, $-5l$. 10. $-2a$, $-2a$.
11. $a-b$, $a+b$. 12. $5a$, $4b$. 13. $14a$, $22b$.
14. $\frac{3a-5b}{3a}$, $-\frac{(3a+10b)}{5b}$. 15. $p=1$.

Exercise 65 b (Pp. 273, 274)

The values of x are placed first and the values of y second.

1. $-3b$, $2a$. 2. $-2l-2m$, $4l+m$. 3. $6d$, c. 4. $a+b$, $b-a$.
5. 1, -1. 6. $7v-4u+30$, $9v-5u+30$. 7. $a+b$, $b-a$.
8. $3l+1$, $-2m-1$. 9. $33a$, $-40a$. 10. $4lm$, -1.
11. $\frac{2a-1}{2a+1}$, $\frac{1-2a}{2(2a+1)}$. 12. $a+b$, $-\frac{2ab}{a+b}$. 13. a^2-b^2. $2ab$.
14. $\frac{a+2b}{5}$, $a-2b$.

EXERCISE 66 A (Pp. 275, 276)

1. $\frac{2c}{3}, -4c.$ **2.** $\frac{2a}{3}, -\frac{a}{2}.$ **3.** $5l, \frac{3l}{2}.$ **4.** $c, -\left(\frac{5c+9}{5}\right).$

5. $-a \pm \sqrt{k}.$ **6.** $l, \frac{4n-5l}{5}.$ **7.** $4c, 5d.$ **8.** $\frac{3}{2a}, -1.$

9. $3l+m, m-3l.$ **10.** $2a, -\frac{a}{3}.$ **11.** $a+1, a-\frac{1}{4}.$

12. $-\frac{a}{13}, \frac{a}{12}.$ **13.** $-2m \pm \sqrt{4m^2+k}.$ **14.** $-l-m-1, -1.$

15. $c, -c-2a.$ **16.** $k-1$ (twice). **17.** $0, \frac{a+b}{2}.$

18. $a, \frac{1-a}{1+a}.$ **19.** $2a, -\frac{13a}{8}.$ **20.** $0, m-l.$

21. $a, -\frac{2ab}{a+2b}.$ **22.** $\frac{2c+4}{3}, \frac{2c-6}{3}.$ **23.** $2l+m, -(l+2m).$

24. $-a, \frac{2a+b}{2}.$ **25.** $\frac{10 \pm \sqrt{20}}{8} = 1{\cdot}81$ or $0{\cdot}69.$

26. $\frac{6 \pm \sqrt{216}}{18} = 1{\cdot}15$ or $-0{\cdot}48.$ **27.** $\frac{7 \pm \sqrt{85}}{6} = 2{\cdot}70$ or $-0{\cdot}37.$

28. $\frac{9 \pm \sqrt{181}}{10} = 2{\cdot}25$ or $-0{\cdot}45.$ **29.** $\frac{-3 \pm \sqrt{-3}}{2}.$

30. $\frac{3 \pm \sqrt{17}}{2} = 3{\cdot}56$ or $-0{\cdot}56.$

EXERCISE 66 B (P. 276)

1. $\frac{a}{5}, -\frac{a}{2}.$ **2.** $7t, \frac{3t}{2}.$ **3.** $\frac{5l}{3}, -\frac{3l}{4}.$ **4.** $\pm\sqrt{3l+4n}.$

5. $a, -\left(\frac{3a+7}{3}\right).$ **6.** $c, -\left(\frac{3c+2l}{3}\right).$ **7.** $\frac{a+b}{2}, \frac{a-b}{2}.$

8. $\frac{3a}{2}, -b.$ **9.** $\frac{2}{5c}, -2.$ **10.** $\frac{9b+8}{3a}, \frac{3b-2}{a}.$

11. $m-3, m+3.$ **12.** $10b, \frac{b}{2}.$ **13.** $2a+1, -3b.$

14. $\frac{2a \pm \sqrt{4a^2+b^2}}{3}.$ **15.** $a, 1.$ **16.** $p, -\left(\frac{3p-2}{3}\right).$

17. $\frac{2}{3b}, -\frac{1}{3b}.$ **18.** $-b, \frac{1+b}{1-b}.$ **19.** $0, 3a.$

20. $\frac{a+1}{a-1}, \frac{a-1}{a+1}.$ **21.** $\frac{2+3c}{4}, \frac{2+5c}{6}.$ **22.** $-b, \frac{ab}{a-3b}.$

23. $-\left(\frac{1}{l}+\frac{1}{m}\right),\ -\frac{2}{l+m}$. 24. $\frac{l+m}{l},\ \frac{l+m}{m-l}$.

25. $\frac{2\pm\sqrt{40}}{6}=1{\cdot}39$ or $-0{\cdot}72$. 26. $\frac{-1\pm\sqrt{21}}{2}=-2{\cdot}79$ or $1{\cdot}79$.

27. $\frac{-6\pm\sqrt{236}}{10}=0{\cdot}94$ or $-2{\cdot}14$. 28. $\frac{1\pm\sqrt{-7}}{4}$.

29. $\frac{-1\pm\sqrt{97}}{16}=0{\cdot}55$ or $-0{\cdot}68$. 30. $\frac{7\pm\sqrt{17}}{4}=2{\cdot}78$ or $0{\cdot}72$.

Exercise 67 A (Pp. 278, 279)

1. $(1, 1), (2\frac{1}{3}, -4\frac{1}{3})$. 2. $(5, -4), (-1, -1)$. 3. $(3, -2), (-2\frac{1}{2}, 3\frac{1}{2})$.
4. $(1, -\frac{1}{2}), (1\frac{1}{2}, -\frac{1}{8})$. 5. $(-3, 1), (\frac{1}{2}, -\frac{3}{4})$. 6. $(3, 2), (-\frac{3}{7}, -\frac{4}{7})$.
7. $(\frac{2}{3}, -\frac{1}{5}), (\frac{6}{17}, -\frac{5}{17})$. 8. $(1, -5), (4\frac{1}{10}, 5\frac{1}{3})$.
9. $(3, 1), (1\frac{4}{5}, \frac{1}{5})$. 10. $(\frac{1}{3}, -1), (\frac{1}{9}, -1\frac{2}{5})$. 11. $(-\frac{3}{4}, -\frac{1}{3}), (\frac{2}{3}, \frac{3}{8})$.
12. $(3, 4), (-1, -2)$. 13. $(-5, -7), (-3, -4)$.
14. $(1, 1), (\frac{1}{2}, \frac{1}{3})$. 15. $(3, 2), (1\frac{2}{3}, 2\frac{2}{3})$.
16. $(3, 2), (2\frac{1}{6}, 2\frac{5}{12})$. 17. $(-\frac{1}{4}, 1\frac{1}{4}), (-\frac{1}{8}, 1\frac{3}{8})$.
18. $(7, 2), (-3\frac{8}{13}, -12\frac{2}{13})$. 19. $(1, \frac{1}{6}), (-1\frac{1}{4}, -\frac{7}{12})$.
20. $(-1, 1), (-\frac{1}{3}, 1\frac{2}{5})$. 21. $(2, 2), (-3, 12)$.
22. $(4, -1), (1\frac{2}{3}, 2\frac{1}{2})$. 23. $(2, 4), (4, 2)$. 24. $(\frac{1}{3}, \frac{1}{2}), (-1\frac{1}{3}, 3)$.
25. $(0, 0), (20, 24)$. 26. $(3\frac{1}{2}, 2), (-1\frac{1}{2}, -3)$. 27. $(3, 5), (-\frac{6}{17}, \frac{9}{17})$.
28. $(-0{\cdot}02, -0{\cdot}38), (-1{\cdot}14, -2{\cdot}62)$.
29. $(1{\cdot}31, 0{\cdot}41), (0{\cdot}19, -1{\cdot}08)$. 30. $(-1{\cdot}71, 0{\cdot}22), (-0{\cdot}29, 1{\cdot}28)$.

Exercise 67 B (Pp. 279, 280)

1. $(2, 1), (2\frac{1}{2}, 2)$. 2. $(6, 2\frac{1}{2}), (-5, -3)$.
3. $(-6, -4), (-\frac{2}{5}, 7\frac{1}{5})$. 4. $(5, 2), (1\frac{1}{4}, -\frac{1}{2})$. 5. $(2, -3)$.
6. $(1, -\frac{1}{5}), (1\frac{1}{7}, -\frac{1}{7})$. 7. $(4, -1), (1\frac{1}{7}, 3\frac{2}{7})$.
8. $(-8, -4), (-1\frac{1}{7}, 1\frac{1}{7})$. 9. $(-2, \frac{1}{3}), (\frac{3}{5}, \frac{41}{45})$.
10. $(2, 2), (\frac{7}{9}, -\frac{3}{4})$. 11. $(1, 2), (-\frac{11}{21}, -\frac{6}{7})$.
12. $(1, -1), (5, -7)$. 13. $(1, 2), (1\frac{18}{19}, \frac{11}{19})$.
14. $(1, 1), (-\frac{3}{4}, -1\frac{1}{3})$. 15. $(1, 2), (2, 4\frac{1}{2})$.
16. $(1\frac{1}{7}, 1\frac{3}{7}), (3\frac{1}{2}, -2\frac{1}{2})$. 17. $(1, 1), (\frac{1}{3}, 1\frac{2}{5})$.
18. $(1, 5), (-5\frac{2}{31}, -4\frac{3}{31})$. 19. $(-5, -3\frac{1}{2}), (-\frac{1}{5}, \frac{1}{10})$.
20. $(12, -1), (-1\frac{1}{2}, 2)$. 21. $(1, -1), (4, -5)$.
22. $(1\frac{1}{2}, 4), (-\frac{1}{54}, -\frac{5}{9})$. 23. $(1\frac{1}{3}, -5), (-1\frac{1}{3}, -1)$.
24. $(3{\cdot}89, 0{\cdot}93), (-2{\cdot}35, -3{\cdot}24)$. 25. $(2, 1), (-1\frac{1}{3}, -4)$.

26. $(2, -1), (6\frac{1}{8}, 1\frac{3}{4})$.
27. $(2, 1), (-0{\cdot}5, -1)$.
28. $(1, 0), (0{\cdot}7, -0{\cdot}3)$.
29. $(0{\cdot}68, 1{\cdot}11), (-0{\cdot}02, -3{\cdot}11)$.
30. $(0{\cdot}45, -0{\cdot}70), (-0{\cdot}45, -1{\cdot}30)$.

Exercise 68 a (P. 281)

1. $x=1, y=-\frac{1}{2}, z=5$.
2. $x=2, y=3, z=-2$.
3. $x=9, y=-2, z=4$.
4. $x=1, y=0, z=-3$.
5. $x=-1, y=2, z=3$.
6. $a=-4, b=-5, c=-6$.
7. $x=8, y=4, z=-1$.
8. No solution. The equations are inconsistent.
9. $x=6, y=-9, z=1$.
10. Any number of solutions. The equations are not independent.
11. $x=1, y=3, z=-1\frac{1}{3}$; $x=5, y=-1, z=0$.
12. $x=1, y=2, z=0$; $x=3, y=4, z=\pm\sqrt{2}$.

Exercise 68 b (Pp. 281, 282)

1. $x=3, y=4, z=-2$.
2. $x=5, y=8, z=-6$.
3. $x=1, y=-2, z=0$.
4. $x=2, y=-1, z=0$.
5. $x=2, y=-2, z=-4$.
6. $x=12, y=-12, z=12$.
7. $x=3, y=-2, z=-\frac{1}{2}$.
8. $x=-11, y=6, z=-15$.
9. No solution. The equations are inconsistent.
10. $x=\frac{1}{2}, y=1\frac{1}{2}, z=-2$; $x=2\frac{1}{2}, y=-\frac{1}{2}, z=0$.
11. $x=4, y=1, z=-1$; $x=2\frac{1}{2}, y=0, z=0$.
12. Any number of solutions. The equations are not independent.

Exercise 69 a (P. 283)

1. $(1\frac{1}{2}, -1), (-1\frac{1}{2}, 1), (\frac{1}{2}, -3), (-\frac{1}{2}, 3)$.
2. $(3, 2), (-3, -2), (0, 4), (0, -4)$.
3. $(2, -1), (-2, 1), (3, -2), (-3, 2)$.
4. $(4, 1), (-4, -1), (10, 3), (-10, -3)$.
5. $(1\frac{1}{2}, 1), (-1\frac{1}{2}, -1), (\frac{1}{2}, 3), (-\frac{1}{2}, -3)$.
6. $(2, -3), (-2, 3), (3, -4), (-3, 4)$.
7. $(2, 1\frac{1}{3}), (2, -1\frac{1}{3}), (-1, \frac{2}{3}), (-1, -\frac{2}{3})$.
8. $(2, 5), (2, -5), (2, -5), (-2, -5)$.
9. $(1, 3), (-1, -3), (\frac{1}{2}, 2), (-\frac{1}{2}, -2)$.
10. $(6, 2), (-6, -2)$.
11. $(-4, 5), (-4\frac{1}{5}, 5\frac{1}{4}), (-3, 4), (-5\frac{1}{4}, 7)$.
12. $(0, 0), (1, 1), (\frac{15}{22}, \frac{9}{44}), (0, 0)$.

13. $(1, -1), (-1, 1), \left(\frac{3}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{3}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

14. $(0, 0), (3, -6), (0, 0), (-1\frac{10}{17}, -3\frac{12}{17})$. **15.** $(2, -3), (-2, 3)$.

16. $(1, -2), (-1, 2), (3\frac{1}{2}, \frac{1}{2}), (-3\frac{1}{2}, -\frac{1}{2})$. **17.** $(5, -9), (-5, 9)$.

18. $(1, -2\frac{1}{5}), (-1, 2\frac{1}{5}), (14\frac{1}{2}, -4), (-14\frac{1}{2}, 4)$.

Exercise 69 b (Pp. 283, 284)

1. $(3, 1), (-3, -1), (-1, 1), (1, -1)$. **2.** $(2, -1), (-2,$ [illegible]

3. $(2\frac{1}{2}, -4), (-2\frac{1}{2}, 4), (1, -3), (-1, 3)$.

4. $(2, 3), (-2, -3), (7\ -12), (-7, 12)$.

5. $(5, -3), (-5, 3), (6\frac{1}{3}, -2\frac{2}{3}), (-6\frac{1}{3}, 2\frac{2}{3})$.

6. $(2, 1), (-2, -1), (1, 3), (-1, -3)$.

7. $(6, 4), (-6, -4), (10, 6), (-10, -6)$.

8. $(5, 1\frac{1}{5}), (5, -1\frac{1}{5}), (-1\frac{1}{2}, \frac{1}{10}), (-1\frac{1}{2}, -\frac{1}{10})$.

9. $(3, 1), (-3, -1), \left(\frac{5}{\sqrt{577}}, \frac{30}{\sqrt{577}}\right), \left(-\frac{5}{\sqrt{577}}, -\frac{30}{\sqrt{577}}\right)$.

10. $(1\frac{1}{2}, 3\frac{1}{2}), (1\frac{1}{2}, -3\frac{1}{2}), (1\frac{1}{2}, 3\frac{1}{2}), (-1\frac{1}{2}, 3\frac{1}{2})$. **11.** $(5, 2), (-5, -2)$.

12. $(0, 0), (3, -2), (0, 0), (-1\frac{10}{17}, -1\frac{4}{17})$.

13. $(0, 0), (\frac{1}{3}, 2), (0, 0), (\frac{5}{22}, \frac{9}{22})$.

14. $(\frac{1}{2}, -5), (-\frac{1}{2}, 5), (16, 10\frac{1}{2}), (-16, -10\frac{1}{2})$.

15. $(1\frac{1}{2}, -2), (-1\frac{1}{2}, 2), \left(\frac{\sqrt{2}}{2}, 4\sqrt{2}\right), \left(-\frac{\sqrt{2}}{2}, -4\sqrt{2}\right)$.

16. $(2\frac{1}{3}, -5), (-2\frac{1}{3}, 5)$. **17.** $(\frac{1}{2}, -9\frac{1}{2}), (-\frac{1}{2}, 9\frac{1}{2}), (5, 4), (-5, -4)$.

18. $(-7, -3\frac{1}{2}), (3, 1\frac{1}{2}), (5, -\frac{1}{2}), (-1, 2\frac{1}{2})$.

Exercise 69 c (Pp. 284, 285)

1. $(1, 1), (\frac{23}{27}, 1\frac{2}{9})$. **2.** $(7, -\frac{1}{2}), (6, -\frac{1}{3})$. **3.** $(-2, 3), (1\frac{2}{3}, -2\frac{1}{2})$.

4. $(3\sqrt{2}, \sqrt{2}), (-3\sqrt{2}, -\sqrt{2}), \left(\frac{\sqrt{6}}{3}, -\frac{4\sqrt{6}}{3}\right), \left(-\frac{\sqrt{6}}{3}, \frac{4\sqrt{6}}{3}\right)$.

5. $(-$[illegible]$, 1), (5, -3)$. **6.** $(1, 1), (-2\frac{1}{6}, -3\frac{2}{9})$. **7.** $(7, -3), (-6, 4\frac{4}{5})$.

8. $(1, -2), (-1, 2), (1{\cdot}40, 1{\cdot}81), (-1{\cdot}40, -1{\cdot}81)$.

9. $(8, -3), (1\frac{1}{9}, 1\frac{16}{27})$.

10. $\left(-3, \frac{3+\sqrt{21}}{2}\right), \left(-3, \frac{3-\sqrt{21}}{2}\right), (2, 2), (-4, 2)$.

11. $(\frac{1}{2}, \frac{1}{2}), (-1\frac{1}{2}, -\frac{7}{10})$. **12.** $(3, 0), (-1\frac{1}{2}, 3)$.

13. $3, -11, -2$. **14.** $-2, 1\frac{1}{2}, -1$. **15.** $2, -2, -10$.

16. $(3, -10\frac{1}{2}), (-2, -33)$. **17.** $(-1, 9), (1, 7)$.

18. $(\frac{2}{3}, 5), (-\frac{2}{3}, -1)$.
19. $(a, -b), (-2b-a, 2a+b)$.
20. $(-6\frac{1}{2}, 9), (4\frac{3}{4}, 1\frac{1}{2})$.
21. $(8, -3), (3\frac{1}{2}, -5\frac{1}{4})$.
22. $(\frac{1}{2}, -2, 1), (-\frac{1}{2}, -\frac{2}{3}, \frac{1}{2})$.
23. $(21, 17, 10), (32, 28, -12)$.
24. $(1, -5\frac{1}{3}), (-\frac{6}{11}, -2\frac{1}{2})$.

TEST PAPERS VI (Pp. 285, 286, 287, 288, 289, 290, 291)

A. 1. $\dfrac{100x}{100-y}$ shillings. **2.** (i) $\dfrac{b-a}{4(a^2+b^2)}$, (ii) $\dfrac{a+6b}{a^2-9b^2}$.
3. $4\frac{1}{3}$. **4.** (i) $1\frac{3}{5}$, (ii) $(\frac{3}{5}, \frac{3}{5}), (1\frac{1}{5}, -1\frac{4}{5})$. **5.** 20.
6. $\dfrac{md-nb}{na-mc}$.

B. 1. (i) $l(l+13)(l-12)$, (ii) $(m-2n+3ln)(m-2n-3ln)$.
2. $(c-b)$ of the 1st to $(a-c)$ of the 2nd.
3. (i) $\dfrac{1}{3x-2}$, (ii) $\dfrac{2}{(2x-3)^2(2x-5)}$.
4. (i) 4, $1\frac{2}{5}$, (ii) $(-1, 1), (\frac{13}{19}, \frac{3}{19})$. **5.** 72 miles.
6. $a=\frac{1}{16500}, b=\frac{1}{300}, c=\frac{1}{25}$; 2·773.

C. 1. (i) $\dfrac{5a}{3}$, (ii) $x=\dfrac{ab(3a+2b)}{a^2+b^2}, y=\dfrac{ab(3b-2a)}{a^2+b^2}$.
2. $x-y-\dfrac{xy}{100}$. **3.** (i) $-1, 2$; (ii) $(5, 3)$; $\left(-\dfrac{1}{13}, -\dfrac{5}{13}\right)$.
4. $\dfrac{8a^2}{a^2-b^2}$. **5.** 326, £2. **6.** 175, 139 yds.

D. 1. $a(a+b)=c(c+d)$.
2. (i) $(x+y+4)(x-y-4)$, (ii) $(1+a+b)(1-4a-4b)$.
3. (i) $x=2, y=-3, z=4$; (ii) $\dfrac{-a\pm\sqrt{2a^2-1}}{a-1}$. **4.** 48, 12, 2.
5. $\dfrac{1-2y}{1-2y+2y^2}$. **6.** $\dfrac{pa+qa-pz}{q}$ shillings per lb.

E. 1. $\dfrac{100(x-y)}{100+x}$. **2.** (i) $1\dfrac{1}{3}$, (ii) $\left(\dfrac{3}{7}, 0\right)$; $\left(-\dfrac{3}{25}, \dfrac{12}{25}\right)$.
3. $-4(x^2+4y^2)$. **4.** (i) $\pm 2·4$; (ii) 3·5 or $-1·5$, $y=1\pm 2\sqrt{x^2+1}$.
5. 0·27, 2, 3·73. **6.** 3·6′, 3·2′.

F. 1. (i) $(5+a-b)(5-a+b)$, (ii) $(l+1)(l+3)(l^2+4l-3)$.
2. $\dfrac{3x-4}{x^2-x-6}$. **3.** (i) $\dfrac{b^2}{6c-b}$, (ii) $\left(\dfrac{(a+b)c}{a^2+b^2}, \dfrac{a^2+b^2}{2c(a-b)}\right)$.
4. 13, 17. **5.** (i) $-\frac{1}{2}, 5\frac{1}{2}$, (ii) $(\pm 5, \pm 2)$; $(\pm 4\sqrt{3}, \mp 3\sqrt{3})$. **6** 198.

G. 1. $\frac{xp+yq}{x+y}, \frac{xz+yz-yq}{x}$. **2.** $v=\frac{Pt+mu}{m}, v=\frac{\sqrt{2Ps+mu^2}}{m}$.

3. (i) $\frac{a}{2}, a$; (ii) $(4, 6), (-1, -4)$. **4.** $\frac{1}{5a(a+1)}$.

5. $a=-\frac{1}{2}, b=\frac{3}{2}, c=4$. $x=y=3\cdot37$ or $-2\cdot37$. **6.** 8, 9, 10.

H. 1. (i) $(a+b)(a+2b)(2a-b)$, (ii) $(a^2+a+1)(a^2-a-1)$.

2. $\frac{a(p-r)}{(q-r)}$. **3.** $\frac{1}{2x}$. **4.** (i) $\left(\frac{a}{a-2}, \frac{a}{2a-2}\right)$, (ii) $\left(\frac{2}{3}, \frac{3}{2}\right)$ twice.

5. $10\cdot22\frac{1}{2}$ a.m., 11 miles. **6.** $a=-0\cdot68, x=-0\cdot2$.

I. 1. $\frac{200a}{100-a}$. **2.** (i) $-4y$, (ii) $\frac{(p-q)(p^2+q^2)}{p+q}$.

3. 5. **4.** (i) 17; (ii) $(7, -3), (4\frac{3}{7}, -1\frac{2}{7})$.

5. $2\cdot2''$, $3\cdot2''$, $3\cdot93''$, $4\cdot5''$, $5''$. **6.** $8\cdot4$ cm.

J. 1. (i) $\frac{2ab}{2a+b}$. Yes, (ii) $\left(b, -\frac{a}{3}\right)$.

2. (i) $(5b-c+z)(5b-c-z)$, (ii) $12(1+a)(2+a)(1-a)$. **3.** 1.

4. $3\frac{23}{35}$. **5.** (i) 3, -3, (ii) $(1\cdot7, 0\cdot4), (0\cdot3, 2\cdot6)$.

6. 145 yd., 67 yd.

K. 1. $x=\frac{4(K-3)}{5}, y=\frac{7-11K}{3}$; $K=-2, x=-4, y=9\frac{2}{3}$.

2. (i) $\frac{a-b}{b}, \frac{b-a}{a}$, (ii) $\left(\frac{1}{4}, -\frac{1}{3}\right)$; $\left(-\frac{2}{9}, \frac{3}{8}\right)$.

3. (i) $\frac{a(100-k)}{100}$, (ii) 12. **4.** $\frac{x-y}{z-x-y}$. **5.** $1\frac{1}{2}''$. **6.** $\frac{1}{4}, \frac{1}{8}, \frac{1}{24}$.

L. 1. $\frac{pa+pb-nb}{n}$. **2.** $y=\sqrt{\frac{7x^2t-2W}{7t}}$; $2''$.

3. (i) $(x^3+5x^2+1)(x^3-5x^2+1)$, (ii) $(a+1)(a-2)(a^2-a-3)$.

4. (i) $x=3\frac{2}{7}, y=-1\frac{2}{7}, z=1\frac{1}{7}$, (ii) $(\pm7, \pm5)$.

5. $\frac{40x(x-3)}{4x^2-25}$. **6.** $4\cdot57''$, $6\cdot57''$.

EXERCISE 70 A (Pp. 294, 295)

1. $\frac{3}{2}x^4+2x^3+\frac{47}{8}x^2-\frac{1}{6}x-\frac{1}{2}$. **2.** $x^4+\frac{23}{12}x^3-\frac{7}{6}x^2+x-\frac{1}{4}$.

3. $\frac{1}{9}x^4+\frac{17}{240}x^2+\frac{1}{25}$. **4.** $1+\frac{4}{5}t-\frac{121}{25}t^2-2t^3+\frac{25}{4}t^4$.

5. $12x^5-25x^4+13x^3+30x^2-83x+35$.

6. $2+5x-2x^2-7x^3+2x^4$. **7.** $5x^5+5x^4+21x^3-2x^2-18x+10$.

8. $14a^5+5a^4-22a^3-3a^2-3a+9$. **9.** $35l^5-34l^3+25l^2+8l-10$.

10. $8c^5d+6c^4d^2-33c^3d^3+9c^2d^4+10cd^5$. **11.** $1-6x+13x^2-14x^3$.

12. $4-4x+21x^2-18x^3+29x^4$. **13.** $1-c-4c^2+4c^3+c^4$.
14. $10+15l^2-4l^3+l^4$. **15.** -1. **16.** -3. **17.** 22. **18.** 34.
19. $a^2+4b^2+9c^2-4ab+6ac-12bc$.
20. $25a^2+4b^2+c^2-20ab-10ac+4bc$.
21. $4a^2+25b^2+9c^2+d^2-20ab+12ac-4ad-30bc+10bd-6cd$.
22. $a^2+16b^2+25c^2+9d^2+8ab-10ac-6ad-40bc-24bd+30cd$.
23. $16x^4-32x^3+24x^2-8x+1$.
24. $81a^8+540a^6b^2+1350a^4b^4+1500a^2b^6+625b^8$.
25. $a^3+b^3+c^3-3abc$.
26. $10x^3+x^2(28y-19z)+x(10y^2-18yz-19z^2)-12y^3-5y^2z+17yz^2+10z^3$.

EXERCISE 70 B (P. 295)

1. $\frac{4}{3}x^4-5x^3+\frac{52}{9}x^2+\frac{5}{6}x-1$. **2.** $\frac{4}{9}x^4-x^2+\frac{4}{5}x-\frac{4}{25}$.
3. $\frac{3}{8}a^4+\frac{1}{16}a^3b+\frac{5}{24}a^2b^2+\frac{1}{36}ab^3+\frac{2}{27}b^4$. **4.** $16-\frac{16}{3}t+\frac{22}{9}t^2-\frac{1}{3}t^3+\frac{1}{16}t^4$
5. $8c^5-26c^4+29c^3+32c^2-109c+18$.
6. $10-9x+21x^2-8x^3+6x^4$. **7.** $6x^4-18x^3-13x^2+27x+6$.
8. $35a^5-43a^3-14a^2-36a-8$.
9. $24d^5-14d^4-30d^3-7d^2+5d+20$.
10. $10x^5-25x^4y+11x^3y^2-2x^2y^3+24xy^4-18y^5$.
11. $16-40a+17a^2+34a^3$. **12.** $25-20x+4x^2-40x^3+16x^4$
13. $6t^2-4t^3-9t^4-t^5$. **14.** $2+5x-3x^2-2x^3$.
15. 13. **16.** 22. **17.** 15. **18.** 41.
19. $4a^2+9b^2+c^2-12ab+4ac-6bc$.
20. $a^2+16b^2+4c^2-8ab+4ac-16bc$.
21. $a^2+9b^2+4c^2+25d^2-6ab-4ac+10ad+12bc-30bd-20cd$.
22. $25a^2+b^2+c^2+49d^2-10ab+10ac-70ad-2bc+14bd-14cd$.
23. $16x^4-160x^3+600x^2-1000x+625$.
24. $16a^4+96a^3b+216a^2b^2+216ab^3+81b^4$.
25. $9a^3+3a^2(2c-5b)+a(3b^2-5bc+7c^2)+2b^3-6b^2c-6bc^2+20c^3$.
26. $2l^3+l^2(7n-m)-2l(8m^2-3mn+2n^2)-15m^3-7m^2n+4mn^2$.

EXERCISE 71 A (P. 297)

1. $c^2-cd+3d^2$. **2.** $\frac{1}{4}x^2+\frac{1}{2}xy+y^2$. **3.** $\frac{1}{9}l^2-\frac{1}{6}lm+\frac{1}{4}m^2$.
4. $\frac{3}{2}t^2+t+\frac{1}{3}$. **5.** $2+6d+9d^2$. **6.** z^2-3z-5.
7. $8-7x+x^2$. **8.** y^2+y-7, rem. $-y-3$.
9. $3x^2y^2-4xy+1$. **10.** $2c^3-3c^2+4c-5$. **11.** $x^2+2xy+2y^2$.
12. $a^2+b^2+c^2+ab-ac+bc$. **13.** $3a+4b-c$.
14. $-a^2-b^2-4c^2-ab+2ac-2bc$. **15.** $8c^3-4c^2d^2+2cd^4-d^6$.
16. $a^6+a^4b^2+a^2b^4+b^6$. **17.** $32+16y+8y^2+4y^3+2y^4+y^5$
18. c^4-5c^2+25. **19.** $16t^4-40t^3s+100t^2s^2-250ts^3+625s^4$.
20. $x^8-x^6y^2+x^4y^4-x^2y^6+y^8$, rem. $-2y^{10}$.

Exercise 71 B (Pp. 297, 298)

1. $a^2-2ab-4b^2$. **2.** $\frac{1}{9}x^2-\frac{2}{3}xy+4y^2$. **3.** $9l^2+\frac{3}{4}lm+\frac{1}{16}m^2$.
4. $\frac{3}{2}x^2-3$. **5.** c^2+3c-5. **6.** $5a^2+11ab+11b^2$.
7. $4c^2+3c-2$. **8.** $5p^2-3p+1$.
9. $4y^2-7y+5$, rem. $3y+5$. **10.** t^2-t. **11.** $a+b-c$.
12. $-c^2-9d^2-4-3cd-2c+6d$. **13.** $4x^2+4y^2+9+4xy-6x+6y$.
14. $ab-ac-bc+c^2$. **15.** $x^6-x^5+x^4-x^3+x^2-x+1$.
16. $81+27a+9a^2+3a^3+a^4$.
17. $32a^5-16a^4b+8a^3b^2-4a^2b^3+2ab^4-b^5$.
18. $-[81x^4+135x^3a+225x^2a^2+375xa^3+625a^4]$.
19. $a^6-a^4b^2+a^2b^4-b^6$. **20.** a^2-ab+b^2.

Exercise 72 A (Pp. 300, 301)

1. 21, 7, $\frac{3}{4}$, 1. **2.** 35, $-5\frac{1}{3}$, 28, $3a^2-2a-5$.
3. 24, 0, $(4a+5)(3a-2)$, $(4x^2+1)(3x^2-5)$.
4. 465, $\frac{1}{2}(n-1)n$, 21, $n+1$. **5.** 63, $-\frac{1}{2}$, $2^{n+1}-1$, $2^{2n-1}-1$.
6. 19, 0, a^3-b^3, $a^3-b^3+3(a^2-b^2)+3(a-b)$. **7.** 3. **8.** -202.
9. -35. **10.** -295. **11.** 8. **12.** $-9\frac{2}{3}$.
13. 15. **14.** $-15\frac{1}{2}$. **15.** 47. **16.** 1.
17. 0, $x-3$ is a factor. **18.** 0, $x+3$ is a factor.
19. 0, $2x-3$ is a factor. **20.** 0, $2x+1$ is a factor.
21. 25. **22.** 3. **23.** 4. **24.** -65.
25. $2x-5$. **26.** $-3x+1$. **27.** $a=-7$, $b=-2$.
28. $a=1$, $b=0$.

Exercise 72 B (Pp. 301, 302)

1. 90, 2, 2, $2b^2-b-1$. **2.** 4, 34, $\frac{2}{25}$, 1.
3. 70, 42, 0, $(3a-7)(5a-6)$. **4.** 2, 26, $3^{2a}-1$, $3^{3n-1}-1$.
5. 95, 55, $(3n+1)(2n-7)$, $(3n+16)(2n+3)$.
6. 9, 16, $(x-y)^2$, $(2a-3b)^2$. **7.** 17. **8.** 11.
9. -35. **10.** -9. **11.** 9. **12.** -8.
13. $-2\frac{1}{9}$. **14.** $-7\frac{2}{25}$. **15.** 214. **16.** -20.
17. 0, $x+4$ is a factor. **18.** 0, $x-3$ is a factor.
19. 0, $2x+3$ is a factor. **20.** 0, $2x-1$ is a factor.
21. 3. **22.** -7. **23.** 5. **24.** -16. **25.** $x-5$.
26. $-5x+2$. **27.** $a=1$, $b=-13$. **28.** $a=4$, $b=-8$.

Exercise 73 A (P. 303)

1. $\pm(a-2b+c)$. **2.** $\pm(3x+4y-2z)$.
3. $\pm(2l-3m+4n)$. **4.** $\pm(5x^2-3y^2-z^2)$.
5. $\pm(2a^2-a+1)$. **6.** $\pm(2a^2-3ab+4b^2)$.

7. $\pm(5x^3 - x^2 - x + 3)$.
8. $\pm(4x^3 - 5x^2 + 3x - 2)$.
9. $\pm(7 - 2x + x^3)$.
10. $\pm(4x^3 - 3x^2y + 2xy^2 - y^3)$.
11. $\pm(3l^3 - 2l^2m + 3lm^2 + 2m^3)$.
12. $\pm\left(x^2 - x + \frac{2}{x} + \frac{1}{x^2}\right)$.

EXERCISE 73 B (P. 303)

1. $\pm(a + 3b - c)$.
2. $\pm(3x - y - 2z)$.
3. $\pm(2l - 5m - n)$.
4. $\pm(2x^2 - 4y^2 + 5z^2)$.
5. $\pm(a^2 - 3a + 2)$.
6. $\pm(l^2 - lm + 2m^2)$.
7. $\pm(2x^3 - x^2 + 3x - 5)$.
8. $\pm(3x^3 - 2x^2 - x + 4)$.
9. $\pm(x^3 - 4x^2y + 2xy^2 - 3y^3)$.
10. $\pm(5 + 2x - 4x^3)$.
11. $\pm(a^3 - 3ab^2 - 7b^3)$.
12. $\pm\left(2x^2 - 3x + \frac{4}{x}\right)$.

EXERCISE 74 A (P. 305)

5. $(x-1)(x^2+x-4)$.
6. $(x+1)(x-2)(x-3)$.
7. $(x+1)(3x^2-3x+5)$.
8. $(x+2)(x+2)(x+3)$.
9. $(a+2)(a+3)(a-5)$.
10. $(l+2)(2l-3)(l+6)$.
11. $(c-2)(c+3)(c-4)$.
12. $(l+1)(l+2)(l^2-2l+2)$.
13. $(x+4)(2x-1)(x^2-x+1)$.
14. $(x+3)(5x+1)(x^2-x-1)$.
15. $(x+2y)(3x+2y)(2x^2-xy+y^2)$.
16. $(2t-1)(3t-1)(4t+1)$.
17. $(2x+1)(3x-1)(5x-1)$.
18. $(3a-2)(2a+3)(5a-2)$.

EXERCISE 74 B (Pp. 305, 306)

5. $(x+1)(x+1)(2x-1)$.
6. $(x+1)(x+2)(x+3)$.
7. $(x-2)(x+2)(x+3)$.
8. $(x-1)(x^2+x-5)$.
9. $(l+2)(l+2)(2l-5)$.
10. $(a+1)(a+2)(a-4)$.
11. $(l+2)(l+4)(l^2-2l+5)$.
12. $(c-1)(c+3)(c+6)$.
13. $(2x+1)(x-4)(x^2-3x+1)$.
14. $(x+5)(4x-1)(x^2-2x-1)$.
15. $(t-1)(2t-1)(3t-1)$.
16. $(x-y)(3x+y)(x+3y)(3x-2y)$.
17. $(3a-2)(5a+2)(2a+5)$.
18. $(2x+1)(3x+1)(5x-1)$.

EXERCISE 75 A (P. 307)

1. $(a+1)(a^2-a+1)$.
2. $(2a-1)(4a^2+2a+1)$.
3. $(1-lm)(1+lm+l^2m^2)$.
4. $(1+2lm)(1-2lm+4l^2m^2)$.
5. $(y-3)(y^2+3y+9)$.
6. $(2y+3)(4y^2-6y+9)$.
7. $(7-n)(49+7n+n^2)$.
8. $(b-5c)(b^2+5bc+25c^2)$.
9. $(2x^2+5yz)(4x^4-10x^2yz+25y^2z^2)$.
10. $b(b+4)(b^2-4b+16)$.
11. $(3l-mn)(9l^2+3lmn+m^2n^2)$.

12. $(7x^2-10a)(49x^4+70ax^2+100a^2)$.
13. $2(l+5m)(l^2-5lm+25m^2)$. 14. $10(7+t)(49-7t+t^2)$.
15. $4(2c-z)(4c^2+2cz+z^2)$. 16. $(8ab+5)(64a^2b^2-40ab+25)$.
17. $(c^2+1)(c^4-c^2+1)$. 18. $(l-m)(l+m)(l^2+lm+m^2)(l^2-lm+m^2)$.
19. $(3c^2-d^2)(9c^4+3c^2d^2+d^4)$. 20. $8(t-5x^2)(t^2+5tx^2+25x^4)$.
21. $(x+5y)(37x^2-50xy+25y^2)$. 22. $(7l+3m)(19l^2+24lm+9m^2)$.
23. $(4x+3y)(79x^2+111xy+39y^2)$. 24. $(20x-y)(688x^2-724xy+217y^2)$.

EXERCISE 75 B (P. 307)

1. $(x-1)(x^2+x+1)$. 2. $(2x+1)(4x^2-2x+1)$.
3. $(1+c)(1-c+c^2)$. 4. $(2-c)(4+2c+c^2)$.
5. $(2a-3b)(4a^2+6ab+9b^2)$. 6. $(z+3)(z^2-3z+9)$.
7. $(a-4)(a^2+4a+16)$. 8. $(3a+bc)(9a^2-3abc+b^2c^2)$.
9. $(a+5)(a^2-5a+25)$. 10. $(7+2x)(49-14x+4x^2)$.
11. $(3x+10y^2)(9x^2-30xy^2+100y^4)$.
12. $(5l^2-2mn)(25l^4+10l^2mn+4m^2n^2)$.
13. $3(2a-b)(4a^2+2ab+b^2)$. 14. $2(x+3y)(x^2-3xy+9y^2)$.
15. $(8-3xy)(64+24xy+9x^2y^2)$. 16. $10(c+4d)(c^2-4cd+16d^2)$.
17. $(a+1)(a-1)(a^2+a+1)(a^2-a+1)$.
18. $(x^2+y^2)(x^4-x^2y^2+y^4)$. 19. $4(p^2+5r)(p^4-5p^2r+25r^2)$.
20. $(2x^2-5y^2)(4x^4+10x^2y^2+25y^4)$. 21. $9(a-b)(13a^2+9ab+3b^2)$.
22. $(5x+y)(25x^2-35xy+19y^2)$. 23. $(6x+y)(84x^2-102xy+37y^2)$.
24. $-x(631x^2+870xy+300y^2)$.

EXERCISE 76 A (P. 309)

1. $(x-1)(2ax-6a-5)$. 2. $(x+1)(3bx-9b-2)$.
3. $(2x-1)(3ax-2a-1)$. 4. $(x-2)(cx-4c-3)$.
5. $(2x+1)(2bx+5b-3)$. 6. $(3x-2)(3xy+4y-2)$.
7. $-(a-b)(b-c)(c-a)$. 8. $(a+b)(b+c)(c+a)$.
9. $-(x+y)(y-z)(z-x)$. 10. $-(a-b)(b-c)(c-a)$.
11. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$.
12. $-(l-m)(m-n)(n-l)(lm+mn+nl)$.
13. $(x+3y)(x+2y+5)$. 14. $(x+y)(x+5y-2)$.
15. $(2x+y)(x+3y-1)$. 16. $(5x-y)(x+y+2)$.
17. $(2x-3y)(3x+2y+4)$. 18. $(3x-2y)(4x-3y+1)$.
19. $(3x-2y)(2x+3y-4)$. 20. $(2x-5)(4xy-5y+2x)$.
21. $(5x-1)(3xy+2y-3x)$. 22. $(5x-2y)(3x-5y-2)$.
23. $(3x-2)(5x-2y+4)$. 24. $(2x+3y)(7x-2y-3)$.
25. $(7x+5y)(2x-5y-5)$. 26. $(4x+1)(2+3y-4x)$.
27. $(7x-2)(2x-2y+3)$. 28. $(5x+3)(2lx-3l-5)$.

EXERCISE 76 B (Pp. 309, 310)

1. $(x+2)(5ax-a-3)$. **2.** $(x-1)(4ax-5a-1)$.
3. $(x-3)(2bx-7b+3)$. **4.** $(2x-3)(5cx+2c-4)$.
5. $(3x+5)(2xy-3y-2)$. **6.** $(2x+7)(3xy-5y-1)$.
7. $(a+b)(b+c)(c+a)$. **8.** $-(a-b)(b-c)(c-a)$.
9. $(a-b)(b-c)(c-a)(a+b+c)$. **10.** $(x-y)(y+z)(z+x)$.
11. $(l-m)(m-n)(n-l)(lm+mn+nl)$. **12.** $(x-y)(y-z)(z-x)$.
13. $(x-3y)(x+y-3)$. **14.** $(x-4y)(x+5y+3)$.
15. $(3x-y)(x+y-5)$. **16.** $(4x-y)(x+5y-1)$.
17. $(4x+3y)(3x+2y-1)$. **18.** $(2x-5y)(4x+3y+7)$.
19. $(5x-4)(3xy-y+4x)$. **20.** $(3x+5y)(5x+2y+2)$.
21. $(7x+2y)(2x-3y+3)$. **22.** $(5x+6)(5x+5y-2xy)$.
23. $(5x+6y)(3x-4y+6)$. **24.** $(3x-7)(2x-6y+3)$.
25. $(5x+7)(x-5y-9)$. **26.** $(7x-5y)(2x+5y+5)$.
27. $(9x-2)(5x-3y-2)$. **28.** $(3y-8x)(7y+2x+1)$.

EXERCISE 77 A (Pp. 313, 314)

1. $(5x-9)(9x-5)$. **2.** $(9x+16)(8x-7)$. **3.** $(4x+3)(8x+9)$.
4. $(5x+27y)(10x-9y)$. **5.** $(3x-20y)(9x+5y)$.
6. $(20x-9)(2x-21)$. **7.** $(4x-25)(25x-4)$. **8.** $3(x-5y)(6x-25y)$.
9. $(6x-11y)(5x+36y)$. **10.** $3(7a+6b)(7a-8b)$.
11. $(6x-5)(9x+25)$. **12.** $(12x+5y)(2x+25y)$.
13. $(10a-9)(25a-21)$. **14.** $(4l-9m)(27l-7m)$.
15. $(5x^2-11y)(16x^2+25y)$. **16.** $(24a-7b)(4a+35b)$.
17. $(20x-21)(8x-49)$. **18.** $(35x-4y)(5x+6y)$.
19. $(30x+49y)(10x-21y)$. **20.** $(9x-16y)(16x-9y)$.
21. $(9x-25)(25x-9)$. **22.** $3(5x-2y)(21x+4y)$.
23. $(32x+27)(3x-5)$. **24.** $(16x-9y)(4x-15y)$.
25. $(a^2+ab-b^2)(a^2-ab-b^2)$. **26.** $(x^2+3xy+3y^2)(x^2-3xy+3y^2)$.
27. $(a^2+ab+5b^2)(a^2-ab+5b^2)$.
28. $(2x^2+2xy-3y^2)(2x^2-2xy-3y^2)$.
29. $(c^2+3cd+4d^2)(c^2-3cd+4d^2)$. **30.** $(a^2+4a+8)(a^2-4a+8)$.
31. $(2x-3y-2)(x+2y-1)$. **32.** $(3x-y+4)(2x+5y-3)$.
33. $(4x+3y-5)(3x-2y-4)$. **34.** $(5x-3y+2)(3x+5y-3)$.
35. $(2x-7y-3)(2x+5y-4)$. **36.** $(6x-y-2)(x-5y+3)$.
37. $(3x+y-5)(2x-5y-2)$. **38.** $(2x+3y-2)(x-2y+1)$.
39. $(5x-3y-3)(3x+5y+2)$. **40.** $(4x-3y-6)(3x+2y-3)$.

EXERCISE 77 B (Pp. 314, 315)

1. $(3x-8)(9x-8)$. **2.** $(9x-8)(8x-9)$.
3. $(10x+21y)(15x-14y)$. **4.** $(9x+4y)(4x+9y)$.

5. $(5x+4y)(25x+24y)$. **6.** $(14x-45y)(7x+9y)$.
7. $(14l+5)(21l+25)$. **8.** $(4c-9d)(27c+7d)$.
9. $(6x+25y)(9x-10y)$. **10.** $(9x-16)(4x+5)$.
11. $(27x-5y)(9x+25y)$. **12.** $(18a+35b)(3a+7b)$.
13. $(32x+15)(16x-3)$. **14.** $(5a^2-7cd)(16a^2-25cd)$.
15. $(4x-9y)(40x-21y)$. **16.** $(4x+27y)(27x-4y)$.
17. $(14x+25y)(21x+5)$. **18.** $(5x^2-6)(35x^2+4)$.
19. $(8c+7d)(49c-15d)$. **20.** $(40x+21y)(16x-49y)$.
21. $(16x-9y)(4x+15y)$. **22.** $(32x-27)(3x-5)$.
23. $(32x+55)(16x-33)$. **24.** $(16x-7y)(40x-21y)$.
25. $(a^2+2ab-b^2)(a^2-2ab-b^2)$. **26.** $(x^2+xy+4y^2)(x^2-xy+4y^2)$.
27. $(x^2+2xy+3y^2)(x^2-2xy+3y^2)$. **28.** $(x^2+2x+2)(x^2-2x+2)$.
29. $(a^2+2ab+4b^2)(a^2-2ab+4b^2)$. **30.** $(x^2+3xy-3y^2)(x^2-3xy-3y^2)$.
31. $(2x-3y-1)(x+2y-2)$. **32.** $(3x-y-3)(2x+5y+4)$.
33. $(4x+3y-4)(3x-2y-5)$. **34.** $(5x+3y-4)(3x-5y-5)$.
35. $(2x-7y-4)(2x+5y-3)$. **36.** $(6x-y+3)(x-5y-2)$.
37. $(3x+y-2)(2x-5y-5)$. **38.** $(2x+3y+1)(x-2y-2)$.
39. $(5x+3y-5)(3x-5y-4)$. **40.** $(4x-3y-3)(3x+2y-6)$.

EXERCISE 78 A (P. 317)

1. $(a+b-c)(a^2+b^2+c^2-ab+ac+bc)$.
2. $(2l-1-y)(4l^2+1+y^2+2l+2ly-y)$.
3. $(2a-3b-c)(4a^2+9b^2+c^2-3bc+2ac+6ab)$.
4. $9(a-3b)(b-2c)(6c-a)$. **5.** $-3a(2b+3c)(a+2b+3c)$.
6. $3(3a-b)(3b-c)(3c-a)$. **7.** $-(a-b)(b-c)(c-a)$.
8. $(a+b)(b+c)(c+a)$. **9.** $-(a-b)(b-c)(c-a)$.
10. $-(a-b)(b-c)(c-a)(a+b+c)$.
11. $-(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)$.
12. $-(x-y)(y-z)(z-x)(xy+yz+zx)$.
13. $-2(a-b)(b-c)(c-a)(a+b+c)$. **14.** $80abc(a^2+b^2+c^2)$.
15. $-9(a-b)(b-c)(c-a)$.
16. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+3ab+3bc+3ca)$.

EXERCISE 78 B (Pp. 317, 318)

1. $(a+b+2c)(a^2+b^2+4c^2-ab-2ac-2bc)$.
2. $(3a-b+c)(9a^2+b^2+c^2+bc-3ac+3ab)$.
3. $(a-2b+3)(a^2+4b^2+9+2ab-3a+6b)$.
4. $-3(5a-4b)(4b-3c)(5a-3c)$. **5.** $6(a-b)(2b-c)(c-2a)$.
6. $3(5a-2b-c)(5b-2c-a)(5c-2a-b)$. **7.** $(a+b)(b+c)(c+a)$.
8. $-(a-b)(b-c)(c-a)$. **9.** $(a-b)(b-c)(c-a)(a+b+c)$.

10. $(l-m)(m-n)(n-l)(lm+mn+nl)$.
11. $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$. 12. $24abc$.
13. $5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca)$.
14. $12abc(a+b+c)$. 15. 0.
16. $3(b-c)(c-a)(a-b)(a-x)(b-x)(c-x)$.

EXERCISE 79 (Pp. 318, 319)

1. $(xz-y)(5y+z)$. 2. $4a(a^2+3b)(a^2-3b)$. 3. $(7a-6b)(6a+5b)$.
4. $5a(a-2)(a^2+2a+4)$. 5. $(a-13)(a^2+5)$.
6. $2(5x+3)(3x-5)$. 7. $(3a^2+l-2m)(3a^2-l+2m)$.
8. $(1-ac^2)(1+b^2c^2)$. 9. $(3a+4b)(2a-9b)$.
10. $3(x-y)(y-z)(z-x)$. 11. $3xy(x+2y)(x-2y)$.
12. $(x+1)(4x-9)$. 13. $(x+3)(x-1)(x-2)$.
14. $(x+1)^2(x+2)^2$. 15. $(a^2+4)(b^2+c^2)$.
16. $(3y-x)(7x^2-6xy+3y^2)$. 17. $(2c+2d-5)(2c-2d-5)$.
18. $(17x-3)(x+12)$. 19. $(a-b+x+1)(a-b-x-1)$.
20. $(2x-3)(3x+1)(x-2)$. 21. $(3x+4y)(6x-5y)$.
22. $2(4a^2-4a+5)(2a+1)(3-2a)$. 23. $2(2a-1)(a+2)(2a-3)$.
24. $(4a-9)(12a+5)$. 25. $5(x-y)(7x^2-11xy+7y^2)$.
26. $(x+1)(x+3)(x^2-3x+9)$. 27. $(a-2b)(a-1)(2b-1)$.
28. $(8x-5)(4x+9)$. 29. $(x+2a)(x-3b)(x+3b)$.
30. $(6x+7)(18x-25)$. 31. $(2x+1)(2x^2+4x-9)$.
32. $(3x+y-2)(3x-y-1)$. 33. $(x+5)(x+4)(x-5)(x-4)$.
34. $(y+3-3x)(y+3+3x)$. 35. $(x-4y)(3x+2y+2)$.
36. $36x^2(9x-2)$. 37. $(2x-y)(2x+y-1)$. 38. $(a-b)(2a-b-c)$.
39. $(3a-4b)(a-2b)$. 40. $(a-b)(a+b-4)$. 41. $(a+1)(a+b-1)$.
42. $(m+n)(l+m-n)$. 43. $(y-x)(xy+4)$.
44. $4x(x+1)(4x+1)(4x+3)$. 45. $(x-2y)(2x-3y)(2x+3y)$.
46. $(3x-1)(3x+1+a)$. 47. $(a-b)(a+2b+1)$.
48. $2a^2(3a+2)(3a-2)(4a^2+1)$. 49. $9(3x-a)(3x+2a)(x+a)(x+2a)$.
50. $(x-a)(x-1)(a-1)$. 51. $(a-2b)(a+b-c)$.
52. $(x+y+l+m)(x+y-l-m)(x-y+l-m)(x-y-l+m)$.
53. $(2x-3)(3x-2y+5)$. 54. $2(x^2+b)\{(x^2+b)^2+3a^2x^2\}$.
55. $(7x+3)(7x+11)$. 56. $(b+c)(c+a)(a+b)$.
57. $(x-3y)(2x-y)(3x-y)(2x+y)$. 58. $(4-x-y)(1-x-y)$.
59. $(xa+ya-x-z)(xa-ya-x+z)$. 60. $(x^2+2x+3)(x^2-2x+3)$.
61. $(a-b+1)(2a-b+1)$. 62. $(x-3)(x+2+4y)$.
63. $(1-4c)^2(1+4c)$. 64. $(x-5y)(x+y+3a)$.
65. $(x-2)(3x-7)(2x+5)$. 66. $(x^2+y^2)(5x^2+4xy-5y^2)$.
67. $-3(y+z)(z+x)(x+y+2z)$. 68. $(x-2y)(x-2y-3a)$.
69. $(y-z)(2v-5z)(2y+7z)$. 70. $(3a+4b)(5a-3b-2)$.

71. $(2x+y)(3x-4y)(4x+3y)$.
72. $(3x+2y)(2x+2y-1)$.
73. $(6x-3y+7)(3x+3y-8)$.
74. $(2x-3y)(3x-4y)(4x+3y)$.
75. $(5x+9)(9x-5)$.
76. $(9x+4y)(4x-9y)$.
77. $(3x+4y)(5x-6y-6)$.
78. $(6x+y-2)(x+5y-5)$.
79. $(4a+27)(27a+4)$.
80. $(10c-3d^2)(25c-63d^2)$.
81. $(2x+7y-3)(2x-5y-5)$.
82. $(2x-7y)(8x+3y-1)$.
83. $(xy-3x-2)(x+y+2)$.
84. $(2xy-4x-5)(3x-y+3)$.
85. $(6x+y-5)(x+5y-2)$.
86. $(4x+9)(3mx-m+8)$.
87. $(xy+5x-3)(x+2y+1)$.
88. $(3xy-2x-4)(5x-2y-3)$.
89. 13.
90. $(3x+5)(x+3)$; $5 \cdot 61 \cdot 103$.
91. 4.
92. -12, -650.
93. $a=-2$, $b=6$.
94. $p=-7$, $q=1$.

Exercise 80 A (Pp. 321, 322)

1. a.
2. $\dfrac{a^2+ab+b^2}{b}$.
3. $\dfrac{x^2+2xy+4y^2}{2x^2}$.
4. $\dfrac{2(16x^2-y^2)z^2}{3x^5}$.
5. 1.
6. x^2-xy+y^2.
7. $\dfrac{x+4}{x^2+2x+4}$.
8. $\dfrac{6x+1}{4x^2-1}$.
9. $4a$.
10. $\dfrac{a^2+5ab+5b^2}{a^2+ab-2b^2}$.
11. $-\dfrac{1}{2(x+y)}$.
12. $\dfrac{a^3}{(1+a^3)(1-a)}$.
13. $\dfrac{1}{1-l^2}$.
14. $\dfrac{2a}{a^2+3b^2}$.
15. 0.
16. $\dfrac{120b^3}{(b^2-a^2)(16b^2-a^2)}$.
17. $\dfrac{2}{x-3y}$.
18. $\dfrac{4d}{c^2-4d^2}$.
19. $\dfrac{6}{x(x+1)(x+2)(x-1)}$.
20. $\dfrac{2}{(x-1)(x-2)(x-3)}$.
21. $\dfrac{9x^2}{(3x-2y)(3x-z)}$.
22. $\dfrac{4}{(a-c)(b-c)}$.
23. 0.
24. $\dfrac{a(m-n)+b(n-l)+c(l-m)}{(m-n)(n-l)(l-m)}$.

Exercise 80 B (Pp. 322, 323)

1. $2x$.
2. a.
3. $\dfrac{1-x}{(1+x)(1+2x^2)}$.
4. $\dfrac{a^2-ab+b^2}{a^2+ab+b^2}$.
5. $2c$.
6. $\dfrac{y^2}{2x+y}$.
7. $-\dfrac{1}{2ab}$.
8. $\dfrac{a^2+ab+b^2}{a(a-9b)}$.

9. $\frac{x-2}{x+2}$. 10. $\frac{1}{2a+b}$. 11. $\frac{x-y}{xy}$. 12. $\frac{2}{(t+2)(t+3)(t+4)}$.

13. $-\frac{x^2}{x^3-y^3}$. 14. $\frac{a^3b}{(a^3-b^3)(a+b)}$. 15. $\frac{1}{1-a^2}$.

16. $\frac{5x}{3(25x^2-y^2)}$. 17. $-\frac{6y^3}{(x+y)(x+2y)(x+3y)(x+4y)}$.

18. $\frac{1}{2(a+1)}$. 19. $\frac{18b^2}{a(a^2-9b^2)}$. 20. $\frac{2}{(x-1)(x-2)(x-3)}$.

21. $\frac{2x+4y}{(x-y)(x-2y)(x+5y)}$. 22. $\frac{2(qr+rp+pq-p^2-q^2-r^2)}{(p-q)(q-r)(r-p)}$.

23. $\frac{y-x}{x^2-2xy+2y^2}$. 24. 1.

EXERCISE 80 C. (Pp. 323, 324)

1. $\frac{8a(a^2-3)}{3(a^2-9)(a^2-1)}$. 2. $-\frac{18c}{25b^2-9c^2}$. 3. $\frac{1+x}{(1-x)^2}$.

4. $\frac{1680x^3}{(9x^2-25y^2)(49x^2-25y^2)}$. 5. $\frac{24ab}{16a^4-81b^4}$. 6. $\frac{(x-1)(2x-1)}{(x+1)(x-2)}$.

7. $\frac{a^2-3}{(a+1)^2}$. 8. $\frac{6}{1-a^6}$. 9. x.

10. $\frac{a+b+c}{abc}$. 11. $\frac{a+b}{a-b}$. 12. $\frac{210-2x^4}{(x^4-25)(x^4-9)}$.

13. $\frac{a^2+3a+1}{a^2-a+1}$. 14. $2(a+b+c)$. 15. 0.

16. 3. 17. $\frac{x^2+x-1}{x+2}$. 18. $-\frac{1}{a+b}$.

19. $\frac{y^3-3y+1}{y^2-y}$. 20. $\frac{1}{a(a-b)}$.

EXERCISE 81 A (P. 328)

1. x^2+3x+2. 2. $8x^2-2x+3$. 3. c^2+3c+7. 4. $x-5$.

5. $4x^2-6x+5$. 6. $a(2a^2-3a-4)$. 7. $\frac{18x^2+14x-6}{8x^2+11x-4}$.

8. $\frac{2a-5}{4a^2-8a+1}$. 9. $2x-3$. 10. $1-c^3-c^4$.

EXERCISE 81 B (P. 329)

1. x^2+2x+1. **2.** $4l^2-26l+5$. **3.** x^2-2x-1. **4.** $3x-5y$.

5. $a(a^2-4a+4)$. **6.** x^2-3x+3. **7.** $\dfrac{x^2-2x-8}{x^2+1}$.

8. $\dfrac{x^3-x^2+x-6}{3x^3-3x^2+8x-8}$. **9.** $3a-2$. **10.** x^2-2x+1.

EXERCISE 82 A (Pp. 329, 330)

1. 3. **2.** 8. **3.** 0 or $7\frac{1}{2}$. **4.** $-1\frac{1}{4}$. **5.** $\frac{1}{2}$.
6. $-1\frac{1}{5}$, $-2\frac{2}{5}$. **7.** $\frac{1}{14}$. **8.** $6\frac{1}{2}$. **9.** $\frac{1}{6}$, $-\frac{1}{14}$. **10.** $5\frac{1}{2}$. **11.** 0.

EXERCISE 82 B (P. 330)

1. 2. **2.** $-1\frac{1}{2}$. **3.** 1, $-2\frac{1}{2}$. **4.** 7.
5. $-\frac{1}{10}$, $-\frac{7}{10}$. **6.** $\frac{5}{14}$. **7.** 0, $-\frac{17}{28}$. **8.** $7\frac{1}{2}$.
9. $\dfrac{3(c+1)}{2(c-1)}$ or $-\dfrac{3(c+1)}{2(c-1)}$. **10.** -6. **11.** $1\dfrac{2}{3}$.

EXERCISE 83 A (Pp. 333, 334, 335, 336)

1. Cloth 1s. 3d., Canvas 6d. per yd. **2.** 20 ft., 14 ft. **3.** 97.
4. 300 yds., 100 yds. **5.** £3. **6.** 3.39 p.m. **7.** 12.30 p.m.
8. 5 m.p.h., 16 miles. **9.** 60 yd. by 48 yd. or 66 yd. by 42 yd.
10. 12 a shilling. **11.** 50 m.p.h., 25 m.p.h. **12.** 1650.
13. 2 hr. 2 min. **14.** 20 ft. **15.** 36 min., 45 min.
16. 24 days. **17.** 89·7. **18.** 30 ($7\frac{1}{7}$ should be rejected) m.p.h.
19. 15. **20.** 144 yd., 42 yd. **21.** 180 m.p.h., 300 miles.
22. 1 yd., 10 yd. **23.** 50 yds. **24.** 12. **25.** 5 m.p.h.

EXERCISE 83 B (Pp. 336, 337, 338)

1. Figs 10d., Currants 4d. **2.** $73\frac{1}{8}$, $36\frac{9}{16}$. **3.** 100, 6d.
4. 960 miles at 32 m.p.h. **5.** 16 yd., 12 yd. **6.** 12 noon.
7. 3.26 p.m. **8.** 4 m.p.h., 14 miles. **9.** $\frac{4}{7}$. **10.** 5, 2.
11. 2s. 10d. per lb. **12.** £2 10s., £2 2s. 6d. **13.** 32 yd. per min.
14. 48 m.p.h. **15.** 30 hr., 50 hr. **16.** 15.
17. 8000 ; 1s. 6d. **18.** 36 m.p.h. and 48 m.p.h.
19. 150 yd., 80 yd. **20.** 83, 38.
21. 5 and 2, or $-8\frac{1}{5}$ and $-6\frac{4}{5}$. **22.** 18 cm., 5 cm., 24 cm., 5 cm.
23. A $9\frac{3}{4}$ secs., B 10 secs. **24.** 12.30 p.m. **25.** 12 m.p.h.

TEST PAPERS VII (Pp. 338, 339, 340, 341, 342, 343, 344)

A. 1. (i) $(a^2-3b+x)(a^2-3b-x)$, (ii) $(7x-8)(2x-7)$,
(iii) $(x+2)(2x-1)(4x+5)$.

2. $\dfrac{(4x+5a)(2x-a)}{4(x+2a)(2x+3a)(x+a)}$. **3.** $u=\dfrac{xz}{x+y}$, $v=\dfrac{ty}{x}$.

4. (i) -1, (ii) $2t$, $\dfrac{1}{2t}$. **5.** 4 m.p.h., $3\frac{1}{3}$ m.p.h.

6. x^2+5x+5 ; $-3{\cdot}62$, $-1{\cdot}38$, each twice.

B. 1. $\dfrac{2a-3b}{a-b}$, $1\frac{3}{5}$.

2. (i) $(9x+2)(18x-5)$, (ii) $x(x+1)(x+2)(x+3)$,
(iii) $(1-x)(3-4c+2cx)$.

3. $A\dfrac{ac}{b+c-a}$ days ; $B\dfrac{ac}{a-b}$ days. **4.** (i) $\dfrac{2c-d}{2}$, (ii) $x=3$, $y=-1$.

5. $2x-3$; $(2x-3)(x+3)(x-2)$; $(2x-3)(2x-1)(x-3)$; $1\frac{1}{2}$, 2, -3.

6. 8 m.p.h.

C. 1. (i) $9(x-3y)(3x^2-12xy+13y^2)$, (ii) $(x^2-3y^2+2z^2)(x^2-3y^2-2z^2)$,
(iii) $(10x-y)(1+5x-3y)$.

2. x^3+3x^2-4x+2. **3.** $\dfrac{3(2a^2+ax-6x^2)}{9x^2-a^2}$ **4.** 24′ ; 16′ 6″.

5. 2, $-2\frac{1}{2}$. **6.** $x=1$, $y=-1$, or $x=\frac{1}{4}$, $y=\frac{1}{8}$.

D. 1. (i) $(x+7y)(6x+35y)$, (ii) $(x+2)(x+6)(x-2)(x-4)$,
(iii) $(x+3)(x^3+x+1)$.

2. (*a*) 105, (*b*) 0. **3.** (*a*) $\dfrac{y(c-x)}{x^2+y-xc}$, (*b*) $1-2a$, $2-2a$.

4. $r^2=q(p+q)$, $qt=rv$. **5.** 45 m.p.h.

6. $x=1$, $y=3$; $x=3\frac{1}{2}$, $y=-\frac{3}{4}$; $x=1$, $y=-2$.

E. 1. £$\dfrac{K}{\left(1+\dfrac{C}{100}\right)^T}$.

2. (i) $(6x+5)(4x-5)$, (ii) $(2x-1)(1+2bx-5b)$,
(iii) $(1+a+b-c)(1+a-b+c)$.

3. $x(3x-2)$. **4.** (i) $\frac{5}{7}$, (ii) $x=2\frac{1}{3}$, $y=-2$ or $x=-2\frac{1}{4}$, $y=\frac{3}{4}$.

5. $\dfrac{2uv}{u+v}$, $\dfrac{u(v-u)}{u+v}$, $\dfrac{v(u-v)}{u+v}$. **6.** 14 m.p.h., 24 m.p.h.

F. 1. $2(4x-3y)(4x^2-6xy+9y^2)$, (ii) $(9x+2y)(6x-3y+4)$,
(iii) $(2x-9y-2)(x+6y+1)$.

2. $\dfrac{x^3-3x^2+6x-4}{2x^2}$.

3. (i) $\frac{5b}{a}$, $-\frac{5a}{b}$, (ii) $x=2, y=-1$; $x=-\frac{1}{9}, y=1\frac{1}{9}$.

4. $\frac{1}{(5x-2)^2}$. **5.** 10 m.p.h.

6. From $x=-3$ to $x=-1\frac{1}{2}$; $-2{\cdot}86$, $-0{\cdot}14$; the line is $y=\frac{1}{2}$.

G. 1. (i) $45(3l-m)(19l^2-36lm+21m^2)$, (ii) $(x+4)(4x-3)(x^2-2x-1)$.

2. $\frac{2x^2+1}{x+1}$. **3.** $a=\sqrt{b^2+x^2}$, $a=\sqrt{b(b+c)}$. **4.** 80.

5. (i) $x=2, y=3$ or $x=0, y=0$. **6.** (b) $a^2+b^2+c^2$.

H. 1. (a) $\frac{1+3a^2}{(1-a)^2}$, (b) 0.

2. (i) $(2a^2-7)(7a^2+8)$, (ii) $(x-1)(8ax-10a-1)$,
(iii) $(a-b)(b+c)(c+a)(a^2+b^2+c^2+ab-bc-ca)$.

3. x^2-2x+3. **4.** $\frac{1}{2}$. **5.** £36.

6. $17\frac{1}{2}$ sec. and $52\frac{1}{2}$ sec. after the start.

I. 1. (a) £$\frac{cX}{A}$, (b) $a=31, b=-12$.

2. (i) $(2a-5b)(109a^2-23ab+7b^2)$, (ii) $(7x-2)(2x-6y+3)$,
(iii) $(2c+d+3x-2y)(2c+d-3x+2y)$.

3. (i) 4, $-3\frac{3}{5}$, (ii) $x=3\frac{1}{2}, y=-2\frac{1}{2}$, or $x=1\frac{1}{7}, y=1\frac{3}{7}$.

4. (1) £$\frac{9(A-310)}{40}$, $\frac{45(A-310)}{2A}$ per cent.;

(2) £$\frac{3(A-220)}{40}$, $\frac{15(A-220)}{2A}$ per cent.; (3) 0, 0.

5. 4 miles, $1\frac{1}{3}$ hrs.

6. $x=\frac{a-b}{2}+\frac{c^2}{2(a-b)}$, $y=\frac{a+b}{2}-\frac{c^2}{2(a-b)}$, $z=\frac{c^2}{2(a-b)}-\frac{a-b}{2}$.

J. 1. (b) $a=17, b=-12$; $(3x-4)(4x+3)$.

2. $3x^3-x^2+4x-7$, $\lambda=-56$.

3. (i) $(30x+7y)(20x-9y)$, (ii) $24(3x+7)(2x+3)(x-1)(x-4)$,
(iii) $(4x-7)(1+6xy+5y)$.

4. (a) $\frac{x-y}{(y+1)z-(x+1)w}$, (b) $4\frac{1}{2}$. **5.** 120.

K. 1. (i) $(4+c+2d)(4-c-2d)$, (ii) $8(3x-2y)(2x-y)$,
(iii) $(9x+2)(5x-3y+2)$.

2. $\frac{125x^2-40x-24}{5x(5x+3)(25x^2-4)}$.

3. (*a*) $\frac{1}{8}-\frac{t}{4}+\frac{17t^2}{48}-\frac{31t^3}{108}+\frac{17t^4}{96}-\frac{t^5}{16}+\frac{t^6}{64}$,

(*b*) $c^2+c(2-a+b)+a^2+b^2+ab-a+b+1$.

4. (i) $\frac{1}{k+1}, \frac{k^2-1}{k}$, (ii) $x=-\frac{1}{2}, y=-1$, or $x=2, y=1$.

5. (*a*) $9a^3+3a^2+33a-26$, (*b*) $a=6, b=13$. **6.** 24.

L. **1.** $5x-3$.

2. (i) $7(4x+37)(x+3)$, (ii) $(2x+1)(2x-1)(x^2+3x+1)$, (iii) $(3x+2y-4)(4x-3y-5)$.

3. $a^2+ab-3b^2$. **4.** 20, 16 or $\frac{1}{4}, \frac{1}{5}$.

5. (i) $a+b, a-c$; (ii) $x=10, y=-7$, or $x=-\frac{2}{35}, y=\frac{19}{35}$. **6.** 2·63.

PART III

EXERCISE 84 (Pp. 348, 349)

1. $a^{\frac{9}{20}}$, $b^{\frac{5}{6}}$, $c^{\frac{1}{3}}$, 1, $h^{\frac{5}{7}}$, l^{-2}, m^3, n, $x^{\frac{5}{6}}$, $y^{\frac{7}{5}}$, $a^{\frac{1}{6}}$, b^5, c, d^3, h^{a-b}, $l^{-\frac{1}{6}}$, m^5, n^{-3x}, x^{-2}, y^{-7}.
2. The cube root of a^5, the ath root of x^b, the square root of c^r, the seventh root of y^3, the sixth root of b^{18}, i.e. b^3, the sth root of z^{3s}, i.e. z^3.
3. a^8, a^8, a^2, b, $x^{\frac{5}{49}}$, $c^{-\frac{4}{3}}$, y^4, d^{-1}, l^{-4}, x^{-3}, a^{-8}, a^8, a^{-2}, b^{-1}, $x^{\frac{5}{49}}$, $c^{\frac{4}{3}}$, y^{-6}. $d^{\frac{1}{2}}$, $l^{-\frac{1}{4}}$, x^{-a}.
4. $a^{\frac{1}{4}}$, $b^{\frac{2}{5}}$, $\frac{1}{c^{\frac{3}{7}}}$, $\frac{1}{x^{\frac{5}{3}}}$, $\frac{1}{l^{\frac{8}{5}}}$, $\frac{1}{z^{\frac{7}{10}}}$.
5. $\sqrt[5]{a^2}$, $\sqrt[7]{b^3}$, $\frac{1}{\sqrt[3]{c^2}}$, $\sqrt[3]{d^4}$, $\frac{1}{\sqrt[3]{l}}$, $\frac{1}{\sqrt[4]{x^3}}$, $\sqrt[8]{y^7}$, $\frac{1}{\sqrt[3]{z^a}}$.
6. $\frac{1}{a^4}$, $\frac{1}{x^5}$, a^3c^2, $b^{\frac{1}{3}}$, $\frac{1}{a^7}$, $a^{\frac{1}{3}}b^{\frac{1}{2}}$.
7. 8, 32, $\frac{1}{125}$, $\frac{1}{4}$, 0·008, $\frac{1}{36}$, 8, $\frac{1}{16}$, $15\frac{5}{8}$, $\frac{1}{8}$, $-\frac{1}{32}$, $\frac{1}{49}$, 0·5, 1, $\frac{1}{64}$, $\frac{1}{128}$, 32, 1, -7, 1000, 9, 0·2, $\frac{16}{81}$, $2\frac{1}{4}$, 8, 1, 25, 5, $\frac{1}{4}$, 27, $\frac{1}{4}$, 1, 4, $\frac{1}{8}$.

8. $\frac{3}{x^{\frac{1}{2}}}$. 9. $\frac{5}{c^{\frac{3}{4}}}$. 10. $7z^3$. 11. $\frac{3a^5}{x^4}$. 12. $\frac{a^7}{3}$.

13. $\frac{c^{\frac{2}{3}}}{4}$. 14. $4x^{\frac{3}{2}}$. 15. $\frac{8}{x^{\frac{5}{2}}}$. 16. $\frac{8}{x^{\frac{5}{2}}}$. 17. $\frac{x^{\frac{3}{2}}}{2}$.

18. $\frac{3b^3d^5}{7a^2c^3}$. 19. $\frac{y^m z^n}{x^l}$. 20. $\frac{10}{a^{\frac{5}{3}}}$. 21. $\frac{x^{\frac{1}{2}}}{4}$. 22. $2x^{\frac{1}{2}}$.

23. $\frac{a}{c^3}$. 24. $\frac{b^3}{3x^2}$. 25. $\frac{9b^3}{x^2}$. 26. $\frac{1}{2x^{\frac{3}{2}}}$. 27. $\frac{3y^{\frac{5}{2}}}{4}$.

28. $x^{\frac{9}{5}}$. 29. $2y^{\frac{3}{2}}$. 30. $\frac{1}{a^2}$. 31. $27y^3$. 32. $\frac{c^2}{ab}$.

33. $\frac{12}{a^{\frac{3}{2}}}$. 34. $\frac{3}{16a^{\frac{3}{2}}}$. 35. $27x^{\frac{7}{6}}$. 36. $\frac{c^2}{3y^4}$. 37. $2z^{\frac{2}{3}}$.

38. $\frac{9c^2}{y^4}$. 39. $\frac{1}{5c^{\frac{1}{2}}}$. 40. $\frac{1}{x}$. 41. $\frac{a^2}{2}$. 42. $\frac{c^2}{4a^2}$.

43. $8^{\frac{2}{3}}$ $8^{\frac{4}{3}}$, 8^0, $8^{-\frac{1}{3}}$, $8^{\frac{1}{6}}$, $8^{\frac{7}{3}}$. 44. $\frac{3}{2}$, $-\frac{1}{2}$, $\frac{7}{6}$, $-\frac{15}{2}$.

f

EXERCISE 85 (Pp. 350, 351, 352)

1. $6x^{\frac{1}{2}} - 14x^{\frac{1}{4}} + 3 + 35x^{-\frac{1}{4}} - 45x^{-\frac{1}{2}}$. **2.** $a^{\frac{2}{3}}b - a^{-\frac{1}{6}}b^{\frac{1}{6}} - 2a^{-1}b^{-\frac{2}{3}}$.

3. $2x + x^{\frac{3}{4}} + 2x^{\frac{1}{4}} + 1$. **4.** $a^{\frac{4}{5}} + a^{\frac{2}{5}}b^{\frac{2}{5}} + b^{\frac{4}{5}}$. **5.** $8a^{-\frac{1}{2}} + 7a^{-\frac{1}{4}} + 6$.

6. $7x^{\frac{2}{5}} - 2x^{\frac{1}{5}} + 1$. **7.** $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$. **8.** $x^{-\frac{2}{7}} - x^{-\frac{1}{7}} - 1$.

9. $\pm(3x - 2 + x^{-1})$. **10.** $\pm\left(2x^{\frac{5}{2}} - 4 + 3x^{-\frac{5}{2}}\right)$. **11.** $\pm\left(\frac{a^{-1}}{4} + 1 - 3b^{\frac{1}{3}}\right)$

12. $\pm\left(3x^{\frac{4}{5}} - 2 - x^{-\frac{4}{5}}\right)$. **13.** x^8y^{20}. **14.** $\frac{x^{\frac{3}{2}}}{y^{\frac{7}{2}}}$. **15.** $\frac{5a^2b}{2}$.

16. $\frac{1}{2x^{\frac{2}{3}}y}$. **17.** $\frac{9}{4c^4d^4}$. **18.** $81x^3y$. **19.** $\frac{x^5}{2y}$. **20.** $\frac{p^{\frac{11}{12}}}{q^3}$.

21. $1 - x$. **22.** $\frac{1}{2}$. **23.** 1. **24.** $1\frac{1}{4}$. **25.** $\frac{8bx^{\frac{1}{2}}}{9}$.

26. $\frac{a^{\frac{1}{9}}b}{2}$. **27.** $4a$. **28.** a^{2+x}. **29.** $\frac{x}{3}$. **30.** $\frac{1}{x}$.

31. $\frac{1}{4x^5}$. **32.** $a^2 - b^2$. **33.** $\frac{1}{c^4}$. **34.** $x^{n^2-n} + x^{2n}$. **35.** 5^{4n^2}

36. 6. **37.** $\frac{b^{\frac{6}{5}}}{a^{\frac{1}{5}}}$. **38.** $\frac{a^{\frac{9}{4}}}{b^{\frac{5}{4}}}$, 0. **39.** $\frac{b^{\frac{4}{3}}}{a^{\frac{1}{3}}}$. **40.** $\frac{c^{11}}{a^4b^6}$, 0.

41. $\frac{c^3}{d^2}$. **42.** k. **43.** $x = \frac{a^{\frac{35}{11}}}{b^{\frac{24}{11}}}$, $y = \frac{b^{\frac{16}{11}}}{a^{\frac{5}{11}}}$. **44.** $x = \frac{l^4}{k^3}$, $y = \frac{k^3}{l^2}$.

45. $x = \frac{b^{\frac{9}{5}}}{a^{\frac{4}{5}}}$, $y = a^{\frac{16}{25}}b^{\frac{9}{25}}$. **46.** 5. **47.** -4. **48.** $\frac{3}{5}$.

49. $x + x^{\frac{1}{2}} - 20$. **50.** $12a - \sqrt{a} - 6$. **51.** $x + 4x^{\frac{1}{2}} + 16$.

52. $1 - 5c^{-2} + 25c^{-4}$. **53.** $a^{\frac{4}{3}} - 6 + 9a^{-\frac{4}{3}}$.

54. $x + 8x^{\frac{1}{2}} + 24 + 32x^{-\frac{1}{2}} + 16x^{-1}$. **55.** $\frac{a^{\frac{1}{2}} + 5}{a^{\frac{1}{2}} - 2}$. **56.** $\frac{x^{\frac{1}{2}} + 2}{x^{\frac{1}{2}} + 3}$.

57. $\frac{1 + 4x^3}{1 + 2x^3}$. **58.** $x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}}$. **59.** $\frac{x^{\frac{1}{2}}}{y}$. **60.** $x^{\frac{3}{2}} + 343$

61. x^6y^{12}. **62.** $\frac{c^{\frac{3}{2}}t}{2}$. **63.** $\frac{b^{\frac{5}{2}}}{a}$. **64.** $\frac{8x^{\frac{15}{4}}}{27y^{\frac{3}{2}}}$.

65. $\frac{l^{\frac{3}{2}}m^3}{81}$. **66.** $\frac{25b^{\frac{4}{3}}}{4a^{\frac{10}{3}}}$. **67.** $\frac{4y^{\frac{4}{3}}}{3x^2}$. **68.** 1.

69. $\frac{1}{x^{\frac{2}{3}}y^{\frac{1}{2}}}$. **70.** b^3. **71.** 4. **72.** 3^{2n^2+4n}.

EXERCISE 86 (Pp. 355, 356)

1. $\sqrt[12]{729}$, $\sqrt[12]{16}$, $\sqrt[12]{49}$, $\sqrt[12]{a^{15}}$, $\sqrt[12]{a^{-10}}$, $\sqrt[12]{x^4y^{-8}}$.

2. (i) $\sqrt[4]{9}$, $\sqrt[3]{5}$, $\sqrt[6]{24}$, (ii) $\sqrt[4]{11}$, $\sqrt[3]{6}$, $\sqrt{3}$.

3. (i) $\sqrt[14]{x^7}$, $\sqrt[14]{x^{12}}$, (ii) $\sqrt[6]{a^8}$, $\sqrt[6]{a^3}$, (iii) $\sqrt[15]{a^6}$, $\sqrt[15]{a^5}$, (iv) $\sqrt[8]{49}$, $\sqrt[8]{16}$, $\sqrt[8]{9}$.

4. $12\sqrt{2}$, $3\sqrt{2}$, $3\sqrt{3}$, $4\sqrt{x}$, $4c\sqrt[3]{2}$, $-2\sqrt[3]{2}$.

5. $7\sqrt{2}$, $11\sqrt{2}$, $12\sqrt{5}$, $2\sqrt[4]{8}$, $-5x\sqrt[3]{3y}$, $80\sqrt{3}$.

6. $2x\sqrt{y}$, $3x^2y\sqrt[3]{xy^2}$, $-xy^2\sqrt[5]{x^3}$, $(a+b)\sqrt{a}$, $15c^3d\sqrt[3]{2c}$.

7. $\sqrt{162}$, $\sqrt{605}$, $\sqrt[3]{375}$, $\sqrt[4]{32}$, $\sqrt{882}$, $\sqrt[3]{320}$.

8. $\sqrt{10}$, $\sqrt{45}$, $\sqrt{\dfrac{3}{14}}$, $\sqrt{\dfrac{5c}{d}}$, $\sqrt[4]{\dfrac{x}{2y}}$. **9.** $14\sqrt{5}$. **10.** $12\sqrt{7}$.

11. $24\sqrt{3}$. **12.** $-10\sqrt{11}$. **13.** $17\sqrt[3]{7}$. **14.** $24\sqrt[3]{3}$.

15. $25\sqrt{3}-42\sqrt{2}$. **16.** $\sqrt[3]{2}+14\sqrt[3]{3}$. **17.** $33\sqrt{3}$. **18.** $3\sqrt{6}$.

19. $6x^2y\sqrt{x}$. **20.** $(3x^2y-5y^2z+6z^3)\sqrt[3]{x}$. **21.** $35\sqrt{2}=49{\cdot}50$.

22. $20\sqrt{2}=28{\cdot}28$. **23.** $21\sqrt{3}=36{\cdot}37$. **24.** $2\sqrt{2}=2{\cdot}83$.

25. $10\sqrt{6}-9\sqrt{3}=8{\cdot}91$. **26.** $64\sqrt{2}-60\sqrt{3}+25\sqrt{5}=42{\cdot}49$.

27. $15\sqrt{6}=36{\cdot}74$. **28.** $40\sqrt{3}=69{\cdot}28$. **29.** $70\sqrt{2}=98{\cdot}99$.

30. $60\sqrt{3}=103{\cdot}93$. **31.** $110\sqrt{6}=269{\cdot}445$. **32.** $\dfrac{9\sqrt{6}}{10}=2{\cdot}20$.

33. $\dfrac{7\sqrt{2}}{2}=4{\cdot}95$. **34.** $5\sqrt{2}=7{\cdot}07$. **35.** $\dfrac{30\sqrt{7}}{7}=11{\cdot}34$.

36. $\dfrac{2\sqrt{6}}{9}=0{\cdot}54$. **37.** $12\sqrt{5}=26{\cdot}83$. **38.** $8\sqrt{2}=11{\cdot}31$.

39. $\dfrac{\sqrt{2}}{20}=0{\cdot}07$. **40.** $\dfrac{5\sqrt{7}}{21}=0{\cdot}63$.

EXERCISE 87 (Pp. 359, 360, 361)

1. $20x-12\sqrt{x}$. **2.** $12\sqrt{l}+21l$. **3.** $70-20\sqrt{10}$.

4. $6x+7\sqrt{xy}-20y$. **5.** $18+19\sqrt{10}$. **6.** -17.

7. $111+60\sqrt{3}$. **8.** $320-70\sqrt{15}$. **9.** $x+2y-\sqrt{x^2-4y^2}$.

10. $4-2\sqrt{4-x^2}$. **11.** $2\sqrt{35}-8$. **12.** $12+12\sqrt{2}$.

13. $\dfrac{9+4\sqrt{3}}{33}$. **14.** $15\sqrt{3}+18\sqrt{2}$. **15.** $\sqrt{5}-1$.

16. $\dfrac{1+\sqrt{5}}{2}$. **17.** $7\sqrt{2}-2\sqrt{7}$. **18.** $\dfrac{24+14\sqrt{3}}{3}$.

19. $27\sqrt{2}+22\sqrt{3}$. **20.** $a+\sqrt{a^2-b^2}$. **21.** $\dfrac{32+x^2+8\sqrt{16+x^2}}{x^2}$.

22. $\frac{\sqrt{5}+1}{8}$. **23.** $7-4\sqrt{3}$. **24.** 1. **25.** 2·82. **26.** 1·34.
27. 0·25. **28.** 19·80. **29.** 0·50. **30.** 7·15. **31.** 0·04.
32. 0·29. **35.** (i) 2, (ii) 24, (iii) 568. **37.** -1. **38.** -24.
39. 1. **40.** 0. **41.** $\sqrt{5}+1$. **42.** $\sqrt{5}+\sqrt{3}$. **43.** $2\sqrt{2}-2$.
44. $\sqrt{6}-\sqrt{3}$. **45.** $3\sqrt{2}-2\sqrt{3}$. **46.** $3\sqrt{7}-2$. **47.** $3\sqrt{5}+2\sqrt{3}$.
48. $\sqrt{\frac{11}{2}}+\sqrt{\frac{5}{2}}$. **49.** $\sqrt{6+x}-\sqrt{4-x}$. **50.** $\sqrt{3x-2}+\sqrt{x-3}$.
51. $\sqrt{2}+1$. **52.** $\sqrt{7}+1$.

In Nos. 53-80, 93-100, roots which must be rejected are given in brackets.

53. 15. **54.** 3. **55.** 22. **56.** 126. **57.** 2. **58.** $1\frac{3}{7}$.
59. 10. **60.** 5. **61.** 9. **62.** 10, $\left(-2\frac{2}{3}\right)$. **63.** $\left(\frac{l}{4}\right)$.
64. 1, (61). **65.** 8. **66.** 6, $(1\frac{13}{17})$. **67.** 8, $(\frac{1}{3})$.
68. $\frac{a}{4}$, $\left(\frac{a}{24}\right)$. **69.** (11). **70.** 1, $\left(-1\frac{1}{3}\right)$. **71.** 0, 2, -2.
72. 7, $\left(-5\frac{3}{5}\right)$. **73.** 441. **74.** 4. **75.** $1\frac{1}{2}$.
76. $1\frac{1}{2}$, 12, (0). **77.** 6, 1, $\left(\frac{21\pm\sqrt{97}}{6}\right)$. **78.** 1, 9, twice.
79. 2, -8, (0, -6). **80.** $2\pm2\sqrt{2}$, i.e. 4·83, -0·83, (6, -2).
81. 0·22. **82.** 22·18. **83.** 0·39. **84.** 0·24. **85.** 0·11. **86.** 5·80.
87. 0·27. **88.** 0·38. **90.** (i) 1, (ii) 34, (iii) 198. **91.** 2. **92.** 0.
93. 19. **94.** -3. **95.** 7, $(\frac{2}{3})$. **96.** 9, (-4). **97.** $2\frac{1}{4}$.
98. 9. **99.** -2, 0. **100.** -3, 5, $(-1\frac{2}{7})$.

EXERCISE 88 (P. 362)

1. 3, -2, 6, -3, -1, 0. **2.** 2, 4, -3, 0·5, -2·5, 1·5.
3. 2, $\frac{1}{3}$, $-\frac{2}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, -1. **4.** 2, 0·5, -1, 1·5, 3, 2·5.

EXERCISE 89 (P. 367)

1. 0·5315. **2.** 0·6812. **3.** 0·7709. **4.** 0·7924. **5.** 0·8451.
6. 0·4518. **7.** 0·9552. **8.** 0·9763. **9.** 0·8274. **10.** 0·8639.
11. 0·5337. **12.** 0·6883. **13.** 0·7733. **14.** 0·7965. **15.** 0·8478.
16. 0·4527. **17.** 0·9545. **18.** 0·9767. **19.** 0·8275. **20.** 0·8640.
21. 0·8539. **22.** 0·9870. **23.** 0·5308. **24.** 0·9297. **25.** 0·6442.
26. 0·8052. **27.** 0·8578. **28.** 0·9890. **29.** 0·1058. **30.** 0·3238.
31. 1·440. **32.** 2·850. **33.** 8·580. **34.** 2·654. **35.** 7·090.
36. 3·409. **37.** 4·999. **38.** 6·147. **39.** 1·045. **40.** 1·798.

41. 1·460. **42.** 2·800. **43.** 8·520. **44.** 2·564.
45. 7·908 or 7·909. **46.** 3·904. **47.** 4·969.
48. 6·715 or 6·716. **49.** 1·053. **50.** 1·784 or 1·783.

Exercise 90 (P. 368)

The answers are, for convenience, given to 4 figures. If antilog. tables are used the answers so obtained may differ slightly in the last figure.

1. 9·333 (or 2). **2.** 9·056 (or 7 or 5). **3.** 8·351 (or 2).
4. 7·263 (or 4). **5.** 7·541 (or 2 or 3). **6.** 7·970.
7. 1·301. **8.** 1·599. **9.** 1·019. **10.** 1·618. **11.** 1·454.
12. 1·526. **13.** 2·395. **14.** 7·635 (or 6). **15.** 7·621 (or 2).
16. 5·696 (or 7). **17.** 3·219. **18.** 1·397.

Exercise 91 (P. 370)

1. 1·5315. **2.** 3·6812. **3.** $\bar{1}$·7709. **4.** $\bar{2}$·7924. **5.** 2·8451.
6. $\bar{3}$·4518. **7.** 4·9552. **8.** 1·9763. **9.** $\bar{1}$·8274. **10.** 3·8639.
11. 1·5337. **12.** $\bar{1}$·6883. **13.** $\bar{2}$·7733. **14.** 5·7965. **15.** 2·8478.
16. 2·4527. **17.** $\bar{1}$·9545. **18.** $\bar{3}$·9767. **19.** 2·8275. **20.** 1·8640.
21. $\bar{1}$·8539. **22.** 2·9870. **23.** $\bar{2}$·5308. **24.** $\bar{1}$·9297. **25.** 4·6442.
26. $\bar{2}$·8052. **27.** $\bar{1}$·8578. **28.** 2·9890. **29.** 3·1058. **30.** $\bar{5}$·3238.
31. 144. **32.** 28·5. **33.** 0·0858. **34.** 0·002654. **35.** 7090.
36. 34090. **37.** 0·0004999. **38.** 614700. **39.** 1045. **40.** 0·1798.
41. 14·6. **42.** 0·028. **43.** 8520. **44.** 0·2564.
45. 0·007909 (or 8). **46.** 39040. **47.** 496·9.
48. 0·6716 (or 5). **49.** 0·01053. **50.** 1784 (or 3).

Exercise 92 (P. 372)

1. $\bar{1}$·41. **2.** $\bar{4}$·61. **3.** $\bar{2}$·79. **4.** 0·2. **5.** $\bar{1}$·2. **6.** 2.
7. 7·03. **8.** $\bar{3}$·73. **9.** $\bar{2}$·87. **10.** $\bar{5}$·38. **11.** 6·1. **12.** 0·74.
13. $\bar{2}$·8. **14.** $\bar{4}$·76. **15.** $\bar{1}$·94. **16.** 2·72. **17.** $\bar{1}$·4. **18.** $\bar{9}$·5.
19. $\overline{31}$·9. **20.** $\bar{5}$·8. **21.** $\bar{1}$·47. **22.** $\bar{2}$·29. **23.** $\bar{2}$·97. **24.** $\bar{1}$·63.
25. $\bar{2}$·57. **26.** $\bar{1}$·59. **27.** $\bar{1}$·97. **28.** $\bar{3}$·86. **29.** $\bar{3}$·42. **30.** $\bar{1}$·676.
31. $\bar{1}$·61. **32.** $\bar{1}$·58. **33.** $\bar{9}$·9. **34.** 3·9. **35.** $\bar{1}$·1. **36.** 0·4.

Exercise 93 (Pp. 373, 374, 375)

(See note at head of Ex. 90.)

1. 981·0. **2.** 10640. **3.** 5991 (or 2). **4.** 251800.
5. 2·675 (or 6). **6.** 30·73 (or 4). **7.** 300·3. **8.** 1·819. **9.** 1875.
10. 751600 (or 751700). **11.** 5·339. **12.** 7·870. **13.** 2653.
14. 9·078 (or 7 or 9). **15.** 19·40 (or 19·39). **16.** 131·6.

17. 13·78 (or 9). **18.** 734·5. **19.** 4·132.
20. 1·337 (or 8). **21.** 342·9 (or 8). **22.** 14·01.
23. 0·1917 (or 6). **24.** 0·1934. **25.** 0·01363. **26.** 16·57.
27. 0·05207 (or 8). **28.** 0·06851 (or 2). **29.** 0·2183 (or 4).
30. 3·926. **31.** 178·5. **32.** 609·0. **33.** 0·03582.
34. 13·07 (or 6). **35.** 21·24 (or 3). **36.** 31·80.
37. 0·4290. **38.** 0·1212. **39.** 0·9242 (or 3). **40.** 0·05224.
41. 0·06163. **42.** 0·006960. **43.** 0·4775 (or 6).
44. 0·6811 (or 2). **45.** 352·0. **46.** 4·207 (or 8).
47. 0·006354 (or 5). **48.** 0·1606. **49.** 129·1.
50. 1·953 (or 4). **51.** 62·18 (or 9). **52.** 34·42 (or 3).
53. 1003 (or 4). **54.** 0·1869 (or 0·1870). **55.** 0·3104.
56. 5·335. **57.** 17·88. **58.** 0·2064. **59.** 6·160. **60.** 5·236.
61. 5·568 (or 7). **62.** (i) 35·62 (or 3), (ii) 353·0.
63. (i) 18·21, (ii) 17·43. **64.** (i) 4·353 (or 4), (ii) 8·838 (or 9).
65. (i) 574·6 (or 7), (ii) 3·782. **66.** (i) 2·158 (or 9), (ii) 90·42 (or 3).
67. (i) 28·42, (ii) 2·177 (or 8). **68.** (i) £573·5 (or 6), (ii) £317·6 (or 7).

EXERCISE 93 C (Pp. 375, 376, 377)

(See note at head of Ex. 90.)

1. 1·411. **2.** 0·1034. **3.** 0·00906. **4.** 0·09990 (or 1).
5. 3·244. **6.** 0·06576 (or 7). **7.** 2·193. **8.** 1·860.
9. 0·2199. **10.** 11·57. **11.** 0·004325. **12.** 0·1634.
13. 0·7667. **14.** 39·66. **15.** 0·5878 (or 9), say 0·59.
16. 7·962. **17.** $\bar{2}$·2772. **18.** 6·415. **19.** 1·375. **20.** 0·3183.
21. 1·145. **22.** 0·2291 (or 2). **23.** 2·452.
24. 4·536 (or 5). **25.** 2·129. **26.** 28·73.
27. 0·5958 (or 9). **28.** 1·634. **29.** 9·067 (or 8)$\div 10^{11}$. **30.** 43·68.
31. 1·681 (or 2)$\div 10^{4}$. **32.** 0·7248 (or 9).
33. (i) 66·24 (or 5) c.c., (ii) 12·03 cm. **34.** 0·1545... .
35. 0·3962. **36.** 22·99. **37.** 1·353. **38.** 0·5310 (or 1).
39. 19·97 (or 8). **40.** 0·9980. **41.** 926·4 (or 5), say 930.
42. 202·1 (or 2). **43.** 3·363. **44.** 7·053 (or 4). **45.** 0·04404.
46. 26·54. **47.** 104. **48.** 10·37 (or 8). **49.** $10^{25} \times$ 9·880 (or 1).
50. 147·94 (or 5 or 6). **51.** 260·0. **52.** 955·3 (or 4).
53. 2·815$\div 10^{6}$. **54.** 3·721. **55.** 47·84. **56.** 350·8.

EXERCISE 94 (Pp. 382, 383)

1. 2. **2.** 2. **3.** 3. **4.** 2. **5.** 3. **6.** 1·2.
7. −3. **8.** 0. **9.** 3. **10.** 2. **11.** 0. **12.** 3.
13. 1. **14.** 1. **15.** 2·5. **16.** 0·75. **17.** 9. **18.** −0·3.

19. $\frac{2}{3}$. **20.** 1·71. **21.** 0·80. **22.** 0·69. **23.** 1. **24.** 2·91.
25. $x=2, y=1$. **26.** $x=3·23, y=1·52$. **27.** $x=2·76, y=2·28$.
28. $x=7, y=3$. **29.** 0·43 or 0·68. **30.** 3. **31.** 6·34.
32. 13·5. **33.** 3. **34.** 3·3. **35.** 28·4. **36.** 6·6. **37.** 2·86272.
38. $\bar{1}$·70927. **39.** 0·518658. **40.** 1·20775. **41.** 0·62518.
42. 0·91079$\dot{6}$. **43.** $\bar{1}$·81259. **44.** $\bar{1}$·84679. **45.** 0·23754. **46.** $\bar{1}$·97360$\dot{6}$.
47. $\bar{2}$·53530. **48.** $\bar{2}$·433895. **49.** $xy^2=7$. **50.** $x^2=11y^3$.
51. $x\sqrt{y}=10$. **52.** $x^4=y^3$. **53.** $2xy^3=5$.
54. $5^x=36^y$ (or 6^{2y}). **55.** $x=3 \cdot 2^y$. **56.** $x^2=100 \cdot 5^y$.
57. $y=2x^{1·5}$. **58.** $ax=a-by$. **59.** $x^2=99y^2$.
60. $(x+y)^2=1000(x-y)$. **61.** 1·26. **62.** 1·05. **63.** 3·24.
64. $-1·32$. **65.** $y=\frac{1}{2}x^3$. **66.** $xy^3=2160$. **67.** $x^2y^3=312·5$.
68. $yx^{0·86}=41·4$. **69.** $y=63x^{1·78}$. **70.** $yx^{1·5}=480$.

EXERCISE 94 C (Pp. 383, 384)

1. $2\frac{1}{2}$. **2.** $-1\frac{1}{2}$. **3.** 1. **4.** 0·003981. **5.** $1\frac{1}{4}$.
6. 3. **8.** 0·1770. **9.** 0·1850. **10.** 10.
11. $x=4-\frac{16}{3}\log 2=2·39$; $y=\frac{4}{3}\log 2=0·40$. **12.** $x=1·17, y=-0·41$.
13. $x=5, y=4$. **14.** 93. **15.** (i) 1·2, (ii) 5·7. **16.** 16.
17. 12. **18.** 236. **19.** (i) $1-2\log 2-\log 3$, (ii) $2\log 3+\log 2-3$.
20. $\dfrac{(3a-1)}{2}$. **21.** (i) $\log\dfrac{15}{2a}$, (ii) 75. **22.** 30.
25. (i) $10^{0·1003}$, (ii) $10^{1·107}$. **27.** 4·61. **28.** 8·66.
29. (i) 1·9965, (ii) 3·883. **30.** $\bar{2}$·1590. **31.** 0·585. **32.** $-9·5$.
33. 0·693. **34.** 2·15. **35.** 1·21. **36.** $-4·75$. **37.** 5·82.
38. 1·24. **39.** $x=\frac{11}{14}, y=\frac{15}{28}$. **40.** $x=\frac{16}{17}, y=\frac{32}{17}$. **41.** 1 or 0·63.
42. 4. **43.** $\sqrt{x} \cdot y^2=5$. **44.** $x^4=13y$. **45.** $xy\sqrt{x}=100$.
46. $x^3=y^5$. **47.** $3^x=8^y$ (or 2^{3y}). **48.** $5x=2y^2$.
49. $10 \cdot 4^x=y^3$. **50.** $x=3\sqrt{y}$. **51.** $y^3(x^2-y^2)=x^2$.
52. $y^3=9x^3$. **53.** 2·81. **54.** 0·67. **55.** 0·69. **56.** 1·002.

EXERCISE 95 (Pp. 388, 389)

1. (i) 2 : 9, (ii) 9 : 1600, (iii) 61 : 90, (iv) 13 : 270, (v) $a^2 : 32b^2$, (vi) 1 : 8.
2. (i) 4 : 45, (ii) 12 : 175, (iii) 10 : 3, (iv) 2 : 7, (v) 7 : 5, (vi) 28 : 45.
3. (i) 4 : 5, (ii) 9 : 4, (iii) 5 : 4 or -5 : 4, (iv) 23 : 20 ; $100+x$: 100.
4. (i) 24 : 45 : 20, (ii) 9 : 6 : 8. **5.** (i) 1 : 3, (ii) 5 : 11, (iii) 2 : 23.
6. (i) 6 : 31, (ii) -11 : 17, (iii) 29 : 74.
7. $(7a+3):(4a+2)$; $a=13$. **8.** 1 : 4. **9.** 2 : 3 or -5 : 4.
10. 5 : 6. **11.** 2·1.

12. (i), (ii), (iv), (vii) are unaltered; (iii), (v) increased in ratio 2 : 1, (vi), (viii) decreased in ratio 1 : 2.

13. $-1 : 20$. **14.** $\frac{xl}{m}$. **15.** 24 years, 32 years.

16. 35, 71. **17.** 10 days. **18.** $37 : 7$. **19.** $\frac{24}{25}, \frac{2}{5}$. **20.** 2·314 to 1.

Exercise 96 (Pp. 394, 395)

1. (i) 24, (ii) $\frac{c^2d}{a}$, (iii) $\frac{9b^3}{2}$. **2.** (i) 64, (ii) a^6, (iii) $\frac{4a^2}{3}$.

3. (i) ± 16, (ii) $\pm a^3b^3$, (iii) $\pm 15ab^2$. **5.** $2\frac{1}{2}$, 1, $1\frac{1}{2}$.

6. (i) 16, 13; (ii) 4, 4. **11.** $c = \frac{ab}{a-b}$.

Exercise 96 c (Pp. 395, 396)

1. $x : y : z = 1 : -26 : -19$. **2.** $x : y : z = 5 : -17 : 14$.

3. $x : y : z = 7 : 19 : -27$. **4.** $x : y : z = -9 : 2 : 7$.

5. $x : y : z = 9 : 11 : -5$.

6. $x : y : z = (br - cq) : (cp - ar) : (aq - bp)$.

7. $x = \pm 2, y = \mp \frac{1}{2}, z = \pm 1$. **8.** $x = 15, y = 41, z = -1$.

9. $x = 22, y = -5, z = -21\frac{1}{2}$. **10.** $x = 3, y = 4, z = -\frac{1}{2}$.

11. $x = \frac{bc}{(a-b)(a-c)}, y = \frac{ca}{(b-c)(b-a)}, z = \frac{ab}{(c-a)(c-b)}$.

12. $x = \frac{b-c}{15}, y = \frac{c-a}{15}, z = \frac{a-b}{15}$. **13.** 0, -2. **14.** $-\frac{2}{3}$.

15. 0, -7. **16.** 0, 3, -3. **20.** $-\frac{1}{2}$.

23. The sign of the inequalities must be reversed, unless $b + x$ is also negative.

Exercise 97 (Pp. 402, 403)

1. $a = kb$. **2.** $c = kd^2$. **3.** $z = ky^3$. **4.** $ab = k$. **5.** $cd^2 = k$.

6. $x^2y^2 = k$. **7.** $(a+b)c^2 = k$. **8.** $t\sqrt{l} = k$. **9.** $c = kr$.

10. $w = kt$. **11.** $t^2\sqrt{s} = k$. **12.** $(l^2 + m^2)n^2 = k$. **13.** $i = kn$.

14. $pv = k$. **15.** $x = kw^2$. **16.** $x^2y^3z^5 = k$. **17.** $h = kv^3$.

18. $t = k\sqrt{d}$. **19.** $t^2 = kd^3$. **20.** $y = 5x$; $x = 7, -2·5, -4$; $y = 20, 45$.

21. $pv = 40$; $p = 4, 25$; $v = 13\frac{1}{3}, 16$. **22.** $y = \frac{1}{4}x^2$; $x = \pm 5, y = 9$.

23. $x^2y = 288$; $x = \pm 5$; $y = 18, 2·88$.

24. $cd = 100$; $c = 10$; $d = 16\frac{2}{3}, 8\frac{1}{3}$.

EXERCISE 98 (Pp. 403, 404, 405)

1. 81, $\pm\frac{1}{3}$. **2.** 3·75. **3.** $6y=5x$, 3·75, 3. **4.** $4\frac{2}{7}$. **5.** 27.
6. 24. **7.** $a=1$; 6. **8.** 3·6. **9.** 50 c. ft. **10.** 6 sec.
11. At a distance of $\sqrt{3}$ ft. **12.** $101\frac{47}{53}=101·9$ cm. approx.
13. 12·8 lb. wt. **14.** £21 17s. 6d. **15.** 30 m.p.h., 12.
16. 1382·4 gallons. **17.** 0·226 sec. **18.** $156\frac{1}{4}$. **19.** 12 years.
20. 1·8. **21.** 25. **22.** $4000\times\frac{161}{6561}$ miles.
23. 192, 20 ft. **24.** 8 sec.

EXERCISE 99 (P. 408)

1. $AC=kB$. **2.** $X=kYZ^2$. **3.** $zy^2=kx^2$. **4.** $z=kx+lx^3$.
5. $xy^2z=k$. **6.** $a=k+lb+\frac{m}{c^2}$. **7.** $E=k+\frac{l}{n}$.
8. $HR=ktV^2$. **9.** $p=kav^2$. **10.** $h=kav^3$.
11. $C=kA+\frac{l}{D^2}$. **12.** $t\sqrt{h}=kl$. **13.** $z=kxy^2$.
14. $C=khr$. **15.** $a=kc^3\sqrt{b}$. **16.** $t\sqrt{h}=kl$.
17. $RA=kL$. **18.** $z=kx+lx^3$.

EXERCISE 100 (Pp. 408, 409, 410, 411)

1. $\frac{128}{27}$, 1944. **2.** $\pm\frac{20}{3}$. **3.** 13. **4.** 6. **5.** $2\frac{1}{4}$.
6. 4. **7.** 47. **8.** 6. **11.** 15.
12. $V=\frac{2}{3}Ah$, $9\frac{3}{8}$. **13.** $10\frac{1}{3}$. **14.** $1\frac{1}{5}$ in. **15.** 4, -2. **16.** 16 or $\frac{1}{16}$.
17. $\sqrt[3]{44}=3·53$ cm. approx. **18.** 2s. 6d. **19.** About 13·5 in.
20. $18a$ shillings. **22.** £3 0s. 6d. **24.** 1·62 ft.
25. $\frac{9}{16}$ in. **26.** $\frac{189}{16}$. **27.** 2 ft., 224 ft. **28.** 84. **29.** 5 cm.

TEST PAPERS VIII (Pp. 411, 412, 413, 414, 415, 416, 417, 418)

A. 1. $\frac{2(b-c)}{2+bc}$. **2.** $x^9+2x^8+3x^7$; $8x^2+9x+10$.

3. (i) $\frac{8a^{\frac{3}{2}}}{b^{\frac{13}{8}}}$, (ii) $\frac{1728}{2197}$, $\frac{1}{10}$, $2^{\frac{1}{3}}$.

4. (i) 0·02213, (ii) 6·766, (iii) 2·185. **5.** (ii) 8 and $-1\frac{1}{2}$.

6. 15 min. 20 sec. approx.

B. 1. £$\frac{20x(2-x+y)}{x+y}$, £$\frac{20y(2-x+y)}{x+y}$. **2.** (i) $\frac{1}{x}$, (ii) 1.

3. (i) $\frac{2x-3}{x-2}$, (ii) 0. **4.** 0·7. **5.** (i) 13, (1); (ii) $-\frac{1}{3}$.

C. **1.** 1. **2.** (i) $\frac{57\sqrt{7}}{7}+\sqrt{5}$, (ii) $\sqrt{3}-\sqrt{2}$. **3.** $\frac{xy}{ab}$.

4. (i) $\frac{x+y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$, (ii) $\frac{2a^{\frac{1}{6}}b^2}{3}$. **5.** (i) 1·734, (ii) 116·2, (iii) 0·397.

6. $k=5$.

D. **1.** 2^{-4}, $2^{-1\cdot5}$, $2^{2\cdot5}$, (ii) 2072. **2.** 3, −1. **3.** (i) $71\sqrt{3}-57\sqrt{2}$, (ii) $\sqrt{3}$.

4. (i) $1\frac{5}{6}$. **5.** $x=20-4z$. **6.** (i) 5, $(-4\frac{1}{3})$; (ii) $\frac{1}{3}$ or $-\frac{1}{3}$.

E. **1.** (i) 3 : 4, (ii) 5 : 3 or −7 : 2.

2. (i) $x^{\frac{3}{5}}+2x^{\frac{2}{5}}-2x^{\frac{1}{5}}-2x^{-\frac{2}{5}}+x^{-\frac{3}{5}}$, (ii) $\sqrt[3]{\frac{ab^{\frac{15}{2}}}{c^9}}$.

3. (i) 1000, (ii) 12, (iii) $\frac{2}{15}$. **4.** 45 ft.

5. (i) 0·6053, (ii) $1\cdot069\div10^6$. **6.** (i) $3\frac{8}{13}$, (ii) ±3, $\pm5\frac{1}{2}$, ±7.

F. **2.** $\frac{-2ab}{(a+b)(a^2+b^2)}$. **3.** (i) $\frac{\sqrt[3]{a+2b}}{\sqrt{a-2b}}$, (ii) 1.

4. (i) 2·672 (or 3), (ii) 0·9914 (or 5). **5.** (i) 18, (ii) 4.

6. $a=1\cdot25$, $n=0\cdot7$.

G. **1.** 2. **3.** (i) $1\frac{1}{4}$, (ii) 1 ; $p=x-y$.

4. (i) 0·1346, (ii) 0·8094 (or 5). **5.** £6 14s.

6. $y=10x^{\frac{5}{2}}$, 2430.

H. **2.** (i) a, (ii) $\frac{247+7\sqrt{5}-66\sqrt{11}}{22}$.

3. (i) $a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{2}}c^{-1}+b^{\frac{1}{2}}c^{-1}$, (ii) $x+2x^{\frac{1}{2}}y^{\frac{1}{3}}-3y^{\frac{2}{3}}$. **4.** 100 ft.

5. 19. **6.** (i) 0·80, −0·96 ; (ii) 0, −1.

I. **1.** 8d., 1s., £48 each time. **2.** 300. **3.** (i) 2700, (ii) −2.

4. (i) 0·3927, (ii) 3·789. **5.** (i) 1, (ii) 1, $-2-\sqrt{3}$.

J. **2.** $2x^{\frac{1}{7}}-1-x^{-\frac{1}{7}}$. **3.** −1. **4.** 1·5 per cent. decrease.

5. (i) 3·22, (ii) 2·63. **6.** 6 : 5 : 2.

K. **1.** 2. **3.** (i) $x^2-\frac{1}{y^2}$, (ii) $a^{\frac{4}{9}}+2a^{\frac{1}{9}}-a^{-\frac{1}{9}}$.

4. (i) 0·0008067 (or 8), (ii) 0·5756 (or 7). **5.** $\pm\frac{2}{3}$, ∓2, ±1.

6. $a=26\cdot5$, $n=-1\cdot32$.

L. **1.** (i) $ab+bc+ca$, (ii) $4x^{\frac{2}{5}}+11x^{\frac{1}{5}}+6$. **2.** $\frac{4\sqrt{5}+5}{11}$, 1·27.

3. $\frac{bc}{(a-c)(a+b)}$. **5.** 8·8 tons.

6. $t=273\left(\frac{v^2d}{45\cdot08\times p\times144}-1\right)$, 16·7.

EXERCISE 101 (Pp. 419, 420)

1. (i) 1, 4, 7, 22, $3r-2$; (ii) 11, 14, 17, 32, $3r+8$; (iii) 1, $\frac{1}{4}, \frac{1}{9}, \frac{1}{64}, \frac{1}{r^2}$;
(iv) $\frac{5}{7}, \frac{7}{10}, \frac{9}{13}, \frac{19}{28}, \frac{2r+3}{3r+4}$; (v) 0, 2, 6, 56, r^2-r ;
(vi) 1, $\frac{1}{3}, \frac{1}{5}, \frac{1}{15}, \frac{1}{2r-1}$; (vii) 7, 14, 28, 896, $7\times 2^{r-1}$;
(viii) $-1, 1, -1, 1, (-1)^r$; (ix) 14, 10, 6, -14, $18-4r$;
(x) 0, $3\frac{1}{2}$, $8\frac{3}{4}$, $2187\frac{1}{128}$, $3^{r-1}-(-\frac{1}{2})^{r-1}$.

2. (i) 10, 13, 16, 19, $10+3(n-1)$; (ii) 10, 6, 2, -2, $10-4(n-1)$;
(iii) 256, 128, 64, 32, $256\div 2^{n-1}$; (iv) $\frac{1}{27}$, $-\frac{1}{9}$, $\frac{1}{3}$, -1, $\frac{1}{27}\times(-3)^{n-1}$.

3. (i) -2, 1 ; (ii) 10, 8 ; (iii) $\frac{9}{11}$, $\frac{11}{13}$; (iv) 10×13, 12×15.

4. (i) 45, (ii) 18, (iii) $\frac{5}{45}$ (or $\frac{1}{9}$), (iv) 36.

5. (i) $19+3n$, (ii) $n+\frac{1}{n+2}$, (iii) $\frac{1}{4}\times(-2)^{n-1}$, (iv) $n\div 10^n$.

6. (i) (*a*) Yes, the 12th, (*b*) No ; (ii) the 8th.

7. (i) 5, (ii) 40. **8.** (i) 3, (ii) 3.

EXERCISE 102 (P. 423)

1. (i) 4 ; (ii) -3 ; (v) $3x$; (vi) -2 ; (vii) $c-d$.

2. (i) 59, 99, $8n+3$; (ii) 22, -43, $113-13n$; (iii) -35, -70, $14-7n$;
(iv) -74, -154, $38-16n$; (v) $-25p$, $-55p$, $17p-6np$;
(vi) $25b^2-5a^2$, $45b^2-10a^2$, $b^2(4n-3)-a^2(n-2)$;
(vii) $20y-35x$, $30y-70x$, $2y(n+3)-7x(n-2)$;
(viii) $18m-3l$, $33m-8l$, $3m(n-1)-l(n-4)$.

3. (i) -12, -44, $16-12n$; (ii) 12, 15, $12n-45$;
(iii) $6p^2$, $29p^2$, $(6n-1)p^2$; (iv) $4x+2y$, $16x+5y$, $4(n-1)x+(2n-5)y$.

4. (i) 54, 61, 68 ; (ii) 92, 87, 82 ; (iii) -70, -66, -62 ;
(iv) -32, -35, -38.

5. (i) 26, (ii) 16, (iii) 21, (iv) 41, (v) 13, (vi) 71.

6. (i) 235, (ii) $-8{\cdot}5$, (iii) $3a^2-4b^2$, (iv) 0. **7.** 14, 17, 20, 23, 26.

8. 31, 27, ... -29, -33. **9.** -39, -36, ... -21, -18.

10. $a-b$, $a-2b$, ... $a-2kb$. **11.** $3n+19$. **12.** 14, $22n+20$.

EXERCISE 102 C (P. 424)

1. (i) 41, (ii) 23, (iii) 33, (iv) 15. **2.** -5, 3. **3.** 2, 8, 14, 20.

4. 25, $-0{\cdot}6$. **5.** $\frac{2x(c-b)+3y(a-c)}{a-b}$. **6.** 7.

8. $4a-3b$, $3a-b$, $2a+b$, $a+3b$. **9.** -2, 2, 6, 10, 14.

10. 35, −3. **11.** $\frac{ar+b}{r+1}, \frac{a+br}{r+1}$.

12. (i) 8, 14, 17, (ii) −3, 1, 9, (iii) 5, 1, −7, −15, (iv) 24, 20, 16, 8, 4.

14. 44, 53, 62, 71, 80. **15.** 15, 27, 39. **16.** 3, 5, 7, 9.

EXERCISE 103 (Pp. 427, 428, 429)

1. 52, 567. **2.** 250, 10541. **3.** $41\frac{1}{3}, 533\frac{1}{3}$. **4.** $8\frac{1}{4}, 276\frac{1}{4}$.

5. $-4, -76\frac{1}{2}$. **6.** $\frac{n+4}{20}, \frac{n(n+9)}{40}$. **7.** −1200.

8. −2780. **9.** 210. **10.** 2675. **11.** $192\frac{1}{2}$.

12. $\frac{n(9n-37)}{8}$. **13.** $na+(n^2-n)x$. **14.** $48b+480$.

15. 6 or 7. **16.** 35. **17.** 73. **18.** 13 or 20.

19. −26400. **20.** £3399. **21.** 76 (495 must be rejected).

22. $\frac{3x}{2}[a(3x+23)-b(12x-24)]$. **23.** 167 ; $\frac{14}{83}$. **24.** 2109.

25. $\frac{n}{2}(5n-1)$. **26.** 114, −2·4, 2764·8. **27.** 16 years old.

28. 212·5 ft. **29.** 1732. **30.** 2. **31.** $\frac{2n}{3n-1}$. **32.** $\frac{n(a+b)}{4}$.

33. (i) $4\frac{4}{5}$, (ii) $-4\frac{4}{11}$, (iii) $\frac{2}{a+b}$. **34.** (i) $-1, \frac{1}{4}, \frac{1}{9}$, (ii) $\frac{6}{19}, \frac{6}{17}, \frac{2}{5}$.

35. $3, 3\frac{3}{4}, 5, 7\frac{1}{2}$. **36.** (i) $\frac{8}{13}, \frac{8}{n+7}$, (ii) $3\frac{1}{2}, \frac{7}{50-8n}$. **37.** $\frac{3}{29}$.

38. $\frac{1}{13}$. **39.** 70. **40.** $1, 4, -2, -\frac{4}{5}, -\frac{1}{2}$; $-45\frac{1}{2}$.

EXERCISE 103 C (Pp. 429, 430, 431)

1. 43, 76, 3268. **2.** 16, 549. **3.** 116. **4.** $367\frac{1}{3}$.

5. 10, 165. **6.** 145, 15 and 16. **8.** 500 ; 25, 250.

9. 17258. **10.** $3, 1\frac{1}{2}$. **11.** 3500. **12.** £8625. **13.** 19.

15. 49 years. **16.** 461 yards, nearly. **17.** 21. **18.** 120. **19.** 36.

20. £3528, £3390, the first makes the better bargain by £$\frac{11n+n^2}{2}$ in n years.

21. 23, 335·8, 653·2. **22.** 570 yards. **23.** 12. **24.** 20.

25. $4n-1$, $n(2n+1)$, 250th term is 1 less than 1000.

26. $(2n+1)(a+nb)$; the (n^2+2n+1)th term is zero.

28. $v=\frac{3uy}{u+2y}, x=\frac{3uy}{2u+y}$.

Exercise 104 (Pp. 433, 434)

1. (i) $\frac{1}{2}$; (ii) $-\frac{1}{3}$; (v) ax; (vi) $-\frac{1}{y^2}$.

2. (i) $\frac{7}{9}$, $\frac{7}{243}$, $\frac{7}{3^{n-3}}$; (ii) $5\frac{1}{2}$, $-\frac{11}{16}$, $\frac{(-1)^{n-1}11}{2^{n-4}}$; (iii) $\frac{3}{4}$, $2\frac{17}{32}$, $\frac{3^{n-4}}{2^{n-3}}$; (iv) $\frac{1}{9}$, $-\frac{8}{243}$, $\frac{(-1)^{n-1}2^{n-5}}{3^{n-3}}$; (v) $\frac{1}{4}$, $-\frac{1}{32}$, $\frac{(-1)^{n-1}}{2^{n-3}}$; (vi) $\frac{x^{12}}{a^4}$, $\frac{x^{18}}{a^{10}}$, $\frac{x^{2n+2}}{x^{2n-6}}$.

3. (i) $-\frac{1}{3}$, $-\frac{11}{243}$, $\frac{(-1)^{n-1}.11}{3^{n-1}}$, (ii) $\frac{1}{2}$, $\frac{1}{40}$, $\frac{1}{5 \cdot 2^{n-3}}$, (iii) $-\frac{1}{x^2}$, $-\frac{1}{x^9}$; $\frac{(-1)^{n-1}}{x^{2n-3}}$, (iv) $-y^2$, $-xy^{10}$, $(-1)^{n-1}xy^{2n-2}$.

4. (i) 512, 256, 128: (ii) $-\frac{1}{64}$, $\frac{1}{32}$, $-\frac{1}{16}$; (iii) 9, 3, 1, or -9, 3, -1.

5. (i) 9, (ii) 11. **6.** (i) 12, (ii) $10\sqrt{6}$, (iii) x^4y^3.

7. ± 9, $\frac{9}{2}$, $\pm\frac{9}{4}$, $\frac{9}{8}$, $\pm\frac{9}{16}$. **8.** -14, $\frac{28}{3}$, $-\frac{56}{9}$, $\frac{112}{27}$.

9. ± 70, 140, ± 280. **10.** $3 \cdot 2^{n-1}$. **11.** $\frac{1}{2^{n-6}}$, or $\frac{(-1)^{n-1}}{2^{n-6}}$.

12. $4\frac{6}{25}$.

Exercise 104 c (Pp. 434, 435)

1. 2, 3. **2.** $\frac{1024}{2187}$. **3.** $2\frac{10}{27}$. **4.** 15, 45.

5. $\pm 27x^{11}$, $9a^2x^7$, $\pm 3a^4x^3$, a^6x^{-1}, $\pm\frac{1}{3}a^8x^{-5}$. **6.** $2\frac{1}{2}$ or $\frac{2}{5}$.

7. $a=4$, $r=1\frac{1}{2}$, or $a=25$, $r=-\frac{3}{5}$. **8.** 1, 0.2.

9. 12, -8, $5\frac{1}{3}$, or 12, -4, $1\frac{1}{3}$. **10.** $\frac{1}{8}$, -2. **11.** $17\frac{7}{9}$.

12. 135, $-\frac{1}{3}$. **13.** $\frac{1}{8}$.

14. 40, ± 20, 10, ± 5, 2·5, taken in either order. **15.** $\pm\frac{1}{8}$.

16. C.R. $=\sqrt[7]{11}=1{\cdot}41$. $T_1=\frac{1}{22}$. **17.** 4.

18. 3, 12, 21, or 63, 12, -39. **20.** 12, 108.

Exercise 105 (Pp. 436, 437)

1. $227\frac{1}{2}$. **2.** $3\frac{681}{2048}$. **3.** $3\frac{31}{144}$. **4.** $1\frac{2933}{11264}$. **5.** $622\frac{58}{81}$.

6. $112\left(1-\frac{1}{2^n}\right)$. **7.** $20\frac{20}{81}$. **8.** $781\frac{31}{125}$. **9.** $\frac{1}{36}[1-(-3)^n]$.

10. $\frac{a^3}{a-b}\left(1-\frac{b^n}{a^n}\right)$. **11.** 6. **12.** 5. **13.** 9. **14.** 6.

15. $8\frac{17}{81}$. **16.** $116\frac{41}{54}$. **17.** 2049. **18.** $\frac{1}{2}$, 4, $8\left(1-\frac{1}{2^{15}}\right)$.

19. $\frac{32}{81}$, $9\left(1-\frac{2^{20}}{3^{20}}\right)$.

EXERCISE 106 (Pp. 443, 444, 445)

1. $20[1{\cdot}05^n - 1]$, 47. **2.** $\frac{2^9}{3^8}$, 0·0780. **3.** $10[1{\cdot}2^n - 1]$, 30.

4. The 15th. **5.** 29·778, 14·134.

7. £1041 (4-fig.), £1042 (7-fig.). **8.** £961 (4-fig.), £962 (7-fig.).

9. £587 (4-fig.), £586 (7-fig.). **10.** £1833 (4-fig.), £1837 (7-fig.).

11. 200 ft. **12.** 244,030. **13.** 9. **14.** $13\frac{1}{2}$. **15.** 6·4.

16. $14\frac{2}{7}$. **17.** 15. **18.** 28, $-\frac{3}{4}$. **19.** $-64[1-(-\frac{1}{2})^n]$, 16.

20. $\frac{1}{3}$. **21.** 5 sec., an infinite time. **22.** £1,503,160, £2,000,000.

23. $\frac{8}{27}$. **24.** $\frac{284}{495}$. **25.** $5\frac{1}{6}$. **26.** $3\frac{379}{990}$. **27.** $\frac{5}{9}$.

28. $2\frac{178}{555}$. **29.** $\frac{3^n - 2^{n+1} + 1}{2}$. **30.** $\frac{7(7^n - 1)}{3} - \frac{n(5n+1)}{2}$.

31. $\frac{16(4^{2n} - 1)}{15} - \frac{3n(n+5)}{2}$. **32.** $n - \frac{2c(1-c^n)}{1-c} + \frac{c^2(1-c^{2n})}{1-c^2}$.

EXERCISE 106 C (Pp. 445, 446, 447, 448)

1. $\frac{7+n}{11+2n}$. **2.** 19 years; £1064. **3.** $\frac{a^{42} - x^{63}}{a^{35}x^8(a^2 - x^3)}$.

4. 9, $2n+7$, A.P. with C.D. 2. **5.** $\frac{1}{2}n(n+1)$, $\frac{1}{2}n(n-1)+1$.

6. 25, 28, 31, ... ; 32, −24, 18, ... ; $48\frac{89}{128}$.

8. 24, $2\frac{1}{4}$; 34th term $= 98\frac{1}{4}$, $2078\frac{1}{4}$. **9.** $-\frac{1}{3}$.

10. C.D. $=\frac{a}{8}$, C.R. $=\frac{3}{2}$, 20th term of A.P. = 4th term of G.P.

11. $\frac{2}{3}$, $\frac{2^n}{3^n}$, a G.P. with C.R. $\frac{2}{3}$. **12.** $\frac{n(3n+1)}{2}$. **13.** $H = \frac{1+A}{3-A}$.

17. 3. **18.** $a=80$, $r=\frac{1}{2}$. **19.** $\frac{4}{3}\log 3$, $\frac{5}{3}\log 3$.

23. From 27 to 113 inclusive.

24. £$(8000-600n)$; $4400\left(\frac{7}{6}\right)^{n-6}$; the 13th.

25. $a=1$, $r=-2$; $a=9$, $r=\frac{2}{3}$; $a=-7\mp 2\sqrt{6}$, $r=\frac{2}{5}(\pm\sqrt{6}-1)$.

27. $4r^2 - 1$.

28. 113, an A.P., C.R. 10, with the exception of the 1st term.

29. $n+1$. **30.** $-\frac{1}{2}$ or 4; $-81\frac{2}{3}$, $773\frac{1}{3}$.

31. $na^2 + 2abc\left(\frac{1-c^n}{1-c}\right) + b^2c^2\left(\frac{1-c^{2n}}{1-c^2}\right)$. **32.** 27.

33. (i) £1197·5 (4-fig.), £1200 (7-fig.); (ii) £1203 (4-fig. and 7-fig.).
34. $a=42$, $r=\frac{1}{7}$ or $a=7$, $r=\frac{6}{7}$. **35.** $8(2+\sqrt{3})=29{\cdot}86$.
36. $2n(n+11)-\dfrac{3a(a^n-1)}{(a-1)}$. **37.** £7500. **38.** 3861.
39. 241 ft., 67. **40.** 26870. **41.** $3\left\{1-\dfrac{2^n}{3^n}\right\}$, 9.
42. $n^2-19n+130$; $n=7$ ($n=12$ must be rejected). **43.** 112·5 sq. ft.
44. £2159 (4-fig.); £2158 (7-fig.).
45. £17,860 (4-fig,); £17,850 (7-fig.).
46. £2613 (4-fig.); £2607 (7-fig.). **47.** £661 (4-fig.); £660 (7-fig.).
48. 2, 15.

Exercise 107 (Pp. 453, 454, 455)

1 and **2.** Rational and unequal. **3.** Real, but irrational.
4. Rational and equal. **5.** Imaginary.
6. Equal in magnitude but opposite in sign; irrational.
7. Real and irrational, (ii) unreal.
8. (i) $1\frac{1}{24}$, (ii) ± 18, (iii) $3, \frac{1}{3}$, (iv) $\frac{3}{5}, -\frac{3}{7}$.
9. (i) $-\frac{4}{81}$, (ii) $-\frac{31}{47}$, (iii) $\dfrac{2a+3}{6a-2}$, (iv) $3l$. **10.** $b=0$ or $b^2 \nless 4ac$. **11.** 36.
13. $5\frac{7}{9}$; $3x^2-26x+3=0$. **14.** 35·2. **15.** $x^2-18x+54=0$.
16. $7x^2-30x+31=0$. **17.** 52, 2702. **18.** $25x^2+234x+676=0$.
19. (i) $\dfrac{b^2-2ac}{a^2}$, (ii) $\dfrac{c(b^2-2ac)}{a^3}$, (iii) $\dfrac{5b^2-13ac}{a^2}$, (iv) $\dfrac{\sqrt{b^2-4ac}}{a}$,
(v) $\dfrac{b^4-4ab^2c+2a^2c^2}{a^4}$, (vi) $\dfrac{b^4-4ab^2c+2a^2c^2}{a^2c^2}$.
21. $qx^2-p(q+1)x+(q+1)^2=0$.
22. $acx^2+2b(2c+a)x+(2c+a)^2=0$. **23.** $\dfrac{a^3+b^3+c^3-3abc}{a^3}$.
24. $k=a(c-a)$, $l=c(a-2c)$. **25.** ± 2. **26.** 17·5.
27. (i) 23, (ii) $-12\frac{6}{7}$. **28.** ± 3. **29.** $k=8$, $l=3$. **30.** 150, -5141.
31. $x^2+52x+1=0$. **32.** $x^2+3px+2p^2+q=0$. **33.** $p=1$, $q=6\frac{11}{16}$.
34. $3x^2-14x+5=0$. **35.** (i) ac, (ii) c^2, (iii) $\dfrac{b^2-2ac}{a^2c^2}$.

Exercise 108 (Pp. 461, 462)

1. Between -5 and 4. **2.** Between $2\frac{1}{2}$ and 6. **3.** None.
4. Between -2 and 5. **5.** None. **6.** Between -7 and $\frac{3}{5}$.
7. (i) $3\frac{1}{8}$, (ii) $4\frac{1}{12}$. **8.** (i) $6\frac{1}{8}$, (ii) 1. **9.** (i) $-5\frac{9}{20}$, (ii) $-\frac{49}{80}$.
10. $10\frac{1}{12}$. **11.** $-3\frac{1}{8}$. **12.** $3\frac{2}{3}$. **16.** $\frac{1}{2}$ and $-\frac{1}{6}$.
17. -1 and 3. **18.** (1, 1); ($1\frac{2}{3}$, -3). **19.** 2 and -3.

EXERCISE 109 (P. 464)

1. $1, -3, \frac{-1\pm\sqrt{-3}}{2}, \frac{3(1\pm\sqrt{-3})}{2}$. **2.** $\pm 4, \pm 5$. **3.** $\pm 3. \pm 4$.

4. $\pm 2, \pm 3$. **5.** $\pm 2, \pm 0\cdot 5$. **6.** $1, 3, \frac{-1\pm\sqrt{-3}}{2}, \frac{3(-1\pm\sqrt{-3})}{2}$.

7. 9 or $\frac{1}{9}$. **8.** 18 or $\frac{2}{25}$. **9.** 4^{2n} or 5^{2n}. **10.** $\frac{1}{6}$ or $-\frac{1}{4}$.

11. $\pm\frac{9\sqrt{3}}{32}$ or $\pm\frac{4\sqrt{6}}{27}$. **12.** 4^n or 3^n. **13.** $4, -2, 1\pm\sqrt{71}$.

14. $-2, -2, \frac{-4\pm\sqrt{10}}{2}$. **15.** $-1, -2, \frac{-3\pm\sqrt{105}}{2}$.

16. $3, -4\frac{1}{2}, \frac{-3\pm\sqrt{65}}{4}$. **17.** $-3, 1\frac{1}{2}, \frac{-3\pm\sqrt{-47}}{4}$.

18. $6, -10, -2\pm 2\sqrt{-6}$. **19.** $1, 1, \frac{-3\pm\sqrt{5}}{2}$.

20. $3, \frac{1}{3}, \frac{-1\pm\sqrt{-35}}{6}$. **21.** $1, 1, \frac{1\pm\sqrt{-15}}{4}$. **22.** $2, \frac{1}{2}, 1\frac{1}{2}, \frac{2}{3}$.

23. $3, \frac{1}{3}, \frac{-1\pm 2\sqrt{-2}}{3}$. **24.** $2\pm\sqrt{3}, \frac{-1\pm\sqrt{-3}}{2}$.

25. $-1, -1, -1, 1, 1$. **26.** $2, -\frac{1}{2}, 4, -\frac{1}{4}$. **27.** $2, -\frac{1}{2}, 5, -\frac{1}{5}$.

28. $-1, 1, 1, \frac{-3\pm\sqrt{5}}{2}$. **29.** $2\frac{1}{2}, \frac{1}{2}, \frac{3\pm 2\sqrt{-7}}{2}$.

30. $0, -4, -2\pm\sqrt{-10}$. **31.** $3, 2, \frac{5\pm\sqrt{-7}}{2}$. **32.** $\frac{1}{2}, -\frac{1}{2}, \frac{\pm\sqrt{-7}}{2}$.

33. $-3, -3$. **34.** $-1, -4$. **35.** $0, 3a, -4a, 7a$. **36.** $0, \frac{126b}{65}$.

EXERCISF 110 (P. 466)

1. 19. **2.** 1. **3.** -47. **4.** 13. **5.** 36. **6.** 17.
7. $3a^2$. **8.** $3ka^2$.
9. (i) $12a^2$, (ii) $6a$, (iii) 5, (iv) $12a^2+6a+5$; (iv) = (i) + (ii) + (iii).
10. (i) $4a$, (ii) -5, (iii) 0, (iv) $4a-5$; (iv) = (i) + (ii) + (iii).

EXERCISE 111 (Pp. 467, 468)

1. $5a^4$. **2.** $8a^7$. **3.** $10a$. **4.** $48a^5$. **5.** $9a^2-2$. **6.** $8a-1$.
7. $16a^3$. **8.** $8a-4$. **9.** $2a^{-\frac{1}{2}}$. **10.** $2a+3a^{-2}$. **11.** a^{-2}.
12. $-10a^{-3}-6a^{-4}$. **13.** $9a^2-5$. **14.** $8a^3+6a$.

15. $6a^2+4a+9$. **16.** $12a+12a^{-3}$. **17.** $9a^2-16a+7-5a^{-2}$.
18. $81a^2+54a+9$. **19.** $160a^9$. **20.** $-6a^{-4}+28a^{-5}$.
21. (8, 7). **22.** (2, 8) and (3, 7); (1, 3) and (4, 12). **24.** 1, $-\frac{2}{3}$, 2.

Exercise 112 (Pp. 472, 473, 474)

1. Min. -2. **2.** Max. $9\frac{1}{12}$. **3.** Max. 7. **4.** No max.; no min.
5. Max. 0; min. -1. **6.** Min. 9; max. 10; min. 9.
7. Max. 14; min. -13. **8.** Max. $3\frac{1}{2}$; min. $-12\frac{1}{2}$.
9. Max. $2\frac{4}{27}$; min. 2. **10.** Max. 53; min. -72.
11. Max. $\dfrac{4a^3}{27}$; min. 0. **12.** Max. 3456; min. 0.
13. 125 cu. in. **14.** $AP=PB=4''$. **15.** $\sqrt{3}\div 3=0{\cdot}577$. **17.** $x=8, y=4$.
18. 4″ from point of suspension. **19.** 614 c.c. **20.** 2·5 sq. in.
21. $9''\times 4{\cdot}5''\times 6''$. **22.** 4″. **23.** 20″, 20″, 10″. **24.** 2.
25. 1·1. **26.** 10″, 10″, 11″. **27.** 2 cu. ft.; 2·55 cu. ft.
28. 0·86 ft. **29.** 156·25 sq. ft. **31.** $r=3''$, $h=6{\cdot}5''$. **32.** 12·6 knots.

Exercise 112 c (Pp. 474, 475)

1. 1·25 cm., 3·75 cm. **2.** Mid-point of AB.
3. $x^2+a^2+(x-b)^2+c^2$; $\left(\dfrac{b}{2}, 0\right)$. **6.** $\dfrac{s}{x}(a+bx^3)$.
10. Max. $b+4a^5$; min. $b-4a^5$. **11.** (i) At C, (ii) 2·5 ft. from B.
12. 4 ft. from the position of lesser C.P.

Exercise 113 (Pp. 478, 479, 480)

1. $-1{\cdot}75$, 0·55, 4·2. **2.** $-3{\cdot}9$, 1·1, 2·8. **3.** $-1{\cdot}13$, $-0{\cdot}26$, 1·65.
4. $-0{\cdot}5$. **5.** 1·9. **6.** 3·84, 0·56, $-1{\cdot}40$. **7.** 2·6.
8. 3·5; $2x^3-12x^2+21x-12=0$. **9.** y lies between 6 and -10.
10. From 0 to 2·87. **11.** 2·9; $-2{\cdot}8$, 0, 2·8. **12.** 2·5.
13. From $x=0$ to 1, and when $x<-1$. **14.** 1. Yes, when $x=5$.
15. 11·95. Read off the values of x where $y=a$. 2·3, 6·1, $-1{\cdot}6$.
16. 0·08 to 1·89. **17.** 2·5. **18.** 10, 1·4.
19. $\frac{1}{5}(x^3-4x-8)$, 2·65.
20. (1) $x=\pm 2$ give the same min. value $-4\frac{1}{2}$. (2) About axis of y. (3) $x=\pm 1$, ± 4. (4) (a) +, (b) +.
21. Max. $x=0$, $y=6$; min. $x=\pm\sqrt{2{\cdot}5}$, $y=-0{\cdot}25$.
22. $(3, -4)$, $(-1\frac{2}{5}, 4\frac{4}{5})$. **23.** $(1, -3)$, $(-\frac{39}{53}, 3\frac{4}{53})$.
24. $(0, 3)$, $(1\frac{4}{5}, -2\frac{2}{5})$.

TEST PAPERS IX (Pp. 480, 481, 482, 483, 484, 485, 486)

A. 2. (i) $2a+3b$, (ii) $\left(\frac{2a-2}{2a-1}\right)^3$. **3.** (i) 0·674, (ii) 16·1.

4. 3, $2\frac{1}{2}$. **5.** $16x^8$. **6.** $\frac{64}{x^2}$ in., $\left(2x^2+\frac{256}{x}\right)$ sq. in.

B. 1. -80. **2.** -190. **3.** (i) 18 lb., (ii) 2·4 cm.
4. $p+q$, $1-p$, $3p+q$, $1\frac{1}{2}$. **5.** ab sq. ft.

6. (i) $\frac{7\pm\sqrt{5}}{2}$, i.e. 4·62, 2·38. (The other roots must be rejected.), (ii) 3130 to 3 sig. figs.

C. 1. $\frac{a}{2a^2-1}$. **2.** (i) $\frac{5}{3}$, -1, (ii) $-\frac{1}{3}$, 1. **3.** 1·42. **4.** $n(10-n)$.
5. £468·559 ; £1000. **6.** 58·4 per cent.

D. 1. $2x^2+\frac{1}{2}$; $(\frac{1}{2}, -\frac{5}{6})$ or $(-\frac{1}{2}, -\frac{3}{2})$; two.
2. $b^2 \geqq 4ac$; a has the same sign as c, but b has the opposite sign.
3. (i) 2·50. **4.** (i) 11, (ii) £410. **5.** 19·83.
6. $1\frac{3}{4}$. The graphs touch.

E. 1. (i) $\frac{p^2-p+1}{p^2+p+1}$, (ii) $\frac{3a^3+6a^2+4a+2}{2a^3+4a^2+6a+3}$. **2.** $\frac{2a^3}{27}$. **3.** 49·37.
4. (i) -2, -2, $-2\pm\sqrt{15}$ (i.e. $-5·87$, 1·87) ; (ii) 13·4. **5.** 6864. **6.** 67.

F. 1. $x^2-x-20=0$, $x^2-12x+46=0$, $x^2+10x+35=0$.
2. (i) 404,250. **3.** £8 17s. **4.** 0·5224.
5. 0·6. **6.** $3x^2-8x+5$; 1, $1\frac{2}{3}$.

G. 1. $6+4x$. **2.** (i) 22722, (ii) 12 years. **3.** 1·983.
4. 0·002 too small ; 1·809 ; 0·691.

H. 1. $d=-1$, $r=-\frac{1}{2}$ or $d=-13$, $r=2\frac{1}{2}$.
2. $x=2^{\frac{3}{2}}y^{-\frac{9}{10}}z^{\frac{3}{4}}$, $y=2^{\frac{5}{3}}x^{-\frac{10}{9}}z^{\frac{5}{6}}$; 1·45. **4.** $x^2-27x+51=0$.
5. (i) $\frac{3\pm\sqrt{17}}{2}$ (i.e. 3·56, $-0·56$), (ii) 3. **6.** Rad. 2 ft., Ht. 1 ft.

I. 1. $\frac{ab(a+b)(b-a)}{(2a+b)(2b+a)}$. **2.** $\frac{100n(1-k)}{k(n-1)}$. **3.** 2 ; $\left(-\frac{2}{3}, 8\frac{4}{27}\right)$.
4. (i) 0·3357, (ii) 2·157. **5.** $\frac{2}{3}$. **6.** $n=\frac{8x^3}{y^2}-xy^2$.

J. 1. (ii) $x=\frac{1}{2}$, $y=\pm5$ or ±3, $z=\mp3$, ∓5. **2.** (i) 2·718, (ii) 9.
3. 1501 ; 752,000. **4.** $acx^2+2b(a+c)x+(a+c)^2=0$.
5. £62·3 ; nearly 33 years.

K. **1.** $x=a^{-\frac{1}{49}}z^{-\frac{6}{49}}$; $y=0{\cdot}498$; $z=12{\cdot}6$.
2. 8 hens per week and 48 during the last week. **3.** 6420.
4. $2^{\frac{n+1}{n}}$, the 15th. **5.** Max. $(5, \frac{1}{2})$; min. $(\frac{1}{5}, 12\frac{1}{2})$. **6.** 2·8.

L. **1.** No pt. on 2nd graph lies between the lines $y=-3\pm2\sqrt{2}$; no such restriction in 1st graph.
3. (i) $1, 1\frac{2}{21}, 1\frac{4}{21}, \ldots$, (ii) $\frac{1}{2}\{2-(\frac{1}{2})^n-(\frac{1}{3})^n\}$. **4.** 45 m.p.h.
5. (i) 1, −1·32, (ii) 0·4557 or −0·8614. **6.** (i) 1·245 ; 3 to 2.

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